

# Meshless Methods in Astrophysics

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# Meshless Methods: Partition of Unity

At any point  $\mathbf{x}$  in space in the domain, we assign a volume partition  $\psi_i(\mathbf{x})$  to each particle  $i$  such that

$$\sum_i \psi_i(\mathbf{x}) = 1 \quad (1)$$

We choose

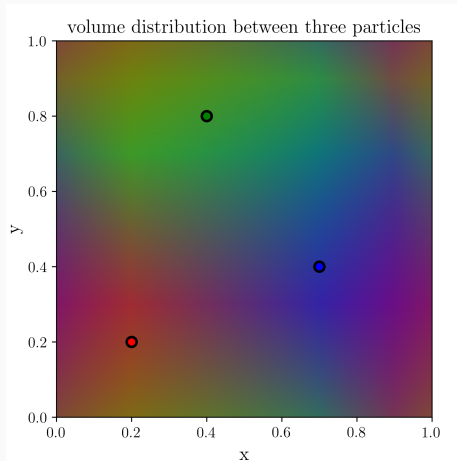
$$\psi_i(\mathbf{x}) = \frac{1}{\omega(\mathbf{x})} W(\mathbf{x} - \mathbf{x}_i, h(\mathbf{x})) \quad (2)$$

$$\omega(\mathbf{x}) = \sum_j W(\mathbf{x} - \mathbf{x}_j, h(\mathbf{x})) \quad (3)$$

$W(\mathbf{x})$  can be any arbitrary function at this point. In practice: use spherically symmetric kernel functions with compact support.

# Meshless Methods: Partition of Unity

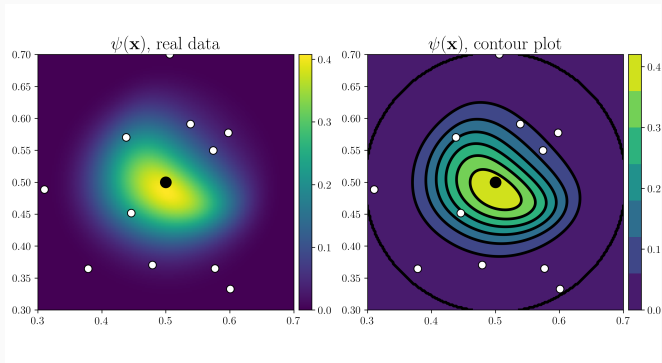
In general, no point in space is assigned to only one particle.



The volume distribution amongst three particles on a periodic two-dimensional domain with side length of unity in arbitrary units and periodic boundary conditions. The colour at each point of the domain is determined by assigning RGB values of  $\psi$  of the red, green, and blue particle at that point.

# Meshless Methods: Partition of Unity

Even for spherically symmetric kernels  $W$ , the volume partitions  $\psi_i(x)$  in general will not be symmetric due to  $\omega(\mathbf{x})$



The partition of unity  $\psi(\mathbf{x}_i - \mathbf{x}, h(\mathbf{x}_i))$  for the black particle  $i$  at the position  $\mathbf{x}_i$  with smoothing length  $h(\mathbf{x}_i)$  using a cubic spline kernel. The white points are neighbouring particles within the compact support radius of particle  $i$ .

# Meshless Methods: Hydrodynamics Equations

To obtain a discrete hydro scheme, we start with the Euler equations in conservative form:

$$\frac{\partial \mathbf{U}_k}{\partial t} + (\nabla \cdot \mathbf{F})_k = 0 \quad (4)$$

for every component  $k$  of the state vector  $\mathbf{U}$  and the flux tensor  $\mathbf{F}$ :

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} \rho \mathbf{v} \\ \rho v_i v_j + P \mathcal{I} \\ (E + P) \mathbf{v} \end{pmatrix} \quad (5)$$

# Meshless Methods: Hydrodynamics Equations

We arrive at the expression

$$\frac{d}{dt}(V_i \mathbf{U}_{k,i}) + \sum_j \mathbf{F}_{k,ij} \cdot \mathbf{A}_{ij} = 0 \quad (6)$$

with associated particle volumes

$$V_i = \int_V \psi_i(\mathbf{x}) dV = \frac{1}{\omega(\mathbf{x}_i)} + \mathcal{O}(h^2) \quad (7)$$

and the flux  $\mathbf{F}_{ij}$  between particle  $i$  and  $j$ .

