Equivalence Principle

The equivalence principle is the physical basis of general relativity.

1) Inertial and gravitational masses are equal

2) Gravitational forces are equivalent to inertial forces. Going to a well-chosen reference frame, one can get rid of the gravitational field.

3) In a freely falling (= moving freely in the presence of whatever gravitational fields are around) accelerated frame of reference, <u>all</u> physical processes run as if there is no gravitational field. This is valid for small enough regions of space.

Energy-Momentum Tensor

Dust: $T_{\mu\nu} = \rho_0 u_\mu u_\nu$

Ideal Fluid: $T_{\mu\nu} = (\rho_0 + P/c^2) u_\mu u_\nu - P g_{\mu\nu}$ $\quad P = \frac{\text{energy}}{\text{volume}}$

Electrodynamics: $T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha} g_{\alpha\beta} F^{\nu\beta} - \frac{1}{4} g^{\mu\nu} F_{\sigma\gamma} F^{\sigma\gamma}\right)$

The energy-momentum tensor is symmetric: $T_{\mu\nu} = T_{\nu\mu}$

Components:

$T^{00} = \frac{\text{energy density}}{c^2}$

$T^{0i} = T^{i0}$ = flux of relativistic mass across the x^i surface; equivalent to the density of the ith component of linear momentum

T^{ii} = normal stress; = pressure if independent of direction
(stress perpend. to surface)

T^{ik} = shear stress (stress parallel to surface)

Newtonian Limit

Assume weak, static gravitational field. Assume slow moving particles: $|u^i| \ll c$

Slow particles: $|u^i| \ll c \Rightarrow c\frac{dx^0}{d\tau} \gg \frac{dx^i}{d\tau}$

Starting with geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \approx -\Gamma^\mu_{00}\left(\frac{dx^0}{d\tau}\right)^2$$

Compute Γ^μ_{00}: $\quad \Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\right)$

For a static field: $\partial_0 g_{\alpha\beta} = 0$

Then $\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\left(\partial_0 g_{\lambda 0} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00}\right)$

$$= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}$$

Ansatz for g: Small perturbation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\Rightarrow \Gamma^\mu_{00} = -\frac{1}{2}\left(\eta^{\mu\lambda} + h^{\mu\lambda}\right)\underbrace{\partial_\lambda\left(\eta_{00} + h_{00}\right)}_{=0}$$

$$\simeq -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{c\,dt}{d\tau}\right)^2$$

Different cases:

i) $\mu = 0$

$$\frac{d^2(ct)}{d\tau^2} = c\frac{d^2t}{d\tau^2} = \frac{1}{2}\eta^{0\lambda}\partial_\lambda h_{00}\, c^2\frac{d^2t}{d\tau^2}$$

$$= \frac{1}{2}\eta^{00}\partial_0 h_{00}\, c^2\frac{d^2t}{d\tau^2} \qquad (\eta^{0i}=0)$$

$$= 0$$

$$\Rightarrow \frac{dt}{d\tau} = \text{const} \equiv 1$$

ii) for $\mu \neq 0$:

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{is}\partial_s h_{00}\left(\frac{c\,dt}{d\tau}\right)^2 = \frac{1}{2}\eta^{ii}\partial_i h_{00}\left(\frac{c\,dt}{d\tau}\right)^2$$

$$= -\frac{c^2}{2}\partial_i h_{00}$$

$$\Rightarrow \frac{d^2\vec{r}}{d\tau^2} = -\frac{c^2}{2}\nabla h_{00}\left(\frac{dt}{d\tau}\right)^2$$

Newtonian physics

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -\frac{c^2}{2}\nabla h_{00} = -\nabla\varphi$$

$$\Rightarrow h_{00} = \frac{2\varphi}{c^2}$$

$$\Rightarrow g_{00} = 1 + \frac{2\varphi}{c^2}$$

Einstein Field Equations

There is no equivalent equation in a local coordinate system; The field equations can't be derived using the covariance principle.

Assumptions:

- Newtonian limit is well confirmed:

$$\Delta \varphi = 4\pi G \rho \quad \text{with} \quad g_{00} \approx 1 + \frac{2\varphi}{c^2}$$

$$\Delta g_{00} = \frac{2\Delta\varphi}{c^2} = \frac{8\pi G \rho}{c^2} \approx \frac{8\pi G}{c^4} T_{00} \qquad T_{ij} \text{ small}$$

Goal: Relate metric $g_{\mu\nu}$ to the energy-momentum tensor $T^{\mu\nu}$. We want to describe the fundamental interaction of gravitation as a result of spacetime being curved by mass and energy

Ansatz: $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

with
1) $[G_{\mu\nu}] = L^{-2}$
2) $G_{\mu\nu}$ tensor, symmetric like $T_{\mu\nu}$
3) $\nabla_\nu T^{\mu\nu} = 0 \Rightarrow \nabla_\nu G^{\mu\nu} = 0$
4) $G_{00} \approx \Delta g_{00}$ for a weak, stationary field
5) "dimension" of $G_{\mu\nu}$ is second derivative
   → assume linear combination of second derivatives or quadrats of first derivatives of $g_{\mu\nu}$

$\Rightarrow$ $G_{\mu\nu}$ is determined uniquely.

$$G_{\mu\nu} = a R_{\mu\nu} + b R g_{\mu\nu}$$

The Ricci tensor is the only tensor made of the metric tensor and its first and second derivatives and is linear in the second derivative.

