Kinetic Theory

- **Motivation - Relaxation Processes**
- **Violent Relaxation**
- **Thermodynamics of self-gravitating system**
  - **negative heat capacity**
  - **the gravothermal catastrophe**
- **The Fokker-Planck approximation**
  - **Master equation**
  - **Fokker-Planck equation**
- **Evolution of stellar systems**

Motivation

- Collision-less system when: *relaxation time is much larger than the age of the system*.
- If relaxation time is smaller than the age of the system, then collision are important for the equilibrium of the system, and the system is likely to evolve with time.

Relaxation Time

$$t_{relax} \sim \frac{0.1N}{\ln N} t_{cross}$$

- N: number of stars
- t_{cross}: time of crossing

- If t_{relax} is equal or smaller than the age of the system, collision will be important, and system will evolve.

System where collision is important

	N	t_{cross}	t_{relax}	Age
globular cluster	10^5,	10^5 yr	10^8 yr	10^{10}yr
open cluster	10^2,	10^6 yr	10^7 yr	10^8yr
galaxy center (1pc)	10^6,	10^4 yr	10^8 yr	10^{10}yr
cluster center	10^3,	10^9 yr	10^{10} yr	10^{10}yr

collision vs. collision-less system

- In a collision-less system: kinetic energy and angular momentum are conserved for every star.
- when collision is important, because of interaction kinetic energy and angular momentum of a star can change (exchange of energy and momentum with other stars).

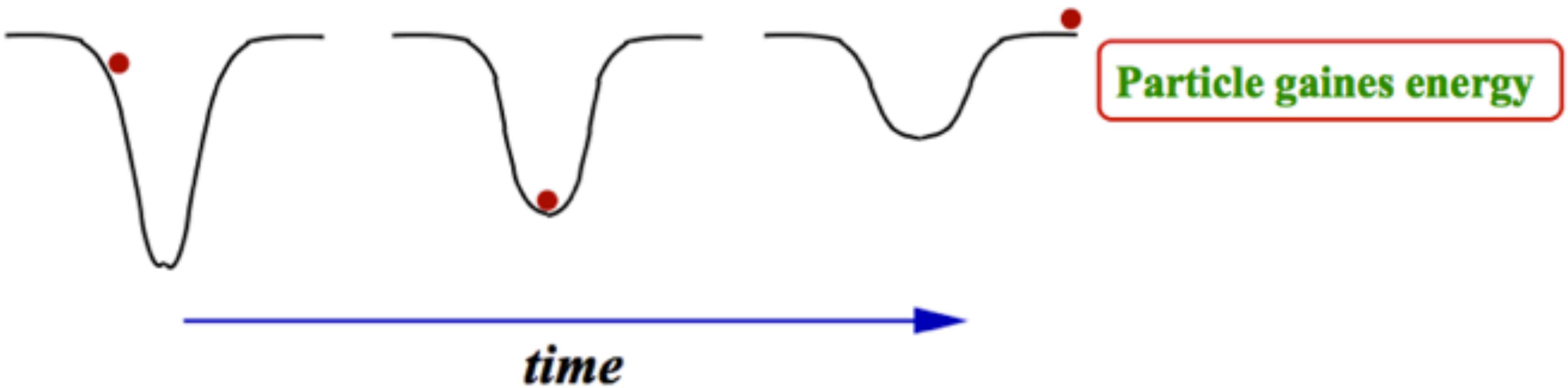
System evolution

- Relaxation
- Equipartition
- Evaporation
- Inelastic encounter
- Binary system formation from 3 body interaction
- Interaction with (primordial) binary system
- **Violent relaxation**

Violent Relaxation

- Elliptical galaxies light distribution is extremely smooth. However relaxation time is very long for a galaxy!
- What process could explain this?
- Elliptical galaxies are thought to be the results of either major merger of gravitational collapse. For these 2 process the variation of the gravitational potential can be faster than the crossing time.

Violent Relaxation



Violent Relaxation

- How the energy of a particle is changing?

$$\frac{dE}{dt} = \frac{1}{2}\frac{d(v^2)}{dt} + \frac{d\Phi}{dt} = \mathbf{v}.\frac{d\mathbf{v}}{dt} + \frac{\partial\Phi}{\partial t} + \mathbf{v}.\nabla\Phi = \frac{\partial\Phi}{\partial t}$$

- A particle can change its energy if the potential is varying, if not its energy is conserved.