# Meshless Methods: Effective Surfaces

There are two expressions for the effective surfaces  $\mathbf{A}_{ij}$  in literature:

- Following Ivanova *et al.* 2013:

$\mathbf{A}_{ij}$  is given by the integral:

$$\mathbf{A}_{ij} = \int_V [\psi_i(\mathbf{x}) \nabla \psi_j(\mathbf{x}) - \psi_j(\mathbf{x}) \nabla \psi_i(\mathbf{x})] dV \quad (8)$$

$$\mathbf{A}_{ij} = V_i \nabla \psi_j(\mathbf{x}_i) - V_j \nabla \psi_i(\mathbf{x}_j) + \mathcal{O}(h^2) \quad (9)$$

- Following Hopkins 2015:

$$\mathbf{A}_{ij}^\alpha = V_i \tilde{\psi}_j^\alpha(\mathbf{x}_i) - V_j \tilde{\psi}_i^\alpha(\mathbf{x}_j) + \mathcal{O}(h^2) \quad (10)$$

# Meshless Methods: Effective Surfaces

The  $\tilde{\psi}(\mathbf{x})$  come from the  $\mathcal{O}(h^2)$  accurate discrete gradient expression from Lanson and Vila 2008:

$$\frac{\partial}{\partial x_\alpha} f(\mathbf{x})|_{\mathbf{x}_i} = \sum_j (f(\mathbf{x}_j) - f(\mathbf{x}_i)) \tilde{\psi}_j^\alpha(\mathbf{x}_i) \quad (11)$$

$$\tilde{\psi}_j^\alpha(\mathbf{x}_i) = \sum_{\beta=1}^{\beta=\nu} \mathbf{B}_i^{\alpha\beta} (\mathbf{x}_j - \mathbf{x}_i)^\beta \psi_j(\mathbf{x}_i) \quad (12)$$

$$\mathbf{B}_i = \mathbf{E}_i^{-1} \quad (13)$$

$$\mathbf{E}_i^{\alpha\beta} = \sum_j (\mathbf{x}_j - \mathbf{x}_i)^\alpha (\mathbf{x}_j - \mathbf{x}_i)^\beta \psi_j(\mathbf{x}_i) \quad (14)$$

# Meshless Methods: Effective Surfaces

Ivanova version:

- + Analytical expression; Allows to demonstrate that conservation laws hold, easier interpretation
- No code is applying it

Hopkins version:

- Expression follows through use of discrete gradient
- + Is implemented in GIZMO

$A_{ij}$  are rather abstract expressions. How can we interpret them, how do they behave? Are there significant differences between the Ivanova and Hopkins version?

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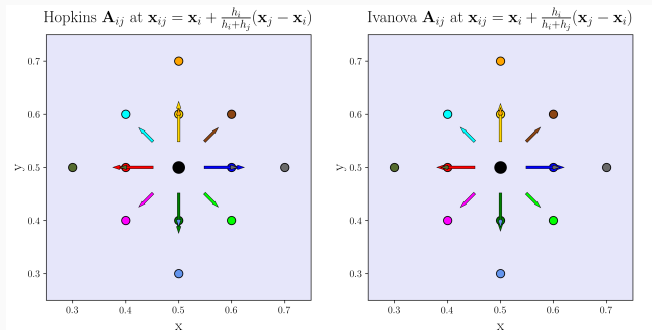
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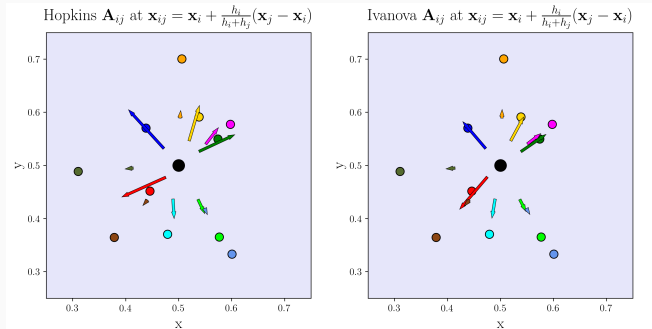
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# Regular Grid Particle Configuration



- $\mathbf{A}_{ij}$  point towards neighbours
- $|\mathbf{A}_{ij}|$  ratio Hopkins/Ivanova vary between 1.06 and 0.33. Ratio tends to lower values with increasing particle distance.

