Differential Geometry

Differential Manifolds

Manifold: topological space that locally looks like the Euclidean $\mathbb{R}^n$ space

Diffeomorphism: a C^∞ (smooth) bijective (invertible) map

Chart: (Coordinate System) bijective map of an open subset $U \subset M$ to an open subset $\Phi(U) \subset \mathbb{R}^n$

Atlas: A collection of C^∞ related charts (smoothly sewn together) such that every point of M lies in the domain of at least one chart.

Maximal atlas: The collection of all C^∞ related charts

Differentiable Manifold: The set (M, A) where M is a space and A is maximal atlas

Tangent Vectors

Let $f: M \to \mathbb{R}$ be a smooth function on M.
Let $\gamma(\lambda): \mathbb{R} \to M$ be a smooth curve on M.

Then $\frac{d}{d\lambda}(f \circ \gamma) = \frac{\partial f}{\partial x^i} \frac{dx^i}{d\lambda}$ (chain rule, summation)
is the rate of change of f along the curve γ.

The tangent vector of a curve γ is a map from the set of $f: p \to \mathbb{R}$:

$$\dot{\gamma}_p : f \mapsto \frac{d}{d\lambda}(f \circ \gamma); \quad \dot{\gamma}_p = \left[\frac{d}{d\lambda}(f \circ \gamma)\right]_p$$

(defined for a point p)

the $\left(\frac{\partial}{\partial x^i}\right)$ span the tangent space T_p. The components of the tangent vector X_p are $X_p(x^i)$ with respect to a given basis:

$$X_p = \underbrace{\frac{dx^\mu}{d\lambda}}_{\text{components}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{basis}}\bigg|_p$$

Dual Space

For every vector space we can define a dual space T_p^* (cotangent space) which is the space of all linear maps from a vector field to $\mathbb{R}$:

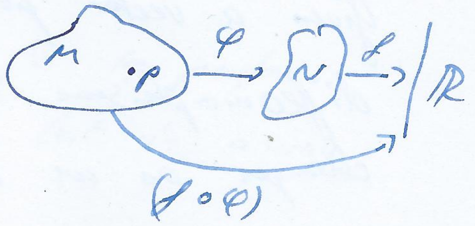
$$\omega \in T_p^* : X \in T_p \mapsto \omega(X) \in \mathbb{R} \quad \text{with } \omega(X) \equiv \langle \omega, X \rangle$$

df form the basis of T_p^*.

Tangent Map

Consider two manifolds $\mathcal{M}, \mathcal{N}$. Let

$\varphi: \mathcal{M} \to \mathcal{N}$ and $f: \mathcal{N} \to \mathbb{R}$



Then $(f \circ \varphi) \equiv \varphi^* f$ is called the pullback. For a covector:
$\langle \varphi^* \omega_N, X_M \rangle = \langle \omega_N, \varphi_* X_M \rangle$

Let $V(p)$ be a vector on $\mathcal{M}$. Then $\varphi_* V(p) = V(\varphi_* f)$
$= V(f \circ \varphi)$ is the pushforward vector $\varphi_* V(p)$.

The action of $\varphi_* V$ is the action of V on the pullback of f.

If φ is a diffeomorphism, both vectors and one-forms can be pulled back or pushed forward. (because φ^* & φ_* invertible)

$$(\varphi_* T_M)(\omega_N, X_N) = T_M(\varphi^* \omega_N, \varphi_*^{-1} X_N)$$

$$(\varphi^* T_N)(\omega_M, X_M) = T_N(\varphi^{*-1} \omega_M, \varphi_* X_M)$$

Flows, generating vector fields

Flow: 1-parametric family of diffeomorphisms: $\varphi_t : \mathcal{M} \to \mathcal{M}$
for which $\varphi_t \circ \varphi_s = \varphi_{t+s}$

→ a point p will be mapped on a curve parametrised for t

→ φ_t is valid for all $\mathcal{M}$ ⇒ curves will fill $\mathcal{M}$

⇒ We can define a vector field to be the set of tangent vectors to each of these curves at every point evaluated at $t = 0$

Now do the reverse: Define 1-parameter family from a vector field.
Let $V^\mu(x)$ be vector field, then integral curves are the curves that satisfy $\frac{dx^\mu}{dt} = V^\mu$. φ_t is then the flow down the integral curve, V^μ is the generator.