See Lynden-Bell 1967

Negative Heat Capacity

- Let's define the local temperature of a self-gravitating system:

$$\frac{1}{2} m < v^2 > = \frac{3}{2} k_B . T$$

- Integrating over the density of the system ρ:

$$K = \frac{3}{2} N k_B . \overline{T}$$

- Virial theorem:

$$E = -K = -\frac{3}{2} N k_B . \overline{T}$$

Negative Heat Capacity

- Hence the heat capacity of the system is then:

$$\frac{dE}{d\overline{T}} = -\frac{3}{2} N k_B$$

- The heat capacity is negative! It means that temperature increases for a loss in energy. This makes the system unstable.
- Example: system in equilibrium with a thermostat, if the stellar system loose energy to the thermostat, then its temperature increase ...

The gravo-thermal catastrophe

- Let's consider a perfect gas with N identical particles, with total mass: $M=Nm$, contained in a reservoir of radius r_b.
- Isothermal sphere truncated at radius r_b.
- Pressure is given by:

$$p(r) = \frac{\rho(r) k_B . T}{m} = \frac{\rho(r)}{m\beta}$$

- where we use the *inverse temperature* β.

The gravo-thermal catastrophe

- We define the potential energy so that:

$$\Phi(r \rightarrow \infty) \rightarrow 0 \qquad \Phi(r_b) = -GM/r_b$$

- The total energy of the system is:

$$E = K + W \qquad K = \frac{3}{2} N k_B . T = \frac{3}{2} \frac{M}{m\beta}$$

- The virial theorem for this system reads:

$$2K + W = 4\pi r_b^3 p(r_b)$$

The gravo-thermal catastrophe

- Hence we can derive the energy of the system:

$$E = (2K + W) - K = 4\pi r_b^3 \frac{\rho(r_b)}{m\beta} - \frac{3M}{2m\beta}$$

- We can express β as a function of the King model radius and density (r_0, ρ_0) which approximate the center of an isothermal sphere:

$$m\beta = \frac{9}{4\pi G \rho_0 r_0^2}$$

The gravo-thermal catastrophe

- rewriting with a-dimensional quantities:

$$\tilde{r} = r/r_0 \quad \tilde{\rho} = \rho/\rho_0 \quad \tilde{M}(\tilde{r}_b) = M/(4\pi\rho_0 r_0^3)$$

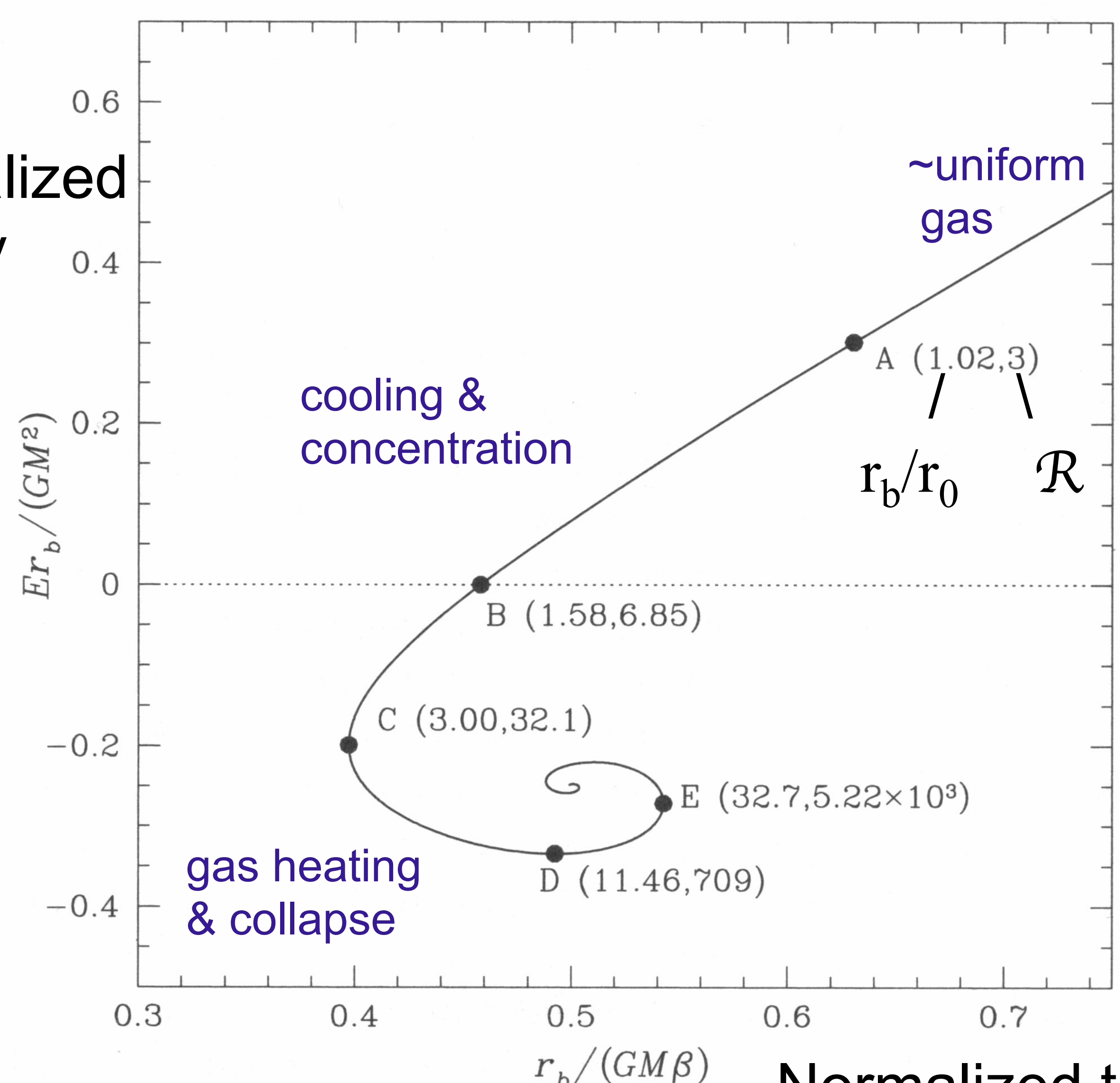
- then:

$$m\beta = \frac{9r_b}{GM}\frac{\tilde{M}(\tilde{r}_b)}{\tilde{r}_b}$$

- and the total energy:
- $$E = \frac{GM^2}{r_b}\left[\frac{\tilde{r}_b^4\,\rho(\tilde{r}_b)}{9\tilde{M}^2(\tilde{r}_b)} - \frac{\tilde{r}_b}{6\tilde{M}(\tilde{r}_b)}\right]$$

The gravo-thermal catastrophe

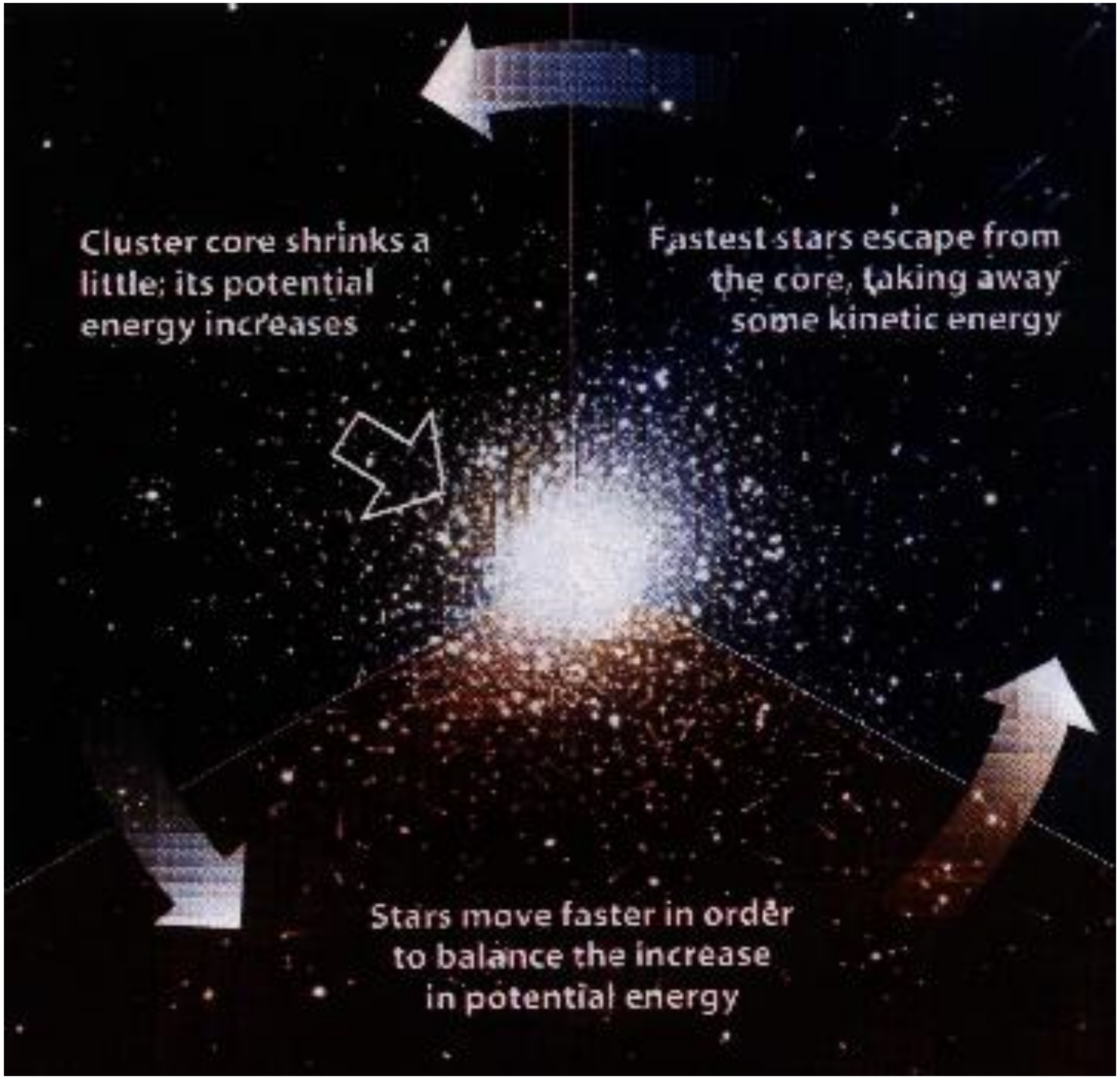
Normalized energy



Density contrast

$$\Re = \frac{1}{\tilde{\rho}(\tilde{r}_b)}$$

Normalized temperature


Cluster core shrinks a little; its potential energy increases
Fastest stars escape from the core, taking away some kinetic energy
Stars move faster in order to balance the increase in potential energy

Fokker-Planck Approximation

- Colission-less Boltzman equation imply that:

$$\frac{df}{dt} = 0$$

- this means that the mean density of stars in the phase-space is always the same.
- When there is strong interaction (collision), stars can be ejected or could enter the local phase-space, and density may change with time.

Fokker-Planck Approximation

- Let's define the probability of transition that a star can be scattered from the phase-space $\mathbf{w}$ to a new volume $d^3\Delta\mathbf{w}$ at $\mathbf{w}+\Delta\mathbf{w}$ during the time interval dt:

$$\Psi(\mathbf{w},\Delta\mathbf{w})d^3\Delta\mathbf{w}\Delta t$$

- Let's call *test stars* the stars that are scattered, and *field stars* the stars that are doing the scattering.

Fokker-Planck Approximation