# Irregular Particle Configuration



- $\mathbf{A}_{ij}$  don't point towards neighbours!
  - Ivanova:  $\psi$  is not spherically symmetric  $\Rightarrow \nabla\psi$  won't be either
  - Hopkins: Matrix multiplication in  $\tilde{\psi}$ .
- $|\mathbf{A}_{ij}|$  ratio Hopkins/Ivanova vary between 1.6 and 0.28.

# Checking Conservation Properties

For a finite volume method: expect  $\sum_j \mathbf{A}_{ij} = 0$

- regular particle grid: Satisfied to machine precision
- irregular particle grid: sum is around the same order of magnitude of a single  $\mathbf{A}_{ij}$ . Ivanova version is smaller than Hopkins
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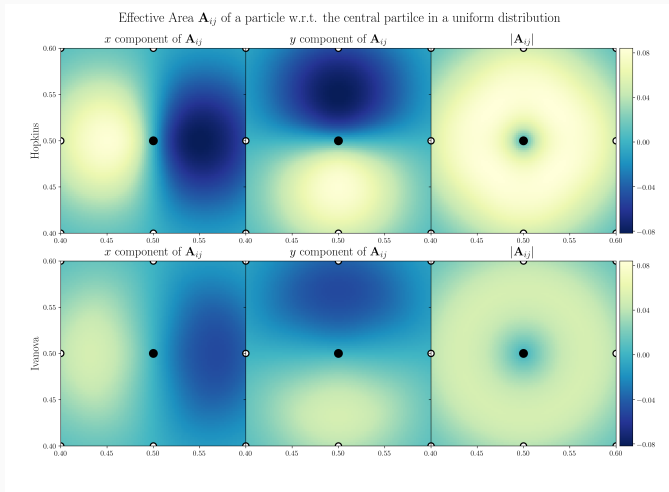
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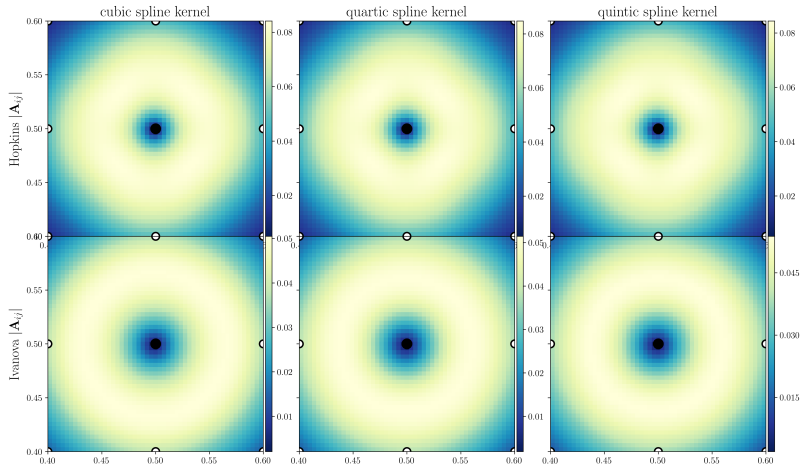
# Dependence on distance



$\mathbf{A}_{ij}$  increase with distance, then drop again. Hopkins: higher peak values; Ivanova: higher relative contribution at distance

# Dependence on Kernels

Effective Area  $\mathbf{A}_{ij}$  of a particle w.r.t. the central particle (black) in a uniform distribution for different kernels



## Summary of Findings

- clear differences in both magnitudes and directions of the  $A_{ij}$  obtained using the Ivanova and Hopkins formulation
- $\sum_j |A_{ij}|$  of the Ivanova < Hopkins version  
⇒ smaller total fluxes possibly allow bigger time step sizes
- Ivanova method displays higher relative contribution with increasing distance
- Ivanova  $A_{ij}$  are always well-defined, even in troublesome particle configurations

