Nyquist frequency on irregular samplings

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Statement of the problem

- Aliasing is a well known effect in frequency analysis
  - it is conspicuous in regular sampling
  - its evidence is less obvious with semi-regular sampling
    - depends on underlying quasi-regularity of gaps or observation clustering
  - there is no aliasing at all with random sampling
- Aliasing restricts the frequency range in harmonic analysis
  - with regular sampling : $\nu_{max} = 1/2\tau$ Nyquist frequency
- From experience : with irregular samplings much higher frequencies recoverable
  - empirical rules applied :
    - inverse of min interval
    - inverse of mean interval
    - no bound
  - key parameter for both the science return and runtime efficiency

Aliasing with regular sampling

Signal: $X(t)$ defined in the time domain

sampling : $X(t_k)$, t_1, t_2, … , t_n

Spectrum : $S(\nu)$ defined in the frequency domain

Power: $P(\nu)$ " " "

$S(\nu) = \Re(T, \exp[2i\pi\nu t])$ with $\Re$ linear operator and $P(\nu) = (S\bar{S})^{1/2}$

if $t_{k+1} - t_k = \tau \Rightarrow \exp(2i\pi\nu t) = \exp(2i\pi\nu t_0).\exp(2i\pi\nu\, k\tau)$

$$\exp(2i\pi(\nu + m/\tau)\, k\tau) = \exp(2i\pi\, km).\exp(2i\pi\nu\, k\tau) = \exp(2i\pi\nu\, k\tau) \quad \forall m \in Z$$

with $\eta = 1/\tau$ one has $\forall \nu$:

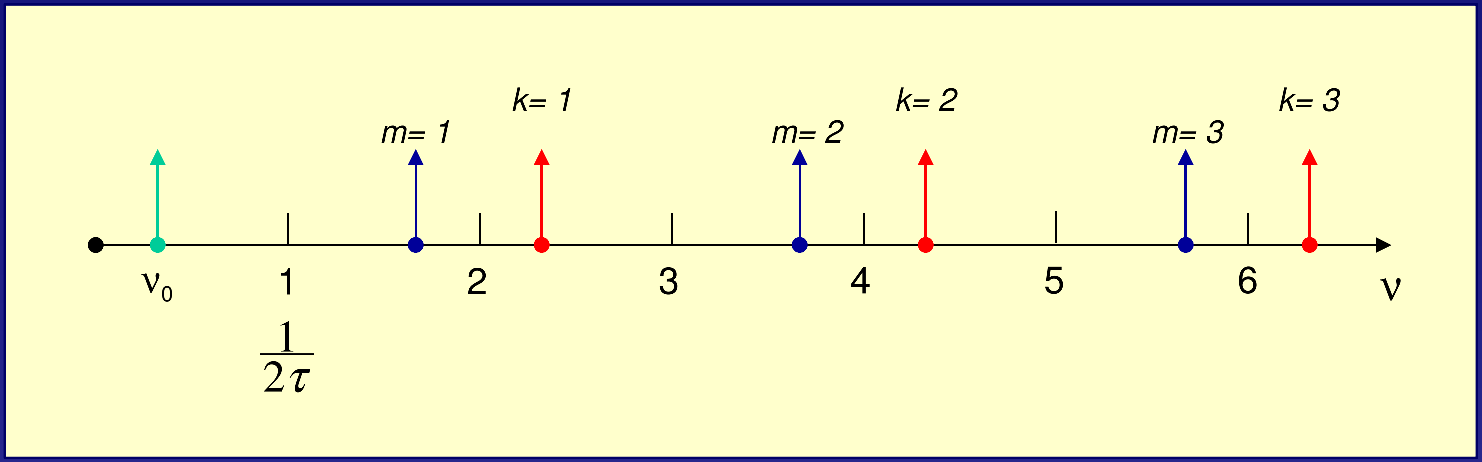
$S(\nu + \eta) = S(\nu)$; $P(\nu + \eta) = P(\nu)$

but : $P(\nu) = P(-\nu) \Rightarrow$ $P(\nu) = P(\eta - \nu)$: symmetry wrt $\eta/2$

