Linearised Field Equations

The field equations are non-linear. There is no standard procedure to solve the equations given a source $T^{\mu\nu}$ for the fields.

3 possibilities for types of solutions:

- Exact solutions assuming simplifying conditions
- Linearised field equations for weak fields
- Systematic expansion of the field equations

(- Numerical methods)

Ansatz: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$

This gives the Ricci tensor to first order:

$$R_{\mu\nu} = \frac{1}{2}\left(\Box h_{\mu\nu} + \partial_\mu \partial_\nu h^\alpha{}_\alpha - \partial_\mu \partial_\alpha h^\alpha{}_\nu - \partial_\nu \partial_\alpha h^\alpha{}_\mu\right)$$

$$= -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right)$$

($T_{\mu\nu}$ is also order h: $\underbrace{R_{\mu\nu}}_{\mathcal{O}(h)} - \frac{1}{2}\eta_{\mu\nu}\underbrace{R}_{\mathcal{O}(h)} = -\underbrace{\frac{8\pi G}{c^4}T_{\mu\nu}}_{\Rightarrow \mathcal{O}(h)}$)

$\rightarrow$ weak field

(= decoupled linearised field equations)

The field equations are covariant → coordinate transformation allowed. $|h_{\mu\nu}| \ll 1$ → only coordinate transformations that deviate slightly from Minkowski are allowed: $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$, $\varepsilon \ll 1$

From $g'_{\mu\nu} = \frac{dx^\alpha}{dx'^\mu}\frac{dx^\beta}{dx'^\nu} g_{\alpha\beta}$ find that $h_{\mu\nu}$ transforms as:

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \varepsilon^\mu}{\partial x_\nu} - \frac{\partial \varepsilon^\nu}{\partial x_\mu}, \quad h'_{\mu\nu} = \eta_{\alpha\mu}\eta_{\beta\nu} h'^{\alpha\beta}$$
$$= h_{\mu\nu} - \partial_\nu \varepsilon_\mu - \partial_\mu \varepsilon_\nu$$

This is called a gauge transformation. We can choose 4 functions ε^μ which give constraints on the "potentials" $h_{\mu\nu}$.

With $2\partial_\mu h^\mu{}_\nu = \partial_\nu h^\mu{}_\mu$ we get:

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right)$$

The solution for h is known from electrodynamics:

$$h_{\mu\nu}(\vec{r}, t) = -\frac{4G}{c^4}\int d^3r' \, \frac{S_{\mu\nu}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}$$

with $S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}$

(retarded potentials

Gravitational Waves

For weak gravitational fields in vacuum ($T^{\mu\nu}=0$):

$$\Box h_{\mu\nu} = 0$$

The simplest solution to this are plane waves.

In electromagnetism, the EM waves follow from $\Box A^{\mu} = 0$, which is exact, whereas gravitational waves are approximate linearisations.

Due to symmetry, $h_{\mu\nu} = h_{\nu\mu}$, 10 of 16 components are independent. With a gauge transformation we can impose 4 additional conditions. Furthermore, for the vacuum case we can perform a further transformation $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \varepsilon_\mu - \partial_\mu \varepsilon_\nu$ with $\Box \varepsilon^{\mu} = 0$. Such a transformation leaves the the wave equation and the gauge condition invariant.
We are left with 2 independent components for $h_{\mu\nu}$.
The solution to the wave equation can be written as

$$h_{\mu\nu} = e_{\mu\nu} \exp[-i k_\alpha x^\alpha] + \text{complex conjugate}$$

where $e_{\mu\nu}$ is the polarisation tensor, that is symmetric like $h_{\mu\nu}$. and $k^{\mu} = (\omega/c, \vec{k}_0)$ so that $k_\mu k^\mu = \eta^{\mu\nu} k_\nu k^\mu = \frac{\omega^2}{c^2} - k_0^2 = 0$

$$\Rightarrow \frac{\omega^2}{c^2} = k_0^2$$

Consider a wave along the x^3 direction:

$$h_{\mu\nu} = e_{\mu\nu} \exp[ik(x^3 - ct)]$$

Inserting this solution into the gauge condition $2\partial_\mu h^\mu{}_\nu = \partial_\nu h^\mu{}_\mu$ gives 4 conditions. Together with the symmetry, $e_{\mu\nu}$ is fully determined by 6 independent components.

With yet another transformation, we can gauge away all redundancies and are left with

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp[ik(x^3 - ct)] + cc$$

Assuming particles at rest at the $x^1 - x^2$ plane on a circle $\left(\frac{dx^i}{d\tau}\Big|_{\tau=0} = 0 \right)$, inserting $h_{\mu\nu}$ into Γ gives $\frac{d^2x}{d\tau^2}\Big|_{\tau=0} = 0$

$\Rightarrow \frac{dx^i}{d\tau} = const = 0 \quad = \quad x^i = const$

The coordinates stay the same, but the particles are no at rest because $g_{\mu\nu}$ varies with τ.

The actual distance ς to the centre of the circle $(x^1)^2 + (x^2)^2 = R^2$ is

$$\varsigma^2 = R^2 \begin{cases} 1 - 2h \cos(2\varphi)\cos(\omega t) & \text{if } e_{11} = h,\ e_{12} = 0 \\ 1 - 2h \sin(2\varphi)\cos(\omega t) & \text{if } e_{11} = 0,\ e_{12} = h \end{cases}$$

where $x^1 = R\cos\varphi$, $r^2 = R\sin\varphi$

$\rightarrow$ ellipse with small eccentricity. The type of oscillation one can infer the polarisation of the wave.

The two independent polarisation states form an angle of $\pi/4$. (+ and X)