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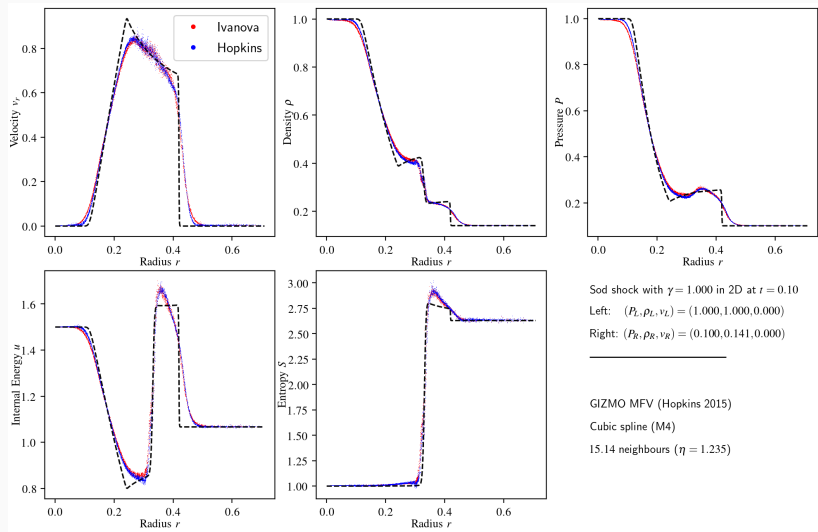
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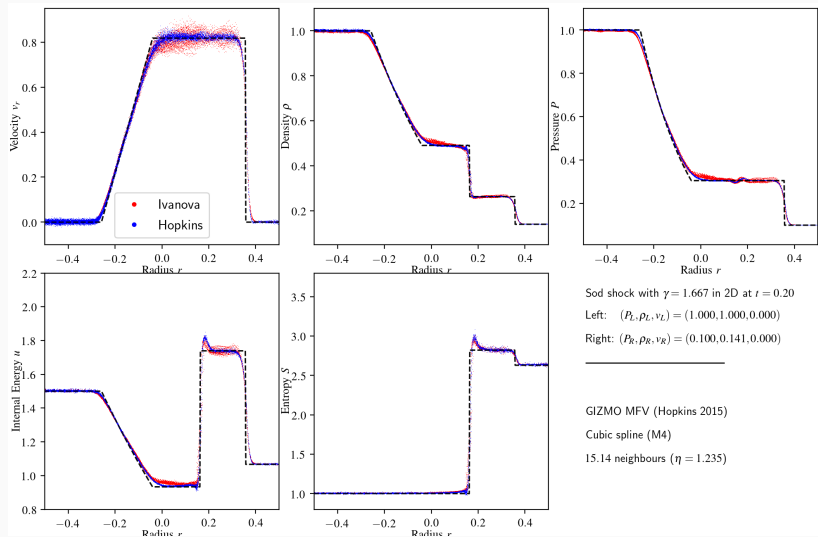
# Preliminary Results of Hydro Tests

## Spherical Sod Shock



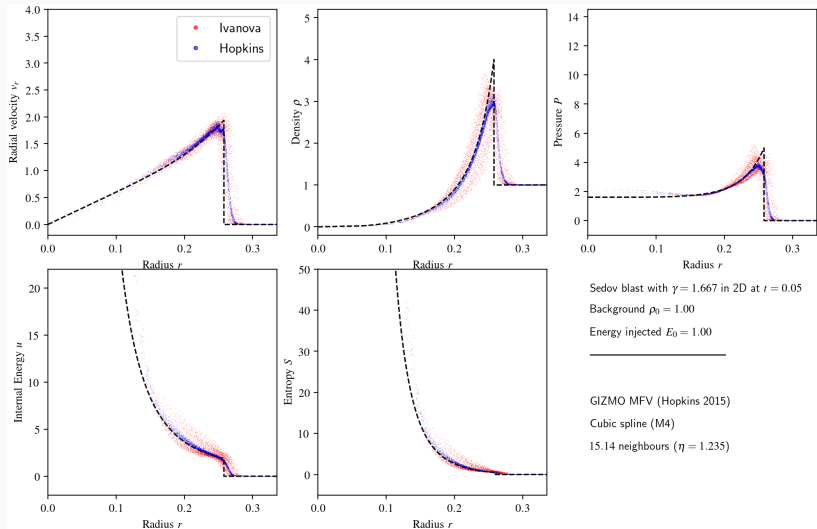
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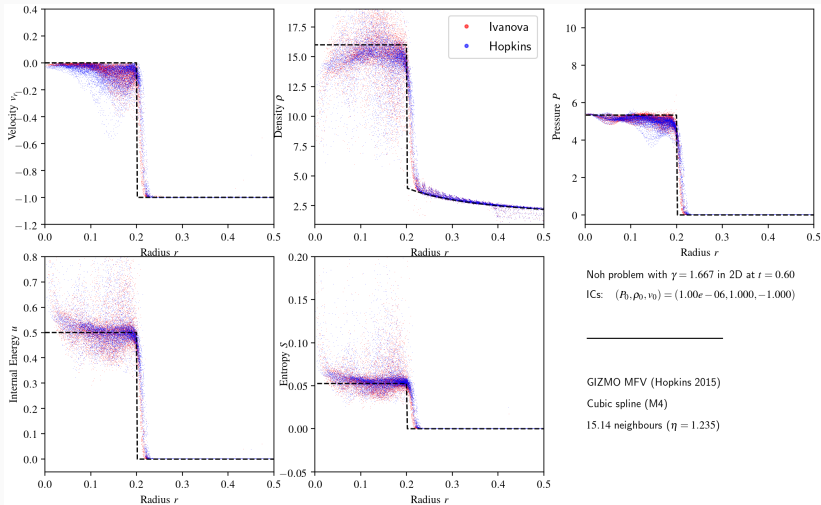
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## Sedov Blast