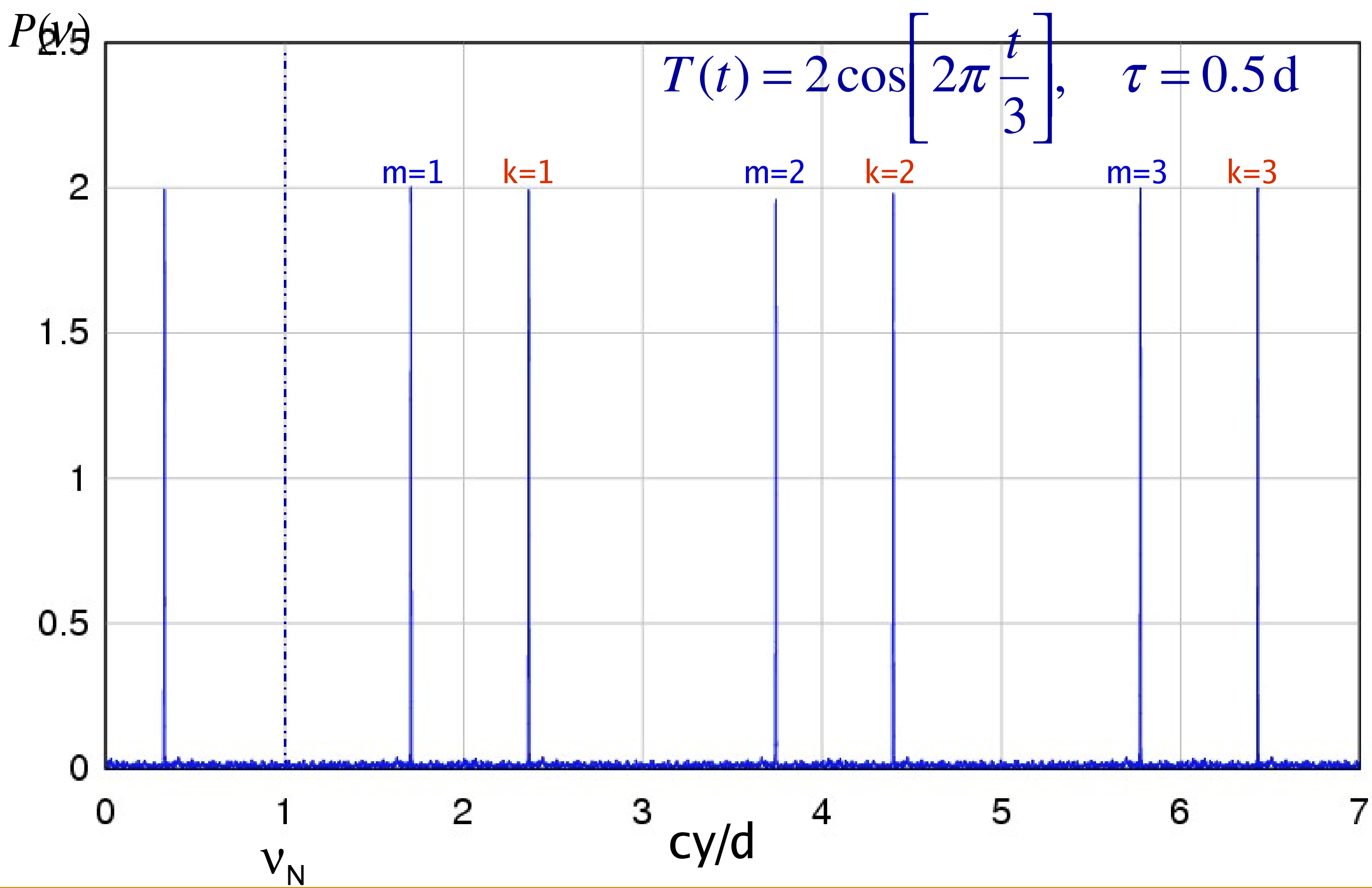
Alias pattern

- Any line in $0 < \nu < \eta/2$ is mirrored infinitely many times at :

$$\nu_+ = \nu + k\eta, \quad k = \cdots, -2, -1, 1, 2, \cdots$$
$$\nu_- = -\nu + m\eta, \quad m = \cdots, -2, -1, 1, 2, \cdots$$



Example with regular sampling



General case

- arbitrary sampling : $t_1, t_2, \ldots, t_n \text{ with } t_{k+1} - t_k = \tau_k$

- aliasing occurs if $P(\nu)$ is periodic

$$\exists\, \eta \;/\; 2\pi\nu t_k = 2\pi(\nu + \eta) t_k + \phi \quad \forall k$$

$$\eta \tau_k \equiv 0 \pmod 1, \quad k = 1, 2, \cdots, n-1$$

System of n-1 equations to be solved for η

No solution in general   no aliasing

Particular cases

- Regular sampling : $\tau_k = \tau$ $\eta\tau \equiv 0 \pmod 1$ $\eta = 1/\tau$ (or m/τ)