From $\nabla_\mu G^{\mu\nu} = 0$: $b = -\frac{a}{2}$

$$\Rightarrow \quad G_{\mu\nu} = a\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Now compute the trace:

$$g^{\mu\nu} G_{\mu\nu} = a\left(R^\mu{}_\mu - \frac{1}{2} R \cdot g^\mu{}_\mu\right) = a\left(R - \frac{4}{2} R\right) = -aR$$

$$\overset{\text{Newt. lim.}}{\simeq} G_{00} = a\left(R_{00} - \frac{R}{2} \underbrace{g_{00}}_{\approx 1 = \eta_{00}}\right) = a\left(R_{00} - \frac{R}{2}\right)$$

$$\Rightarrow \quad R_{00} = -\frac{1}{2} R$$

Compute Ricci tensor from Riemann tensor up to first order:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \underbrace{\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\lambda} - \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\alpha\nu}}_{\substack{\text{each } \Gamma \text{ is } \mathcal{O}(h) \\ \to \Gamma\Gamma \text{ is } \mathcal{O}(h^2) \approx 0}}$$

$$\Rightarrow \quad R_{00} = \partial_i \Gamma^i_{00} - \underbrace{\partial_0 \Gamma^i_{i0}}_{\text{static: } \partial_0 = 0}$$

$$= \partial_i \Gamma^i_{00} \overset{\text{Newt. lim.}}{=} \partial_i\left(\frac{1}{2} \frac{\partial g_{00}}{\partial x^i}\right) = \frac{1}{2} \Delta g_{00}$$

$$\Rightarrow \quad G_{00} = a\left(R_{00} - \frac{R}{2}\right) = 2a R_{00} = 2a \frac{1}{2} \Delta g_{00} = a\Delta g_{00}$$

$$\overset{!}{=} \Delta g_{00} \quad \text{(requirement for weak stationary limit)}$$

$$\Rightarrow \quad a = 1$$

$$\Rightarrow \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Other form of Field Equations:

By contracting:

$$G^\mu{}_\mu = R^\mu{}_\mu - \frac{1}{2} \mathcal{R} g^\mu{}_\mu = \mathcal{R} - \frac{4}{2}\mathcal{R} = -\mathcal{R}$$

$$= \frac{8\pi G}{c^4} T^\mu{}_\mu = \frac{8\pi G}{c^4} T$$

$$\Rightarrow \mathcal{R} = -\frac{8\pi G}{c^4} T$$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}\frac{8\pi G}{c^4} T g_{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

The Einstein equations constitute a set of 10 algebraically independent second order DEQ for $g_{\mu\nu}$ because $R_{\mu\nu}$ is symmetric.

The Einstein equations are generally covariant, so they can at best determine the metric up to coordinate transformation (4 functions), leaving 6 independent generally covariant equations.

Indeed, see contracted Bianchi identity: $\nabla_\mu G^{\mu\nu} = 0$

→ 4 relations

non-linear, coupled, partial differential equations

Cosmological Constant:

As a generalisation, one can relax the condition that $G_{\mu\nu}$ is exclusively made of derivatives of $g_{\mu\nu}$ and add a linear term in $g_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

when Λ is a constant with $[\Lambda] = L^{-2}$

Using the Newtonian limit:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) - \Lambda g_{\mu\nu}$$

Applying limit: $T \simeq T_{00}$, $g_{\mu\nu} \approx \eta_{\mu\nu}$, $R_{00} = \Delta g_{00} = \frac{\Delta\varphi}{c^2}$

Then

$$R_{00} \simeq \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2} T_{00}\right) - \Lambda =$$

$$= \frac{4\pi G}{c^4} T_{00} - \Lambda \overset{\text{N.L. static}}{\simeq} \frac{4\pi G \rho}{c^2} - \Lambda$$

$$= \frac{\Delta\varphi}{c^2}$$

$$\Rightarrow \Delta\varphi = 4\pi G\rho - \Lambda c^2 = 4\pi G(\rho - \rho_{vac}) \text{ with } \rho_{vac} = \frac{c^2\Lambda}{4\pi G}$$

Einstein-Hilbert action

The field equations can be obtained from a covariant variational principle:

$$S_D[g] = \int_{D} R(g) \underbrace{\sqrt{|g|}\, d^4x}_{\text{volume element}}$$

compact region in spacetime

Get Euler-Lagrange equations from $\delta S_D[g] = 0$

Physical laws with Gravitation

Mechanics

In a local inertial system:

$$m \frac{du^\alpha}{d\tau} = f^\alpha \xrightarrow[\text{principle}]{\text{covariance}} m \frac{Du^\alpha}{D\tau} = f^\alpha$$

$$\Rightarrow m \frac{du^\mu}{d\tau} = f^\mu - m \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda$$

$$m \frac{d^2 x^\mu}{d\tau^2} = f^\mu - m \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau}$$

Electrodynamics

Maxwell equations in SRT:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta \qquad \varepsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$\Rightarrow \nabla_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta \qquad \varepsilon^{\alpha\beta\gamma\delta} \nabla_\beta F_{\gamma\delta} = 0$$