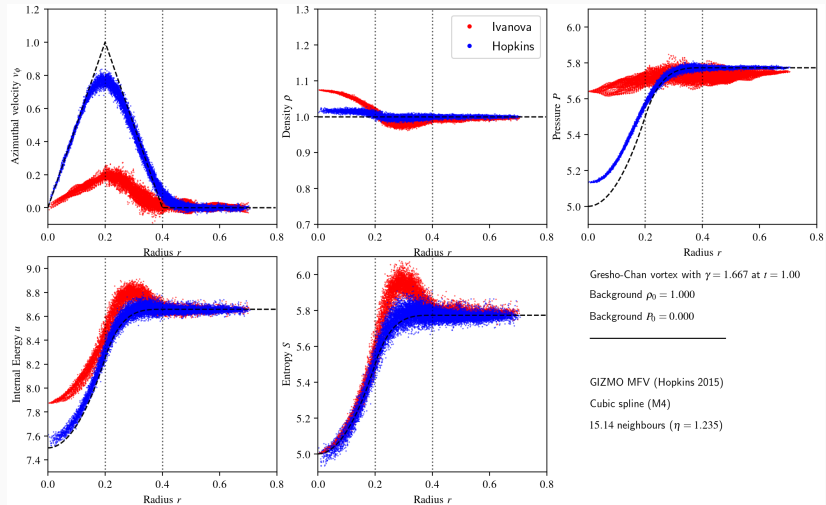
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## Noh Implosion test



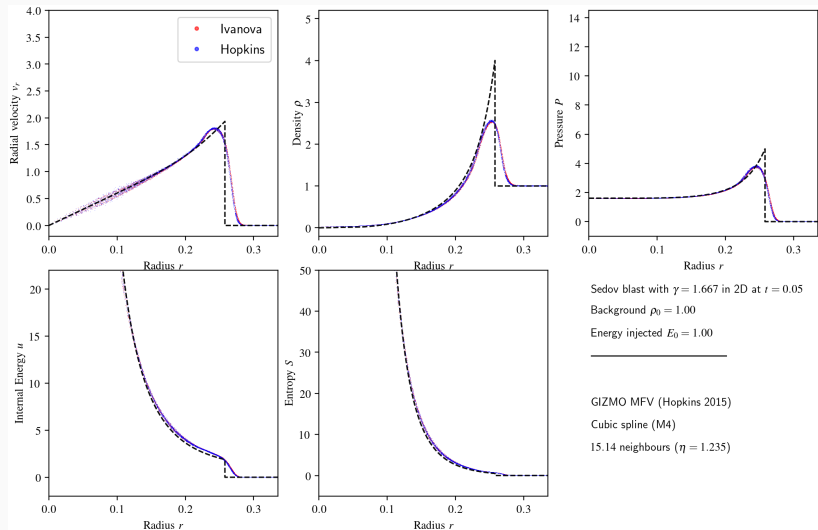
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## Gresho-Chan Vortex



# Preliminary Results of Hydro Tests

## Sedov Blast with fixed particles



# References



P. F. Hopkins, *Monthly Notices of the Royal Astronomical Society* **450**, 53–110, ISSN: 0035-8711, 1365-2966 (June 2015).



N. Ivanova *et al.*, *The Astronomy and Astrophysics Review* **21**, ISSN: 0935-4956, 1432-0754, DOI: 10.1007/s00159-013-0059-2, arXiv: 1209.4302 (Nov. 2013).



N. Lanson, J.-P. Vila, *SIAM J. Numer. Anal.* **46**, 1912–1934, ISSN: 0036-1429 (Apr. 2008).