- Pseudo-regular sampling : $\tau_k = p_k\tau$, $k = 1, 2, \ldots, n-1$

  - selecting the the largest possible τ $(p_1, p_2, \ldots, p_{n-1}) = 1$

$$\eta \tau p_k \equiv 0 \pmod 1, \quad k = 1, 2, \cdots, n-1$$

$$\eta = \frac{1}{\tau}$$

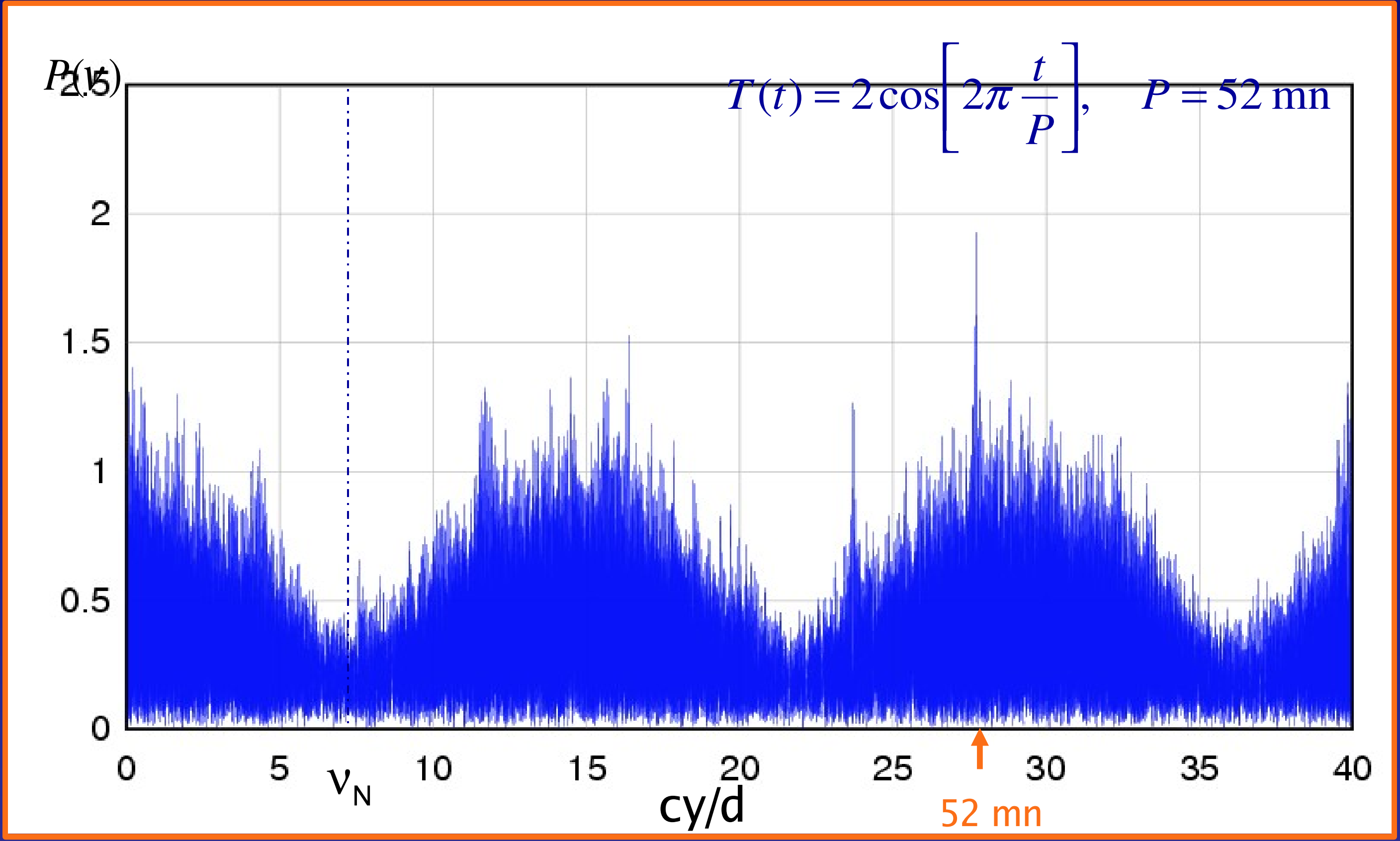
(see also : Eyer & Bartholdi, 1999)

