Advection Schemes

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Piecewise Constant Method: Donor-Cell Advection

Assume cell state within cell is constant.
For $u = 1$:

$$\rho_i^{n+1} = \rho_i^n + \frac{\Delta t}{\Delta x}(f_{i-1/2}^{n+1/2} - f_{i+1/2}^{n+1/2})$$

$$f_{i\pm 1/2}^{n+1} = u_{i\pm 1/2} \quad \rho_{i-1/2\pm 1/2}$$

First order accurate in time and space.

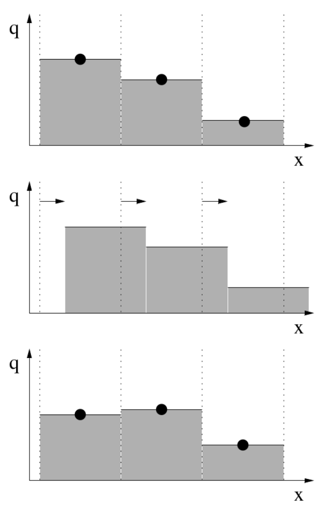
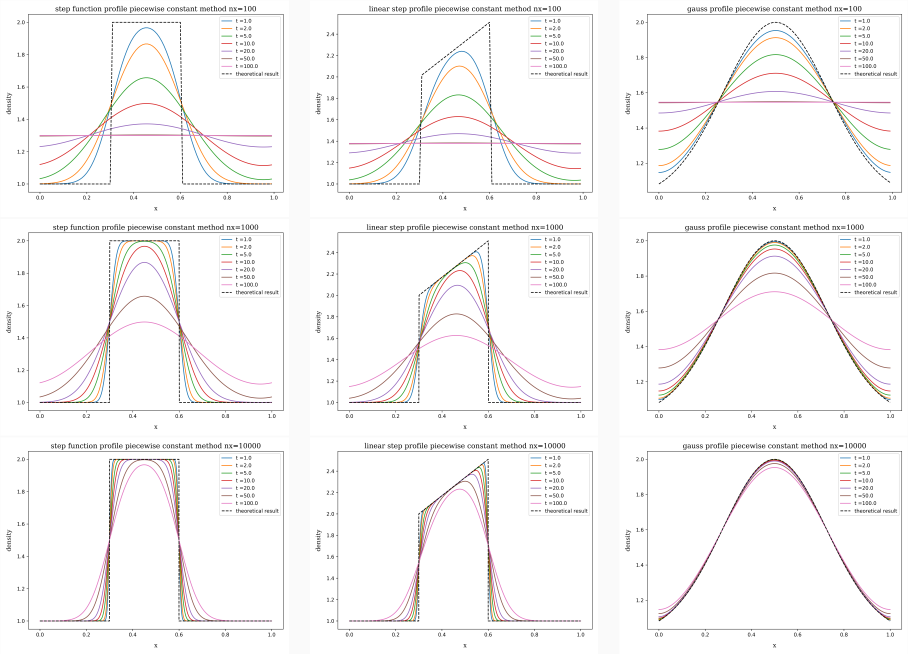


Image adapted from "Lecture Numerical Fluid Dynamics", Lecture given by C.P. Dullemond and H.H. Wang at Heidelberg University, 2009


step function profile piecewise constant method nx=100
linear step profile piecewise constant method nx=100
gauss profile piecewise constant method nx=100
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t =1.0
t =2.0
t =5.0
t =10.0
t =20.0
t =50.0
t =100.0
theoretical result
density
x

Piecewise Linear Method

Assume that the state in the cell is piecewise linear with some slope s. This gives a second-order accurate method.

For $\quad x_{i-1/2} < x_i < x_{i+1/2}: \quad \rho(x, t = t_n) = \rho_i^n + s_i^n(x - x_i)$