Continuity equation: $\nabla_\mu j^\mu = 0$

Applications of the Newtonian Limit

1) Time dilation

Proper time: $d\tau \equiv \frac{ds}{c} = \frac{1}{c}\sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$

i) Without gravity: $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$$d\tau = \frac{1}{c}\sqrt{c^2 dt^2 - d\vec{r}^{\,2}} = dt\sqrt{1 - d\vec{r}^{\,2}/dt^2 c^2} = dt\sqrt{1 - v^2/c^2}$$

$$\Rightarrow d\tau = \sqrt{1 - v^2/c^2}\, dt \qquad \Longrightarrow d\tau < dt$$

ii) Clock at rest, gravitational field (Newtonian limit) present:

$dx^i = 0$

$$d\tau = \frac{1}{c}\sqrt{g_{\mu\nu}\, dx^\mu dx^\nu} = \frac{1}{c}\sqrt{g_{00} c^2 dt^2} = dt\sqrt{1 + \frac{2\varphi}{c^2}}$$

$\varphi < 0$: $\quad d\tau < dt$

↳ A clock in a gravitational field goes slower than a clock outside.

2) Redshift

Define redshift $z \equiv \frac{\nu_S - \nu_O}{\nu_O}$

The proper times at the source and observer are:

$$d\tau_O = \sqrt{g_{00}(r_O)}\, dt_O , \qquad d\tau_S = \sqrt{g_{00}(r_S)}\, dt_S$$

Now consider $d\tau$ to be the time interval between two peaks of the emitted/received EM wave:

$$d\tau_O = 1/\nu_O , \quad d\tau_S = 1/\nu_S$$

The EM wave propagates at constant speed: $dt_S = dt_O$

$$\Rightarrow \frac{\nu_S}{\nu_O} = \frac{d\tau_O}{d\tau_S} = \sqrt{\frac{g_{00}(r_O)}{g_{00}(r_S)}} = \left(1+\frac{2\varphi(r_O)}{c^2}\right)^{1/2}\left(1+\frac{2\varphi(r_S)}{c^2}\right)^{-1/2} \simeq$$

$$\simeq \left(1+\varphi(r_O)/c^2\right)\left(1-\varphi(r_S)/c^2\right) \simeq 1+\frac{\varphi(r_O)}{c^2}-\frac{\varphi(r_S)}{c^2}$$

$$\Rightarrow z = \frac{\nu_S - \nu_O}{\nu_O} = \frac{\nu_S}{\nu_O} - 1 = \frac{\varphi(r_O)-\varphi(r_S)}{c^2}$$

Simple Derivation of Geodesic equation

In a local coordinate system, a free particle follows the equation:

$$\frac{d^2 x^\mu}{d\tau} = \frac{du^\mu}{d\tau} = 0$$

$$\frac{du^\mu}{d\tau} = \frac{\partial u^\mu}{\partial x^i}\frac{dx^i}{d\tau} = \frac{\partial u^\mu}{\partial x^i} u^i = 0$$

Use covariant principle: $\partial_i \to \nabla_i$

$$(\nabla_i u^\mu) u^i = 0 = \partial_i u^\mu u^i + \Gamma^\mu_{ij} u^i u^j =$$

$$= \frac{\partial u^\mu}{\partial x^i}\frac{dx^i}{d\tau} + \Gamma^\mu_{ij} u^i u^j$$

$$= \underline{\underline{\frac{du^\mu}{d\tau} + \Gamma^\mu_{ij} u^i u^j}}$$

Static Isotropic Metric

1) Standard form

Solving spherically symmetric, static problem for the metric.

Requirement: as $\varphi = -\frac{GM}{r} \xrightarrow{r\to\infty} 0$, the metric should become

Minkowskian: $ds^2 = c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

Ansatz: $$ds^2 = B c^2 dt^2 - A\, dr^2 - C r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The problem is isotropic and time independant:

Thus A, B, C may only depend on r.

Furthermore, we're free to choose our coordinates and can absorb C into r with a coordinate transformation: $Cr^2 \to r^2$

This gives us:
$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2 d\Omega^2$$

with $A, B \to 1$ for $r \to \infty$

2) Robertson expansion

Expansion for the metric for weak fields, outside of the mass distribution.

The metric is unitless and depends only on G, M, c, r.

Only unitless quantity that can be formed is $\frac{GM}{c^2r}$.

For $\frac{GM}{c^2r} \ll 1$:
$$B(r) = \overbrace{1 - \frac{2GM}{c^2r}}^{g_{00}} + 2\underbrace{(\beta - \gamma)}_{1-1}\left(\frac{GM}{c^2r}\right)^2$$
$$A(r) = 1 + 2\gamma \frac{GM}{c^2r}$$

Schwarzschild Metric

Assume a static, spherically symmetric mass distribution with finite extension: $\varrho(r) \begin{cases} \neq 0 & r \leq r_0 \\ = 0 & r > r_0 \end{cases}$

Outside of the distribution, the pressure P vanishes and the inside of the mass distribution is static.
$\Rightarrow \partial_t T^{\mu\nu} = 0$

Ansatz: Standard form for the metric.

$$ds^2 = Bc^2dt^2 - Adr^2 - r^2 d\Omega^2$$

Solving for $r > r_0$:

$R_{\mu\nu} = 0$ for $\mu \neq \nu$ for the standard form in general.
$R_{\mu\mu} = 0$ needs to be set because