- The stars that are scattered out can be defined by:

$$\left.\frac{\partial f(\mathbf{w})}{\partial t}\right|_{-} = -f(\mathbf{w}) \int \Psi(\mathbf{w}, \Delta\mathbf{w}) d^3 \Delta\mathbf{w}$$

- The stars that are scattered in can be defined by:

$$\left.\frac{\partial f(\mathbf{w})}{\partial t}\right|_{+} = \int \Psi(\mathbf{w}-\Delta\mathbf{w}, \Delta\mathbf{w}) d^3 f(\mathbf{w}-\Delta\mathbf{w}) \Delta\mathbf{w}$$

Fokker-Planck Approximation

- The collision term is thus:

$$\Gamma[f] = \frac{df}{dt} = \left.\frac{\partial f(\mathbf{w})}{\partial t}\right|_{-} + \left.\frac{\partial f(\mathbf{w})}{\partial t}\right|_{+}$$

- also called the **"master equation"**

Jean-Paul Kneib - Astro III

Fokker-Planck Equation

- In the simplify estimate of the relaxation time it was shown that for each crossing-time, encounters give rise to a mean-square velocity perturbation:

$$\Delta v_{\perp}^2 \sim \frac{8v^2 \ln(R/b_{min})}{N}$$

- N: number of stars
- v: star velocity
- $b_{min}=Gm/v^2$: impact parameter at which the velocity change is ~ v
- R: characteristic radius of the system

Fokker-Planck Equation

- Because of the logarithmic dependence, the equal logarithmic intervals of impact parameter contribute equally to $\Delta v_{\perp}^2$
- As a consequence most of the scattering is due to weak encounters that is those with $\delta v << v$. For weak encounter $\Delta \mathbf{w}$ is small so we can Taylor expand the master equation.