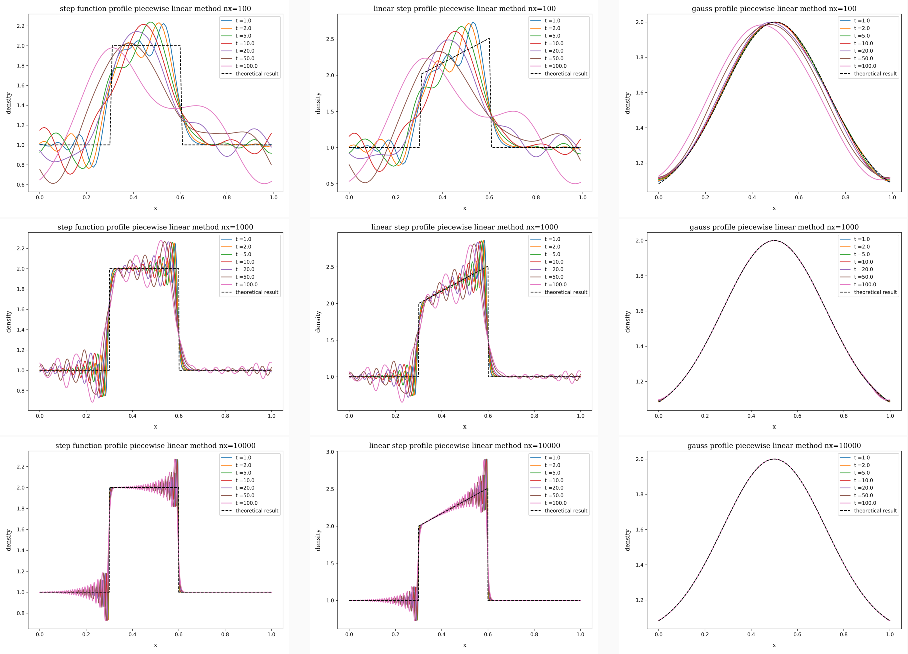
The flux over the interface is then

$$\begin{aligned} f_{i-1/2}(t) &= u\rho(x = x_{i-1/2}, t) \\ &= u\rho_{i-1} + us_i^n({\Delta x}/2 - u(t - t_n)) \end{aligned}$$

Finally averaging the fluxes over a time step gives:

$$\rho_i^{n+1} = \rho_i^n - \frac{u\Delta t}{\Delta x}(\rho_i^n - \rho_{i-1}^n) - \frac{u\Delta t}{\Delta x}\frac{1}{2}(s_i^n - s_{i-1}^n)(\Delta x - u\Delta t)$$

Choice of slope: $s_i^n = \frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}$ (Lax-Wendroff method)



step function profile piecewise linear method nx=100
linear step profile piecewise linear method nx=100
gauss profile piecewise linear method nx=100
step function profile piecewise linear method nx=1000
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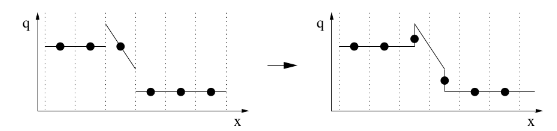
New Problem: Oscillations

The piecewise linear elements can have overshoots, leading to the oscillations seen in the previous plots.



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Godunov's theorem: *any linear algorithm for solving partial differential equations, with the property of not producing new extrema, can be at most first order.*

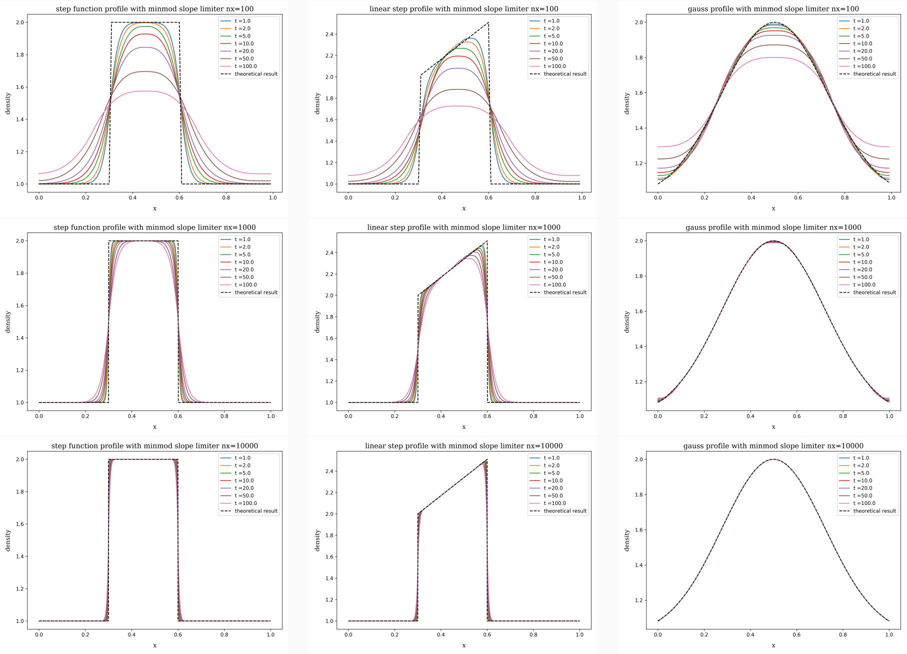
$\Rightarrow$ use non-linear conditions (slope limiters) to modify the slope s_i^n to prevent overshoots. Requirement: total variation diminishing: $TV(\rho^{n+1}) \leq TV(\rho^n) \equiv \sum |\rho_i - \rho_{i-1}|$. Such a scheme will not develop oscillations near a jump, because a jump is a monotonically in/decreasing function and a TVD scheme will not increase the TV.