τ could be << the smallest interval resulting in η very large

- Irregular sampling : diophantine approximations $q\tau_k/\tau_m \approx p_k$ $\eta \approx q/\tau_m$

Gaia astro sampling

- Smallest interval t = 100 mn $\nu_N \sim 7.2$ cy/d
  - but no aliasing visible
  - very small periods can be recovered



Conclusions

- Better understanding of the aliasing
- Further developments involve higher arithmetics
- Gaia period search puts on safer grounds

Details and proofs in :

F. Mignard, 2005, About the Nyquist Frequency, Gaia_FM_022

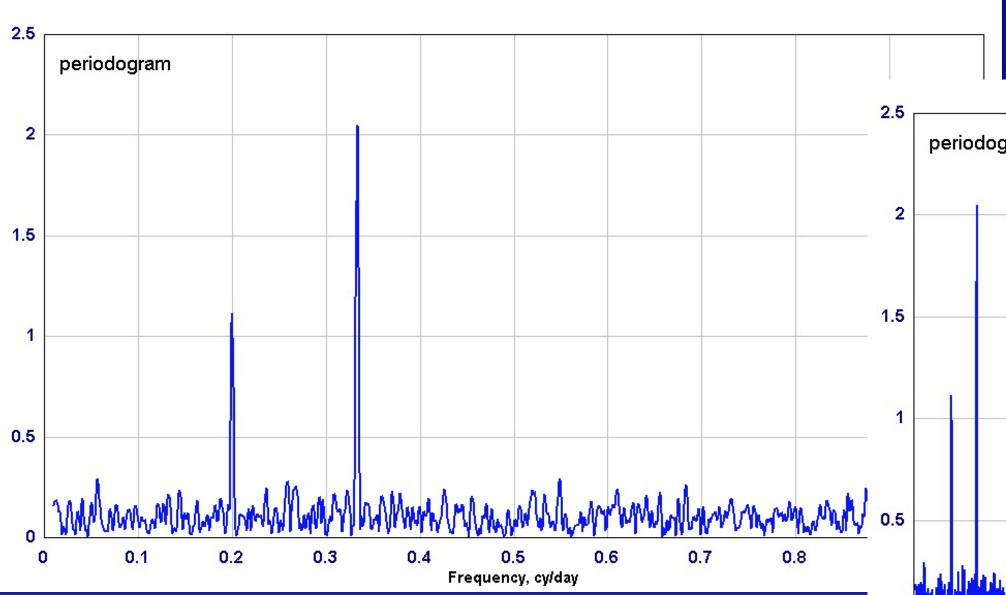
Examples : random sampling

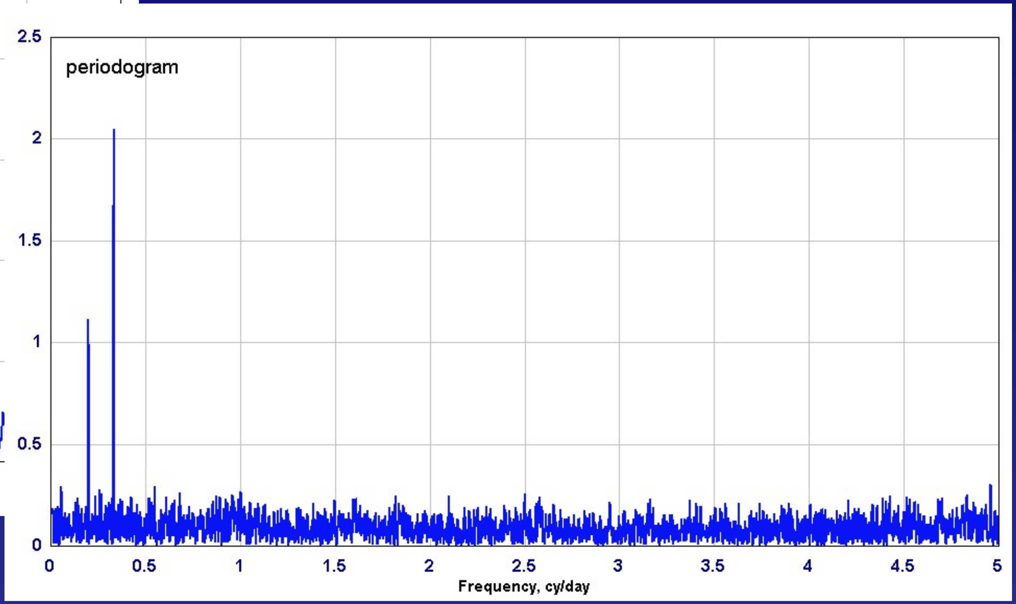
p_1 : 3 d
p_2 : 5 d

$$S(t_k) = 2\cos(2\pi t_k / P_1) + \cos(2\pi t_k / P_2)$$

$<\tau> = 0.5$ d $\sigma = 0.1$

Exponential waiting time sampling of 1000 data points over 500 days.