For $A(r \to \infty) = B(r \to \infty) = 1 \quad \Rightarrow AB = 1$
(Comes from $AB = \text{const}$ when solving for A, B with components of the Ricci tensor)

Solving gives: $B = 1 - \frac{2a}{r}$, $A = \frac{1}{1 - \frac{2a}{r}}$

Comparing with Newtonian limit:

$$g_{00} = B = 1 + \frac{2\Phi}{c^2} = 1 - \frac{2GM}{c^2 r} = 1 - \frac{2a}{r}$$

$\Rightarrow r_S = 2a = \frac{2GM}{c^2}$ Schwarzschild radius

Kruskal Coordinates

Let $T = \sqrt{\frac{r}{2GM} - 1}\, e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$

$R = \sqrt{\frac{r}{2GM} - 1}\, e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right)$

with Schwarzschild metric:

$$ds^2 = \frac{32 G^3 M^3}{r} e^{-r/2GM} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

$\to$ no singularity at $r = r_s = 2GM$, but still one at $r = 0$:

Using $\sinh^2 - \cosh^2 = -1$: $[\cosh^2 - \sinh^2 = 1]$

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}}$$

$\Rightarrow$ Event horizon is at $T^2 - R^2 = 0 \Rightarrow T = \pm R$

Real singularity $r = 0$:

$T^2 - R^2 = 1 \quad \to T^2 = 1 + R^2$

(all surfaces of constant r are hyperbolas on $T-R$ diagram)

The allowed regions are:

$$-\infty < R < \infty \quad , \quad T^2 < R^2 + 1$$

(At $T^2 = R^2 + 1$, $r = 0$, which is not allowed. For all other $r > 0$,
$T^2 - R^2 < 1 \Leftarrow \left(1 - \frac{r}{2GM}\right) e^{r/2GM} \Rightarrow T^2 < R^2 + 1$)

Surfaces of constant t:

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right) = \text{const} \quad \rightarrow \text{straight lines}$$

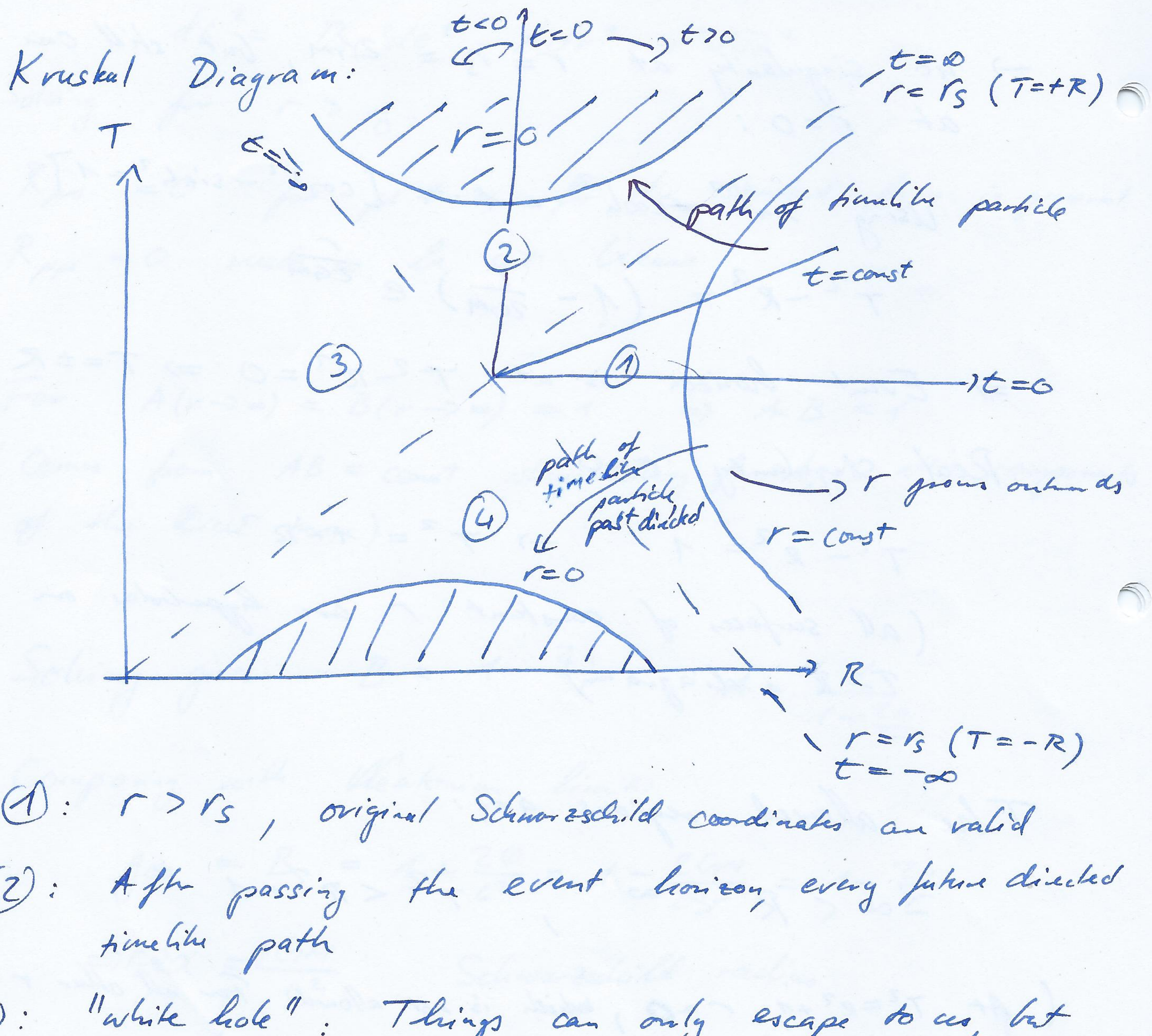
Surfaces of constant r:

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}} = \text{const} \quad \rightarrow \text{hyperbolas}$$

Light cones:

$$ds^2 = 0 \quad \Rightarrow \quad dR^2 - dT^2 = 0 \quad \Rightarrow \quad T = \pm R + \text{const}$$
$$\Rightarrow \quad 45^\circ \text{ between them}$$

Kruskal Diagram:



①: $r > r_s$, original Schwarzschild coordinates are valid

②: After passing the event horizon, every future directed timelike path

④: "white hole": Things can only escape to us, but nothing can reach it.

③: Can only be reached by following a spacelike path. The same as our spacetime, asymptotically flat.

General Equations of Motion

For the relativistic orbit $x^{\mu}(\lambda)$ we have:

$$\frac{d^2x^{\mu}}{d\lambda^2} = -\Gamma^{\mu}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}$$

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \left(\frac{ds}{d\lambda}\right)^2 = c^2\left(\frac{dt}{d\tau}\right)^2 = \begin{cases} c^2 & m \neq 0,\ \lambda = \tau \\ 0 & m = 0 \end{cases}$$

$$ds^2 = B(r)\,c^2dt^2 - dr^2 A(r) - r^2 d\Omega^2$$

These equations define the relativistic Kepler problem.

Solution:

Choose $\theta = \frac{\pi}{2}$ = constant $\qquad \frac{d^2\theta}{d\lambda^2} = 0 \Rightarrow$ angular momentum is conserved

$$r^2\frac{d\varphi}{d\lambda} = \text{const} = \ell$$

$$B\frac{dx^0}{d\lambda} = \text{const} = F$$

$$A\left(\frac{dr}{d\lambda}\right)^2 - \frac{F^2}{B} + \frac{\ell^2}{r^2} = \text{const} \equiv -\varepsilon = \begin{cases} c^2 & m \neq 0 \\ 0 & m = 0 \end{cases}$$

$$\Rightarrow \quad \frac{dr}{d\lambda} = \sqrt{\frac{1}{A}\left(\frac{F^2}{B} - \frac{\ell^2}{r^2} - \varepsilon\right)}$$

or using $r^2\frac{d\varphi}{d\lambda} = \ell = r^2\frac{d\varphi}{dr}\frac{dr}{d\lambda}$:

$$\frac{d\varphi}{dr} = \frac{\ell}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1} = \frac{\ell}{r^2}\sqrt{A\left(\frac{F^2}{B} - \frac{\ell^2}{r^2} - \varepsilon\right)^{-1}}$$

In the Schwarzschild metric:

$$A\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} - \frac{F^2}{B} + \varepsilon = 0$$

$$\underbrace{AB}_{=1}\left(\frac{dr}{d\lambda}\right)^2 + B\frac{\ell^2}{r^2} - F^2 + \varepsilon B = 0 =$$

$$= \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2a}{r}\right)\frac{\ell^2}{r^2} - F^2 + \varepsilon\left(1 - \frac{2a}{r}\right)$$

$$= \left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} - \frac{2a\ell^2}{r^3} - \frac{2a\varepsilon}{r} + \varepsilon - F^2 = \text{const}$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 \underbrace{- \frac{a\varepsilon}{r} + \frac{\ell^2}{2r^2} - \frac{a\ell^2}{r^3}}_{\equiv V_{eff}} = \frac{F^2 - \varepsilon}{2} = \text{const}$$

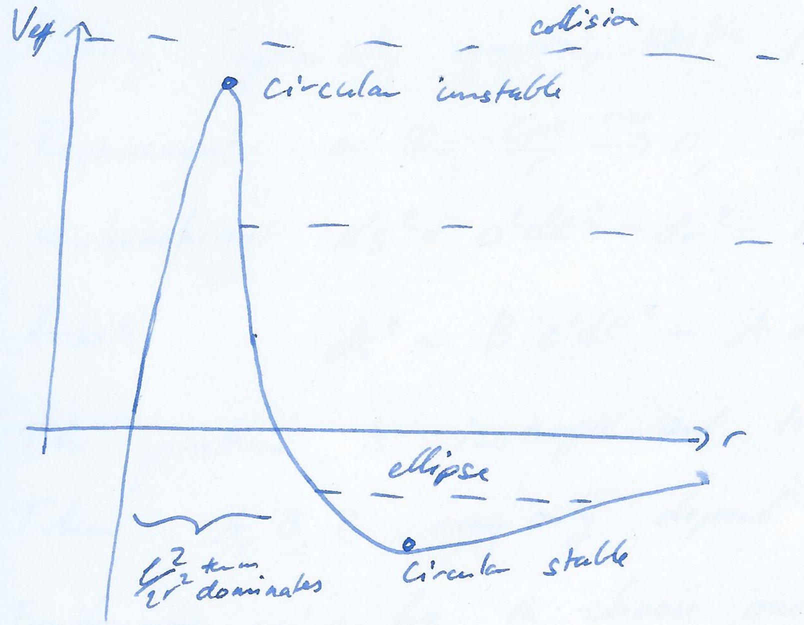
$$\rightarrow V_{eff} = \begin{cases} -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{c^2 r^3} & m \neq 0 \\ \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{c^2 r^3} & m = 0 \end{cases}$$