Fokker-Planck Equation

- Taylor expansion:

$$
\begin{aligned}
&\Psi(\mathbf{w}-\Delta\mathbf{w},\Delta\mathbf{w})f(\mathbf{w}-\Delta\mathbf{w}) = \Psi(\mathbf{w},\Delta\mathbf{w})f(\mathbf{w}) \\
- \quad &\sum_{i=1}^{6} \Delta w_i \frac{\partial}{\partial w_i}\left[\Psi(\mathbf{w},\Delta\mathbf{w})f(\mathbf{w})\right] \\
+ \quad &\frac{1}{2}\sum_{i,j=1}^{6} \Delta w_i \Delta w_j \frac{\partial^2}{\partial w_i \partial w_j}\left[\Psi(\mathbf{w},\Delta\mathbf{w})f(\mathbf{w})\right] + \mathcal{O}(\Delta\mathbf{w}^3)
\end{aligned}
$$

First order

Second order

Fokker-Planck Equation

- Introducing the diffusion coefficients:

Deviation systematic in the phase-space

$$D[\Delta w_i] \equiv \int d^6(\Delta \mathbf{w})\Psi(\mathbf{w},\Delta \mathbf{w})\Delta w_i$$

$$D[\Delta w_i \Delta w_j] \equiv \int d^6(\Delta \mathbf{w})\Psi(\mathbf{w},\Delta \mathbf{w})\Delta w_i \Delta w_j$$

Brownian diffusion in the phase-space

- the master equation becomes: $\Gamma[f] = \frac{df}{dt}$

$$\Gamma[f] = -\sum_{i=1}^{6} \frac{\partial}{\partial w_i}\left(D[\Delta w_i]\, f(\mathbf{w})\right) + \frac{1}{2}\sum_{i,j=1}^{6} \frac{\partial^2}{\partial w_i \partial w_j}\left(D[\Delta w_i \Delta w_j]\, f(\mathbf{w})\right)$$

This corresponds to the Fokker-Planck equation

Fokker-Planck Equation

- Local encounters: *most of the diffusion are due to small scale encounters* $b<<R$
- The encounter time is thus short compared to the crossing time: *only the speed is affected*.
- Hence we can assume that the probability of transition is null except when $\Delta\mathbf{x}=0$, thus we can simplify:

$$\Gamma[f] = -\sum_{i=1}^{3} \frac{\partial}{\partial v_i} \left(D\left[\Delta v_i\right] f(\mathbf{w})\right) + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2}{\partial v_i \partial v_j} \left(D\left[\Delta v_i \Delta v_j\right] f(\mathbf{w})\right)$$

Evolution of Spherical Stellar System:
Mass loss due to stellar evolution

- When stars explode they release part of their mass into gas that can leave either the stellar system:
  – velocity ejection larger than escape velocity
  – gas wiped out by collision with galactic/ intergalactic gas
- stellar evolution is on a long timescale (larger than the crossing time), so stellar orbits should not be affected.

Evolution of Spherical Stellar System:

Mass loss due to stellar evolution

- In the case of a globular cluster, let's consider a mass loss of ψ
- One can show that the orbits are then dilated by $1/\psi$
- Impact on the mean density (assuming a Plummer law):

$$\overline{\rho} = \frac{M(r)}{4/3\pi r^3} = \frac{3M}{4\pi b^3}\left(1+\frac{r^2}{b^2}\right)^{-3/2}$$

Evolution of Spherical Stellar System:
Mass loss due to stellar evolution

- The mass loss imply that:

$$M' = \psi M \qquad\qquad b' = b/\psi$$

- Thus replacing in the mean density

$$\overline{\rho} = \frac{3M'}{4\pi b'^3}\left(1 + \frac{r^2}{b'^2}\right)^{-3/2} = \frac{3\psi^4 M}{4\pi b^3}\left(1 + \frac{\psi^2 r^2}{b^2}\right)^{-3/2}$$

- this is a change of ψ^4 in the mean density!