Periods and amplitudes found

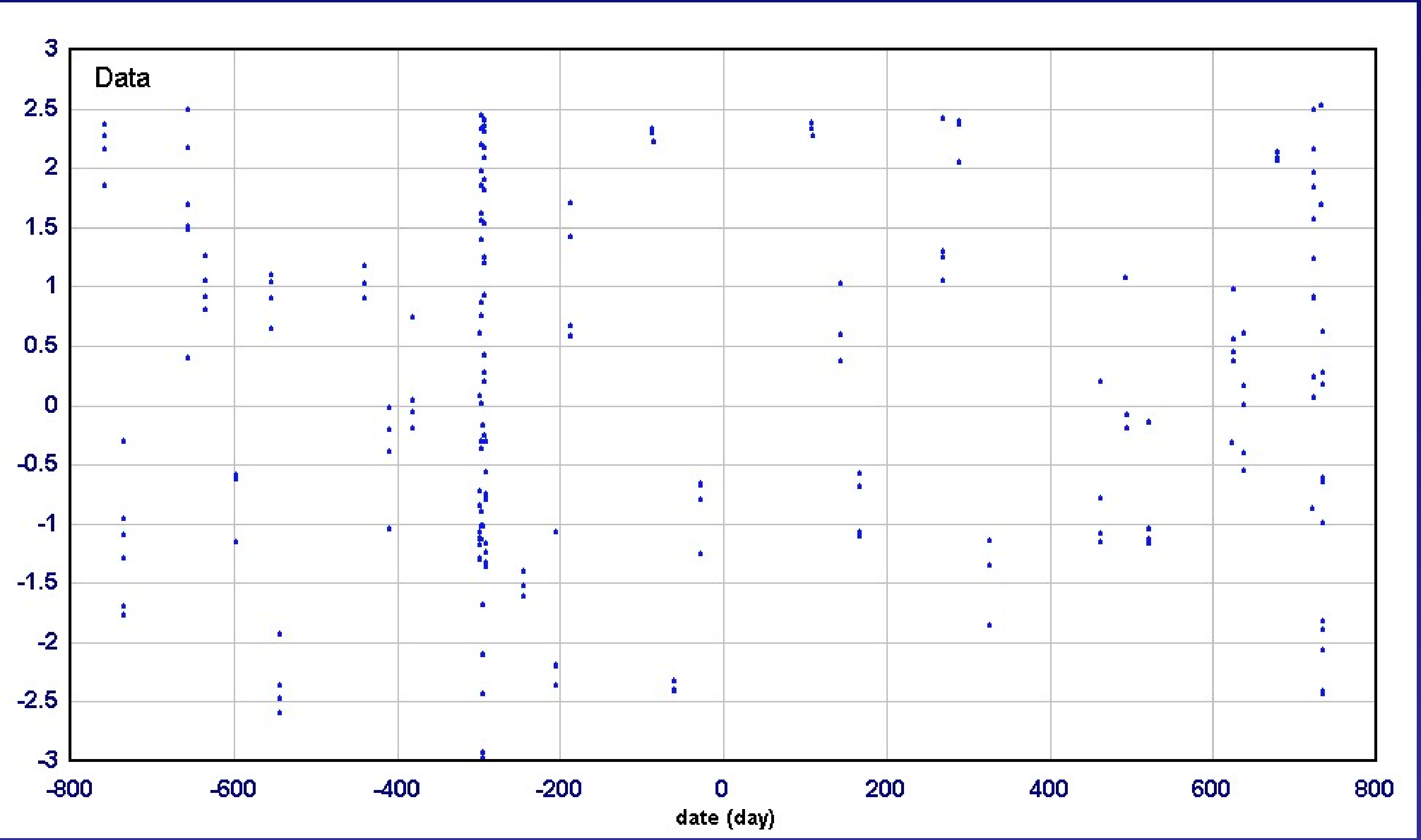
2.99999 +/- 0.00002	2.017 +/- 0.005
4.99994 +/- 0.0001	0.988 +/- 0.005