Minmod Slope Limiter

$$s_i^n = \mathrm{minmod}\left(\frac{\rho_i^n - \rho_{i-1}^n}{\Delta x}, \frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\right)$$

with

$$\mathrm{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |a| > |b| \text{ and } ab > 0 \\ 0 & \text{if } ab < 0 \end{cases}$$


step function profile with minmod slope limiter nx=100
linear step profile with minmod slope limiter nx=100
gauss profile with minmod slope limiter nx=100
step function profile with minmod slope limiter nx=1000
linear step profile with minmod slope limiter nx=1000
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Van Leer Slope Limiter

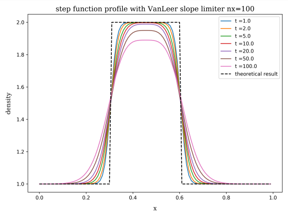
Rewrite flux (assuming $u = 1$) as

$$f_{i-1/2}^{n+1/2} = u\rho_{i-1} + \frac{1}{2}u\left(1 - \frac{u\Delta t}{\Delta x}\right)\phi(r_{i-1/2}^n)(q_i^n - q_{i-1}^n)$$
$$r_{i-1/2}^n = \frac{q_{i-1}^n - q_{i-2}^n}{q_i^n - q_{i-1}^n}$$

Here, ϕ is the slope/flux limiter.

The Van Leer flux limiter is defined as

$$\phi(r) = \frac{r - |r|}{1 - |r|}$$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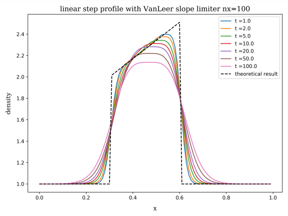
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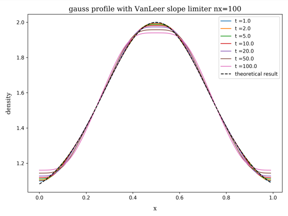
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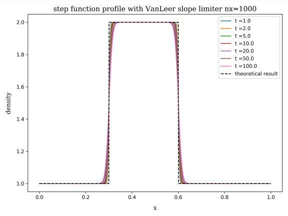
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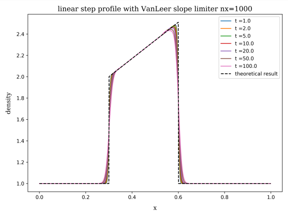
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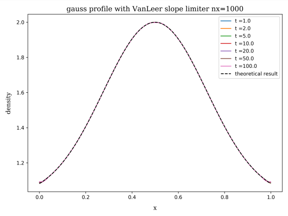
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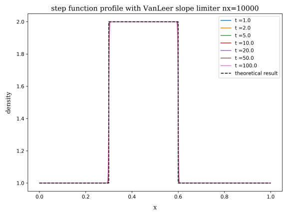
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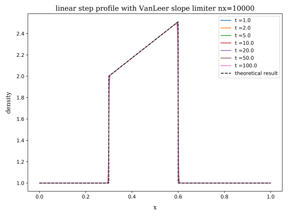
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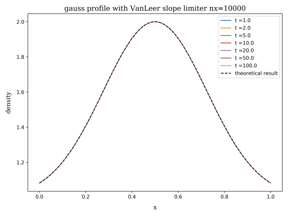
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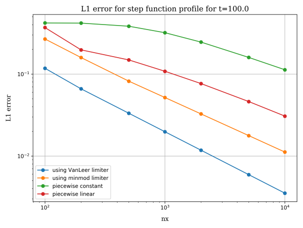
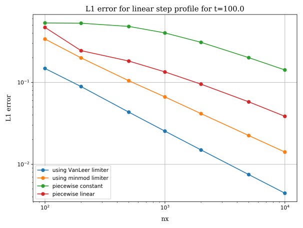
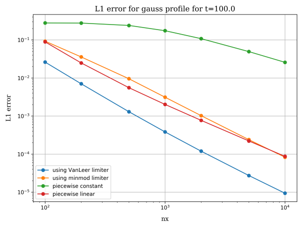
Precision of the Algorithms

Quantify error through $L1$ error norm: $L1 = \frac{1}{N} \sum_i |\rho_i - \tilde{\rho}(x_i)|$, where $\tilde{\rho}(x_i)$ is the analytical solution.

For $t = 100$:







Program, plotting scripts and this presentation available on
`https://bitbucket.org/mivkov/computational_astrophysics`