Evolution of Spherical Stellar System:

Mass loss due to stellar evolution

- The radius r_J of the globular cluster is limited by the tidal forces imposed by the galaxy hosting the globular cluster so that:

$$\overline{\rho}\,(r_J) \sim \overline{\rho_h}$$

- where $\overline{\rho_h}$ is the mean density of the galaxy within the radial distance of the globular cluster to the galaxy center
- How r_J is modified by the mass loss? assuming that:

$$\overline{\rho'}\,(r'_J) = \overline{\rho}\,(r_J)$$

Evolution of Spherical Stellar System: Mass loss due to stellar evolution

- This means:

$$r'^{\,2}_J = \psi^{2/3} r_J^2 + b^2(\psi^{2/3} - 1/\psi^2)$$

- if $r_J >> b$ then $r'_J \sim \psi^{1/3} r_J$
- small impact
- if $r_J \sim b$ there is no solution for $\psi \sim 0.7$
- then the globular cluster may vanish because of the stellar mass loss

Evolution of Spherical Stellar System: Evaporation and ejection

- **Ejection**: when a star escape after one encounter
- **Evaporation**: when a star slowly escape after many encounters
- Ejection rate has been computed to be (Henon 1960, 1969)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -1.05 \times 10^{-3} \frac{N}{t_{rh} \ln(\lambda N)}$$

t_{rh} half-mass relaxation time

$\lambda \sim 0.1$

Evolution of Spherical Stellar System:
Evaporation and ejection

- The typical ejection time is then:

$$t_{ej} = -\left(\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}\right)^{-1} = 1 \times 10^3 \ln(\lambda N)\, t_{rh}$$

- Based on Fokker-Planck computation one can show that the evaporation time is:
-

$$t_{evap} = -N\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)^{-1} = f\, t_{rh}$$