Examples : Gaia-like sampling

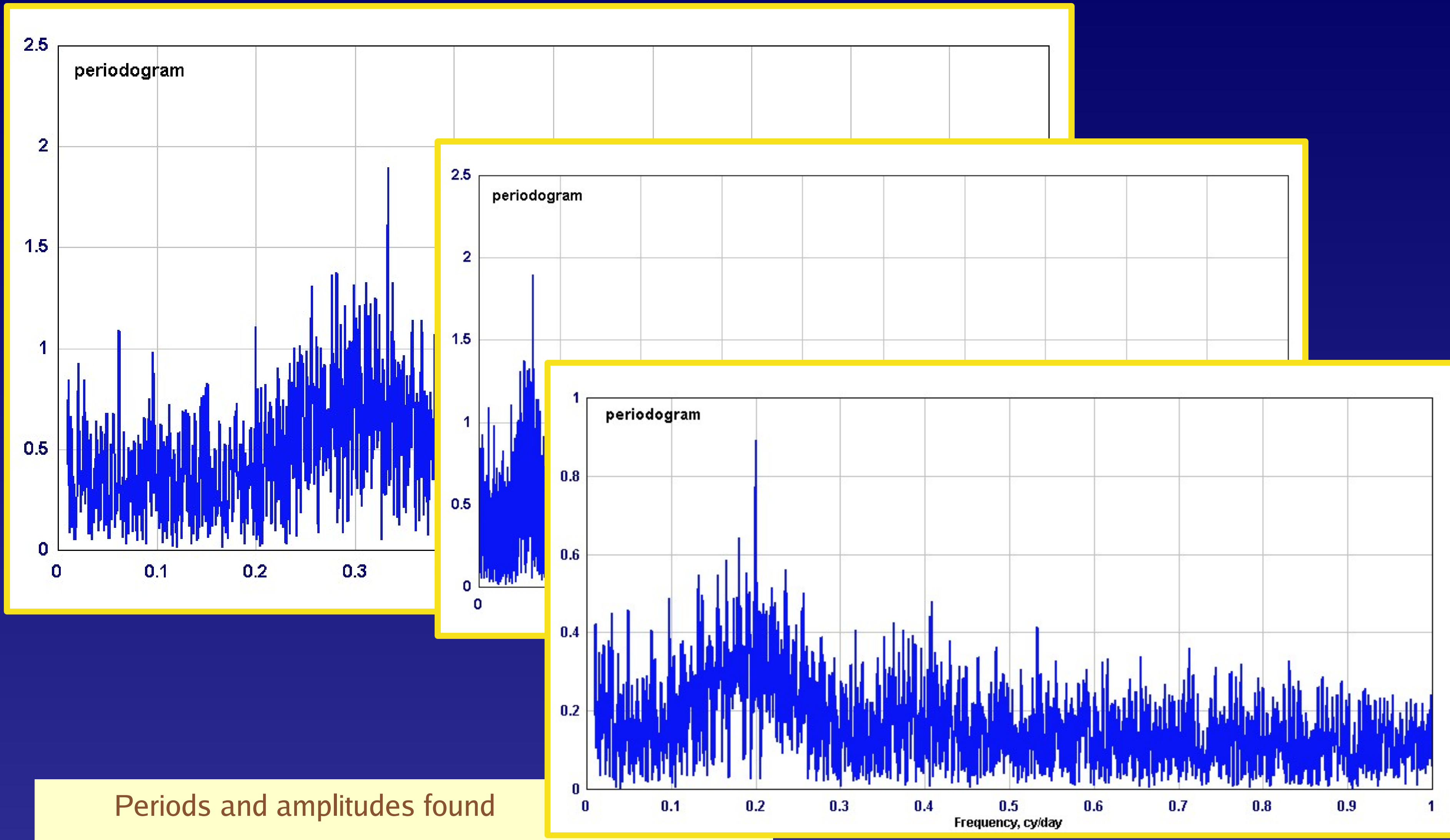
p_1 : 3 d
p_2 : 5 d

$$S(t_k) = 2\cos(2\pi t_k / P_1) + \cos(2\pi t_k / P_2)$$

$\sigma = 0.1$
220 samples over 1600 days.



Examples : Gaia-like sampling



Periods and amplitudes found

3.000023 +/- 0.000015	2.006 +/- 0.015
5.000036 +/- 0.00008	0.998 +/- 0.015