$\Rightarrow$ Formal solution:

$$\lambda = \pm \int \frac{dr}{\sqrt{2(\text{const} - V_{eff})}}$$

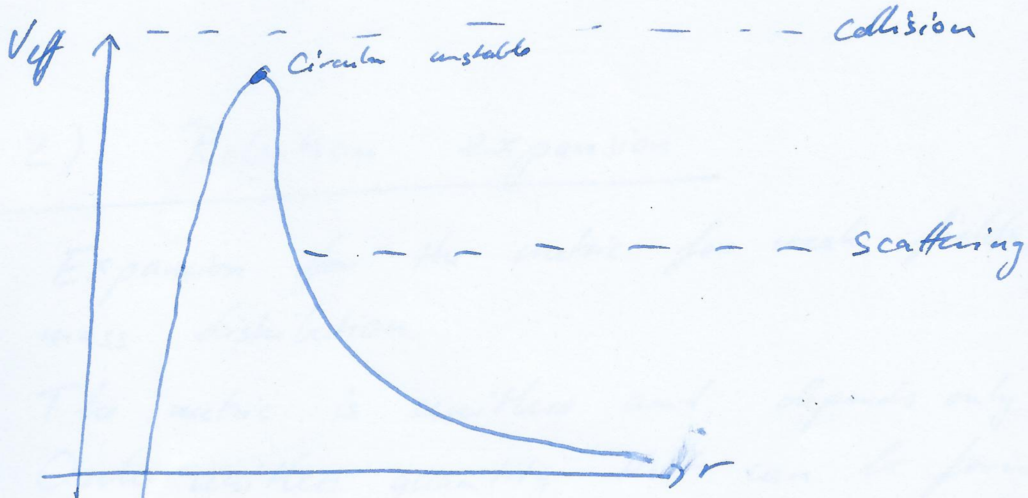
Effective Potential

i) For massive particles



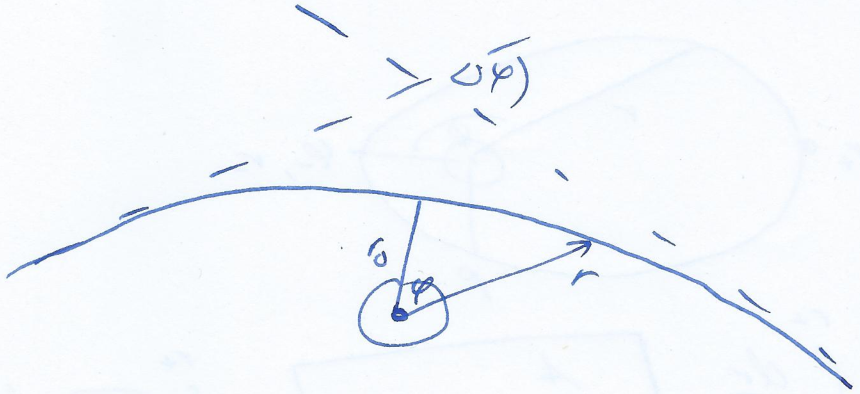
The $\frac{1}{r^3}$ term is new, it doesn't exist in the non-relativistic case.

ii) Massless particles



Both terms $\propto l^2$, Shape independant of l

Light Deflection



In static, spherically symmetric metric:

$$\varphi(r) = \varphi(r_0) + \int_{r_0}^{r} \frac{d\tilde{r}}{\tilde{r}}\, \ell \sqrt{\frac{A}{F^2/B + \frac{\ell^2}{r^2} - \underbrace{\varepsilon}_{=0}}}$$

Choose $\varphi(r_0) = 0$, and because a minimum: $\left.\frac{dr}{d\varphi}\right|_{r_0} = 0$
(point closest to sun)

Use $\left.\frac{dr}{d\varphi}\right|_{r_0}$ to relate F^2/B, ℓ^2/r^2

Then use Robertson expansion up to first order in a to solve the integral

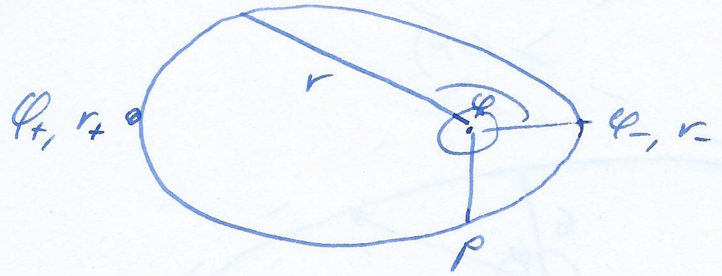
Find the deflection by $\Delta\varphi = 2\varphi(\infty) - \pi$

$$= \frac{2a}{r_0}(1+\gamma)$$

$$= \frac{2 r_S}{r_0} = \frac{r_S(1+1)}{r_0}$$

$\gamma = 1$ for GR

Perihelion Precession



$$\varphi_+ - \varphi_- = \int_{r_-}^{r_+} \frac{dr}{dr^2} \sqrt{\frac{A}{F^2/BL^2 - 1/r^2 - C^2/L^2}} \equiv \int_{r_-}^{r_+} \sqrt{\frac{A}{K}} \frac{dr}{r^2}$$