Lie Derivative

Given a vector field $V^\mu(x)$, we can find a family of 1-param diffeomorphisms φ_t. Now we can ask how fast a tensor changes as we travel down the integral curves x^μ with $\left.\frac{dx^\mu}{dt} = V^\mu\right|_{t=0}$

→ For any vector field V^μ, we can find a map and define

$$L_V R = \frac{d}{dt} \varphi^*_t R \Big|_{t=0} = \lim_{t\to 0} \frac{\varphi_t^* R - \varphi_0 R}{t} = \lim_{t\to 0} \frac{\varphi_t^* R - R}{t}$$

→ pull $R_{p'}$ back to p, where you can differentiate

→ It's the Lie derivative along (in the direction of) a vector field V^μ

Properties:

L_X is a linear map from tensor fields to tensor fields of the same type

$$L_X f = X f$$

$$L_X (T \otimes S) = (L_X T) \otimes S + T \otimes (L_X S)$$

$$L_X Y = XY - YX = [X Y]$$

$$(L_X R)^i{}_j = R^i{}_j \partial_k X^k - R^\alpha{}_j \partial_\alpha X^i + R^i{}_\beta \partial_j X^\beta$$

Covariant Derivative

Define a covariant derivative operator ∇ to perform the function of the partial derivative, but in a way independent of coordinates.

Ansatz: ∇X is tensor $\Rightarrow$ transforms like tensor.
∇X should have effect of partial derivative:

$$\Rightarrow \nabla_\mu X^\nu = \partial_\mu X^\nu + \text{correction}$$
$$= \partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha$$

Examples:
$$\nabla_k X^i = \partial_k X^i + \Gamma^i_{kl} X^l$$
$$\nabla_k X_i = \partial_k X_i - \Gamma^l_{ki} X_l$$
$$\nabla_r T^i_k = \partial_r T^i_k + \Gamma^i_{rl} T^l_k - \Gamma^l_{kr} T^i_l$$

∇_X is a linear map from tensor fields to tensor fields of the same type.

$$\nabla_X f = X f$$
$$\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S)$$

Covariance Principle

- Write appropriate SRT equations that hold in absence of gravitation
- $\eta_{\alpha\beta} \to g_{\alpha\beta}$, $\partial_\alpha \to \nabla_\alpha$
- Resulting equations are then generally covariant.

Parallel Transport

= transporting a vector through space so that it keeps its length and direction. The vector doesn't change, but the coordinates do! → Components need to be adjusted properly.

A geodesic is a path which parallel transports its own tangent vector.

If $\nabla_X Y = 0$, then Y is parallel transported along X.

If for a vector field X $\nabla_X X = 0$, then the integral curves ($\frac{dx^\mu}{dt} = X^\mu$ is satisfied) are geodesics.

Geodesic equation:

$$\nabla_X X = \nabla_{\frac{\partial^i}{\partial t}\partial_i}(X^k \partial_k) = 0$$

$$= \frac{\partial x^i}{\partial t}\nabla_{\partial_i}(X^k\partial_k) = \frac{\partial x^i}{\partial t}\left(\frac{\partial X^k}{\partial x^i}\partial_k + \Gamma^k_{ij}X^j\partial_k\right)$$

$$= \left(\frac{dX^k}{dt}\partial_k + \Gamma^k_{ij}X^iX^j\partial_k\right)$$

$$= \left(\frac{dX^k}{dt} + \Gamma^k_{ij}X^iX^j\right)\partial_k$$

$$= \left(\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)\partial_k = 0$$

$$\Rightarrow \frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

Curvature and Torsion

Torsion:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$T^k{}_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

Riemann tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^\alpha_{lj} \Gamma^i_{k\alpha} + \Gamma^\alpha_{kj} \Gamma^i_{\alpha l}$$

$$R^i{}_{jkl} = - R^i{}_{jlk} \quad \text{antisymmetric in } l,k$$

Ricci tensor:

$$R_{jl} = R^i{}_{jil} = \partial_i \Gamma^i_{lj} - \partial_l \Gamma^i_{ij} + \Gamma^\alpha_{jl} \Gamma^i_{i\alpha} - \Gamma^\alpha_{ij} \Gamma^i_{\alpha l}$$

Scalar curvature:

$$\mathcal{R} = R^l{}_l = g^{lj} R_{jl}$$

With a Riemannian connection the Riemann tensor has $C_n = \frac{n^2(n^2-1)}{12}$ independent components.

$C_4 = 20$, $C_3 = 6$

Riemannian Connection

The metric tensor g at a point p in M is a symmetric $(0,2)$ tensor. It assigns a magnitude $\sqrt{|g(X,X)|}$ to each vector X and defines the angle between any two vectors $\neq 0$.