with $f \sim 300$

Evolution of Spherical Stellar System: Evaporation and ejection

- Then $t_{ej} >> t_{evap}$

- we can neglect the process of ejection compared to the evaporation

Evolution of Spherical Stellar System: Core collapse

- Evolution of a globular cluster derived from N-body numerical simulation:
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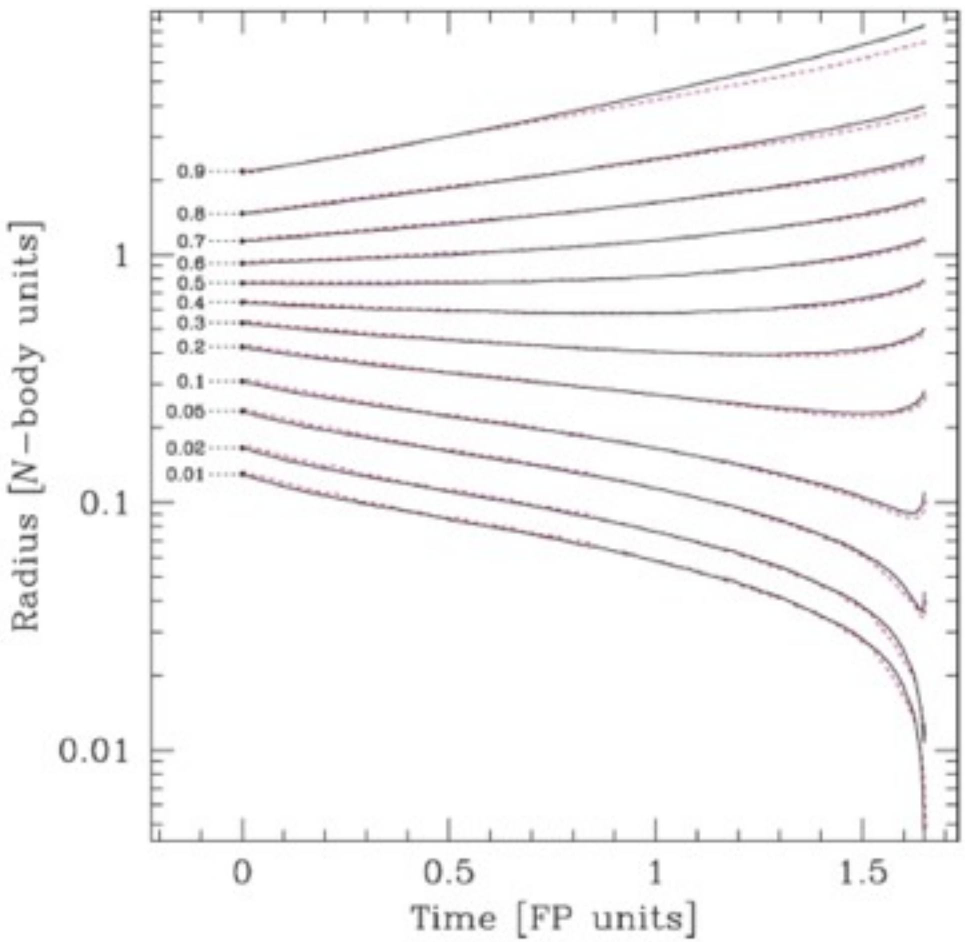
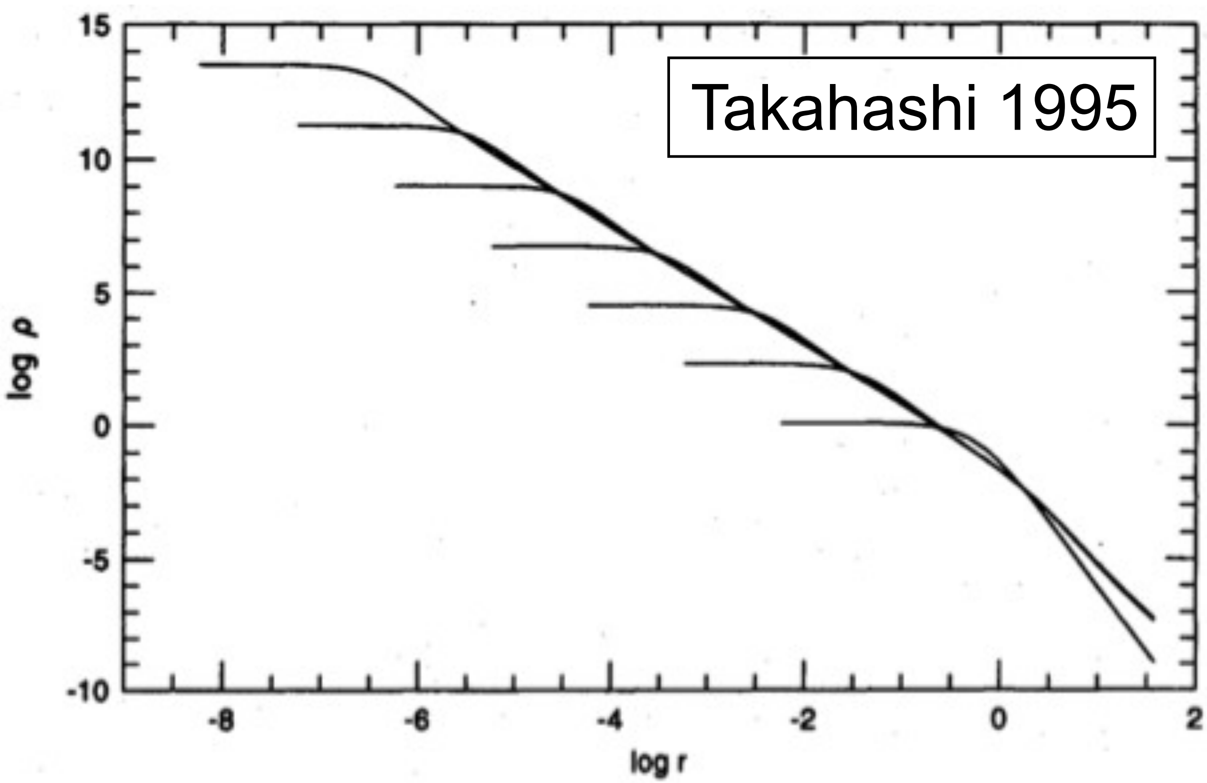


N-body simulation with 65k particle

Evolution of Spherical Stellar System: Core collapse

- Evolution of a globular cluster derived from N-body numerical simulation:
- 

Evolution of Spherical Stellar System: Core collapse

- During the core collapse: the core contracts and could lead to a singularity
- In reality one has to take into account the role of binaries and close encounters (star merging) that will stop the collapse.
- The core collapse is a manifestation of the gravo-thermal catastrophe
- Core collapse time is generally much smaller than the evaporation time.
- However is some cases (when tidal field from the host galaxy are strong) evaporation time can be smaller than core collapse time.