Define $\Delta\varphi = \text{orbit} - 2\pi = 2(\varphi_+ - \varphi_-) - 2\pi$

Find relations for $\frac{F^2}{BL^2} - \frac{1}{r^2} - \frac{C^2}{L^2} \equiv K$ in dependence of $B, r, B_\pm, r_\pm$ from minimum/maximum requirements:

$$\left.\frac{dr}{d\varphi}\right|_{r_\pm} = 0$$

Use Robertson expansion and clever substitution to solve the integral.

Solution (for GR): $\Delta\varphi = \frac{6\pi a}{p}$ with $p = \frac{1}{r_+} + \frac{1}{r_-}$

Killing Vectors

An infinitesimal coordinate transformation is a symmetry of the metric if $\mathcal{L}_K g_{\mu\nu} = 0 \iff \nabla_\varsigma K_\sigma + \nabla_\sigma K_\varsigma = 0$

Any 4-vector $K_\sigma(x)$ satisfying this equation will form a Killing vector.

Proof of equivalence:

$$\mathcal{L}_K g_{\mu\nu} = \partial_\lambda g_{\mu\nu} K^\lambda + g_{\mu\lambda} \partial_\nu K^\lambda + g_{\lambda\nu} \partial_\mu K^\lambda$$

Use $g_{\lambda\nu} \partial_\mu K^\lambda = \partial_\mu (g_{\lambda\nu} K^\lambda) - K^\lambda \partial_\mu g_{\lambda\nu}$

$$= \partial_\mu K_\nu - K^\lambda \partial_\mu g_{\lambda\nu}$$

$$\Rightarrow \mathcal{L}_K g_{\mu\nu} = \partial_\lambda g_{\mu\nu} K^\lambda + g_{\mu\lambda} \partial_\nu K^\nu + g_{\nu\lambda} \partial_\mu K^\lambda$$

$$= \partial_\lambda g_{\mu\nu} K^\lambda + \partial_\mu K_\nu - K^\lambda \partial_\mu g_{\lambda\nu} + \partial_\nu K_\mu - K^\lambda \partial_\nu g_{\lambda\mu}$$

$$= \partial_\mu K_\nu + \partial_\nu K_\mu + (\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu}) K^\lambda$$

$$= \partial_\mu K_\nu + \partial_\nu K_\mu - (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) g^{\lambda\alpha} K_\alpha$$

$$= \partial_\mu K_\nu + \partial_\nu K_\mu - 2\Gamma^\alpha_{\mu\nu} K_\alpha$$

$$= \nabla_\mu K_\nu + \nabla_\nu K_\mu$$

If K_1 and K_2 are Killing vectors, then $[K_1, K_2]$ is also a Killing vector: $[\mathcal{L}_{K_1}, \mathcal{L}_{K_2}] g_{\mu\nu} = \mathcal{L}_{[K_1, K_2]} g_{\mu\nu} = 0$

From symmetries of the metric conserved quantities follow:

If x^μ is a geodesic, then $K_\mu \dot{x}^\mu$ is constant along the metric:

$$D_{\dot{x}}(K_\mu \dot{x}^\mu) = \dot{x}^\nu D_\nu (K_\mu \dot{x}^\mu) =$$

$$= D_\nu K_\mu \dot{x}^\mu \dot{x}^\nu + K_\mu \underbrace{D_\nu \dot{x}^\mu \dot{x}^\nu}_{=0:\ \text{geodesic}}$$

$$= D_\nu K_\mu \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}(\underbrace{D_\nu K_\mu + D_\mu K_\nu}_{=0}) \dot{x}^\mu \dot{x}^\nu = 0$$

$J^\mu = T^{\mu\nu} K_\nu$ is a covariantly conserved current:

$$D_\mu J^\mu = D_\mu (T^{\mu\nu} K_\nu) = \underbrace{D_\mu T^{\mu\nu}}_{=0} K_\nu + T^{\mu\nu} D_\mu K_\nu$$

$$= T^{\mu\nu} \frac{1}{2}(D_\mu K_\nu + D_\nu K_\mu) = 0$$

Maximally Symmetric Spaces

A maximally symmetric space is a space with a metric with a maximal number of Killing vectors $\frac{n(n+1)}{2}$

A set of Killing vectors is linearly independent if $\sum_i c_i K_\mu^{(i)} = 0 \Leftrightarrow c_i = 0$

A Killing vector is completely determined everywhere by the values $K^\mu(x_0)$ and $\nabla_\mu K_\nu(x_0)$ at a single point x_0.

$\nabla_\mu K_\nu$ is antisymmetric: $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \Rightarrow \nabla_\mu K_\nu = -\nabla_\nu K_\mu$

Therefore, we can have at most $\frac{n(n+1)}{2}$ independent Killing vectors:

- n translational Killing vectors at every point (homogeneous space)
- $\nabla_\mu K_\nu$ antisymmetric $\rightarrow \frac{n(n-1)}{2}$ Killing vectors (isotropic space; rotation)

$\Rightarrow$ together: $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ Killing vectors

The Riemann curvature tensor simplifies for a maximally symmetric space.