Metric connection: $\nabla g = 0$ (connection is metric)

For every pseudo-Riemannian manifold, there exists a unique affine connection such that ∇ is metric and torsion free ($=$ symmetric). This metric is called the Riemann connection. The Christoffel symbols for this connection are $\Gamma^l_{ij} = \frac{1}{2} g^{lk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$

$\Rightarrow$ the inner product of any two vectors remains constant upon parallel transporting them along any curve.

$\Rightarrow$ ∇_μ commutes with raising/lowering indices because $\nabla_\mu g_{\alpha\beta} = 0$

Symmetries:

$$R^i{}_{jkl} = -R^i{}_{jlk}$$

$$\sum_{(jkl)} R^i{}_{jkl} = 0 \qquad \text{1st Bianchi}$$

$$\sum_{(klm)} \nabla_m R^i{}_{jkl} = 0 \qquad \text{2nd Bianchi}$$

$$R_{ijkl} = -R_{jikl}$$

$$R_{ijkl} = R_{klij}$$

$$R^k{}_{i;k} = \frac{1}{2} R_{;i}$$

$$G^k{}_{i;k} = 0$$

Showing Tensor Properties

1) Ricci Tensor Symmetric

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^\alpha_{lj}\Gamma^i_{\alpha k} - \Gamma^\alpha_{jk}\Gamma^i_{\alpha l}$$

$$R^i{}_{jil} = \partial_i \Gamma^i_{jl} - \partial_l \Gamma^i_{ji} + \Gamma^\alpha_{lj}\Gamma^i_{\alpha i} - \Gamma^\alpha_{ji}\Gamma^i_{\alpha l} = R_{jl}$$

$$g^{\mu i} R_{\mu j i l} = g^{\mu i} R_{i l \mu j} = R^\mu{}_{l\mu j} = R_{lj}$$

$$\Rightarrow R_{jl} = R_{lj}$$

Using $R_{\alpha\beta\gamma\sigma} = R_{\gamma\sigma\alpha\beta}$ for Riemannian Connection

2) Einstein Tensor Symmetric

$$G_{ik} = \underbrace{R_{ik}}_{\text{symmetric}} - \frac{1}{2} R \underbrace{g_{ik}}_{\text{symmetric}} = R_{ki} - \frac{1}{2} R g_{ki} = G_{ki}$$

3) $\nabla_k R^k{}_i = \frac{1}{2} \nabla_i R$

$$R_{ik} = g^{jl} R_{lijk} = g^{jl} R_{jkli}$$

Using 2nd Bianchi identity: $\sum_{(klm)} \nabla_m R^i{}_{jkl} = 0$

$$\nabla_m R^i{}_{jkl} + \nabla_l R^i{}_{jmk} + \nabla_k R^i{}_{jlm} = 0$$

Take (ik) trace ($k=i$, sum up)

$$\rightarrow \nabla_m R_{jl} + \nabla_l R^i{}_{jmi} + \nabla_i R^i{}_{jlm} =$$

$$= \nabla_m R_{jl} - \nabla_l R_{jm} + g^{ik} \nabla_i R_{kjlm}$$

$$= \nabla_m R_{jl} - \nabla_l R_{jm} - g^{ik} \nabla_i R_{jklm}$$

$$\Rightarrow \nabla_m R^j{}_l - g^{ik}\nabla_i R^j{}_{klm} - \nabla_l R^j{}_m = 0$$

Now take (jm) trace:

$$\nabla_m R^m{}_l - g^{ik}\nabla_i R^m{}_{klm} - \nabla_l R^m{}_m =$$

$$= \nabla_m R^m{}_l + g^{ik}\nabla_i R^m{}_{kml} - \nabla_l \mathcal{R} =$$

$$= \nabla_m R^m{}_l + \nabla_k R^k{}_l - \nabla_l \mathcal{R}$$

$$= 2\nabla_m R^m{}_l - \nabla_l \mathcal{R}$$

$$\Rightarrow \nabla_k R^k{}_i = \frac{1}{2}\nabla_i \mathcal{R}$$

4) Show $\nabla_k G^k{}_i = 0$

$$\nabla_k G^k{}_i = \nabla_k\left(R^k{}_i - \frac{1}{2}g^k{}_i \mathcal{R}\right)$$

$$= \nabla_k R^k{}_i - \frac{1}{2}g^k{}_i \nabla_k \mathcal{R}$$

$$= \nabla_k R^k{}_i - \frac{1}{2}\underbrace{g_{ki}\, g^{ki}}_{\delta^i_k}\nabla_k \mathcal{R}$$

$$= \nabla_k R^k{}_i - \frac{1}{2}\nabla_i \mathcal{R}$$

$$= \frac{1}{2}\nabla_i \mathcal{R} - \frac{1}{2}\nabla_i \mathcal{R} = 0$$