Consider a maximally symmetric subspace $d\sigma^2 = A\,dr^2 + r^2 d\Omega^2$ which is spherically symmetric and homogeneous on each "plane of constant time".

Applying the standard form for the metric with $B=0$, we get $d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$

where k is a constant: $k = \begin{cases} +1 & \text{sphere, positive curvature} \\ -1 & \text{hyperbola, negative curvature} \\ 0 & \text{plane, zero curvature} \end{cases}$

($\mathcal{R} = n(n-1)k$ for maximally symmetric space)

The full metric then has the form:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

where $a(t)$ is the cosmic scale factor.

Friedmann Equation

Use the FRLW metric: $ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$

Calculate the Ricci tensor and scalar to obtain Einstein tensor:

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \quad G_{0i} = 0, \quad G_{ij} = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a^2} + \frac{k}{a^2}\right) g_{ij}$$

Treat the Universe as a perfect fluid: $T_{\mu\nu} = (\rho + \frac{p}{c^2}) u_\mu u_\nu - p g_{\mu\nu}$ with an equation of state: $p = p(\rho) = w\rho$

Non-interacting particles (dust) : $p = 0 = w$

Radiation: $p = 1/3\, \rho$

In a comoving frame, we set $u^\mu = (c, 0, 0, 0)$

The first Friedmann equation is obtained by the conservation law $\nabla_\mu T^{\mu\nu} = 0$ giving

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

which gives us a in dependence of ρ:

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} = -3(\rho + w\rho)\frac{\dot{a}}{a} = -3\rho(1+w)\frac{\dot{a}}{a}$$

$$\Rightarrow \frac{\dot{\rho}}{\rho}\frac{1}{-3(1+w)} = \frac{\dot{a}}{a} \Rightarrow \frac{da}{a} = \frac{d\rho}{\rho}\frac{1}{-3(1+w)} \Rightarrow \ln a = \frac{\ln \rho}{-3(1+w)} + \text{const}$$

$$\Rightarrow \ln(a) = \ln(\rho^{-3(1+w)}) + c \Rightarrow a \propto \rho^{-3(1+w)}$$

The other two equations can be obtained through the Field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}$$

For the 00 component:

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho + \Lambda$$

For the ij components:

$$\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) g_{ij} = \left(-8\pi G p + \Lambda\right) g_{ij}$$

for a perfect fluid in a comoving system.

Using the 00 equation in the ij equation gives:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) - \Lambda$$

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

For $\Lambda = 0$: $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \Rightarrow 1 = \frac{8\pi G}{3H^2}\rho - \frac{k}{a^2H^2}$

Then define $\rho_{crit} \equiv \frac{3H^2}{8\pi G}$, such that

$\rho > \rho_{crit}$: $k > 0 \Rightarrow k = 1$

$\rho = \rho_{crit}$: $k = 0 \Rightarrow$ flat universe

$\rho < \rho_{crit}$ $k < 0 \Rightarrow k = -1$ open universe

Now assume that $\rho = \rho_m + \rho_r$ and that matter and radiation are decoupled, meaning $\rho_m \propto a^{-3}$, $\rho_r \propto a^{-4}$

Then $K_r \equiv \frac{8\pi G}{3}\rho_r a^4$ and $K_m \equiv \frac{8\pi G}{3}\rho_m a^3$ are constant.

Insert them into the Friedmann equation:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda}{3}$$

$$\dot{a}^2 - \frac{8\pi G}{3}\rho_m a^2 - \frac{8\pi G}{3}\rho_r a^2 - \frac{\Lambda}{3}a^2 = -k$$

$$= \dot{a}^2 - \frac{K_m}{a} - \frac{K_r}{a^2} - \frac{\Lambda}{3}a^2 = -k$$

$$= \dot{a}^2 + V(a) = -k$$

with $V(a) = -\frac{K_m}{a} - \frac{K_r}{a^2} - \frac{\Lambda}{3}a^2$.

For $a \to 0$, K_m and K_r dominate the behaviour of a.

If K_m dominates: $\dot{a}^2 \approx \frac{K_m}{a} \Rightarrow \frac{3}{2}a^{3/2} = \sqrt{K_m}\,t \Rightarrow a \propto t^{2/3}$

If K_r dominates: $\dot{a}^2 \approx \frac{K_m}{a^2} \Rightarrow \frac{1}{2}a^2 = \sqrt{K_r}\,t \Rightarrow a \propto \sqrt{t}$

For $a \to \infty$, the $\frac{\Lambda}{3}a^2$ term dominates:

$$\dot{a}^2 \approx \frac{\Lambda}{3}a^2 \Rightarrow \ln a = \sqrt{\frac{\Lambda}{3}}t + \text{const} \Rightarrow a \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)$$

$\Lambda > 0$: No bound solutions

$\Lambda = 0$: $k = 1$: bounded solution

$k = 0$: expansion velocity $\to 0$

$k = -1$: expansion velocity $\to$ const.

$\Lambda < 0$: Bounded solutions. Periodical between $a = 0$, $a = a_{max}$