Mechanical Equilibrium of a Star

Stars must be in mechanical equilibrium most of their life: Mechanical equilibrium is a necessary condition for stable luminosity and temperature over long periods of time.

It is a fundamental property of stars. The gravity pointing inward and the thermal pressure pointing outwards are in exact balance.

Any departure from this equilibrium is restored quickly through a strong reaction.
[For the sun: ~1/2 hour to restore.]
The mechanical equilibrium of a star governs all its properties and is always satisfied except in very short phases.

Momentum and Continuity Equations

Basic equations of hydrodynamics:

- Continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0}$$

expresses mass conservation

- Navier-Stokes equation:

$$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{a} - \frac{1}{\rho}\vec{\nabla} P + \nu \nabla^2 \vec{v}}$$

$\vec{a}$: acceleration due to external forces

P: pressure

ν: viscosity coefficient

expresses equation of motion.

We can simplify for a spherical star:

$$\frac{d\vec{v}}{dt} = (\ddot{r}, 0, 0), \quad \nabla P = \left(\frac{\partial P}{\partial r}, 0, 0\right), \quad \vec{a} = \vec{g} = (-g, 0, 0)$$

with $g = \dfrac{G M_r}{r^2} = \dfrac{G}{r^2} 4\pi \displaystyle\int_0^r \rho(r') r'^2 dr'$

and $M_r = 4\pi \displaystyle\int_0^r \rho(r') r'^2 dr'$ is the mass interior to radius r.

This simplification gives us (neglecting viscosity)

$$\boxed{\ddot{r} = -g - \frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{G M_r}{r^2}}$$

This is the momentum equation of hydrodynamic models.

The acceleration is due to two effects:

- pressure gradient $\frac{\partial P}{\partial r}$, which is always negative: Negative slope from center to surface
- Gravity, always pointing inward towards the center

If there is no fast radial motions, we have a situation of <u>hydrostatic equilibrium</u>:

The internal pressure gradient and the gravity are balancing each other perfectly:

$$\vec{\nabla} P = \rho \vec{g}$$

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = - \frac{G M_r}{r^2}$$

At any level in a star in equilibrium the gradient of pressure sustains the matter against the gravity force.

As for the mass conservation in spherical symmetry, we can write

$$d M_r (r, t) = \underbrace{4\pi r^2 \rho \, dr}_{\text{due to radius variation}} - \underbrace{4\pi r^2 \rho v \, dt}_{\text{mass escaping radius } r \text{ with positive velocity (pointing outward)}}$$

$$= \left.\frac{\partial M_r}{\partial r}\right|_t dr + \left.\frac{\partial M_r}{\partial t}\right|_r dt$$

$$\Rightarrow \left.\frac{\partial M_r}{\partial r}\right|_t = 4\pi r^2 \rho , \qquad \left.\frac{\partial M_r}{\partial t}\right|_r = -4\pi r^2 \rho v$$

Since dM_r is an exact integral, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r} M_r\right) = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} M_r\right)$$

$$\frac{\partial}{\partial t}\left(4\pi r^2 \rho\right) = \frac{\partial}{\partial r}\left(-4\pi r^2 \rho v\right)$$

$$4\pi r^2 \frac{\partial \rho}{\partial t} = -4\pi \quad \frac{\partial}{\partial r}(r^2 \rho v)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0$$

which is again the continuity equation, but in spherical coordinates.

Note that $\vec{\nabla} f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f)$ in spherical coordinates. [Divergence, not gradient!]

In a static situation, we have

$$v = 0$$

$$\frac{d}{dt} = 0$$

$$\Rightarrow \boxed{\frac{\partial M_r}{\partial r} = 4\pi \rho r^2}$$

The coordinate r is not always convenient as an independent variable. Instead, M_r is usually a better choice:

$$\frac{\partial}{\partial r} = \frac{\partial M_r}{\partial r} \frac{\partial}{\partial M_r} = 4\pi \rho r^2 \frac{\partial}{\partial M_r}$$

$$\frac{\partial}{\partial M_r} = \frac{\partial r}{\partial M_r} \frac{\partial}{\partial r} = \left(\frac{\partial M_r}{\partial r} \right)^{-1} \frac{\partial}{\partial r} = \frac{1}{4\pi \rho r^2} \frac{\partial}{\partial r}$$

If we insert it in the equation of motion:

$$\ddot{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{G M_r}{r^2} = -\frac{1}{\rho} 4\pi \rho r^2 \frac{\partial P}{\partial M_r} - \frac{G M_r}{r^2}$$

or $$\boxed{\frac{\ddot{r}}{4\pi r^2} = -\frac{\partial P}{\partial M_r} - \frac{G M_r}{4\pi r^4}}$$

Giving the hydrostatic equilibrium equation for $\ddot{r} = 0$

$$\boxed{\frac{\partial P}{\partial M_r} = -\frac{G M_r}{4\pi r^4}}$$

These equations already allow us to make order-of-magnitude estimates for pressure, temperature and timescales.

Pressure

Using hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{-GM_r}{r^2}$$

S for surface, C for center

$$\rightsquigarrow \frac{1}{\bar{\rho}}\frac{P_S - P_C}{R_S - R_C} = \frac{-G\, M_S/2}{(R_S/2)^2} = -2\frac{GM_S}{R_S^2} = -\frac{P_C}{R_S}\frac{1}{\bar{\rho}}$$

when we approximate with average values

$$M_r \approx \frac{M(\text{Surface}) + M(\text{center})}{2} = \frac{M_S}{2}$$

$$r \approx \frac{R(\text{Surface}) + R(\text{center})}{2} = \frac{R_S}{2}$$

with $R_{center} = 0$, $M_r(\text{center}) = 0$

If we now approximate $\bar{\rho} = \frac{M_S}{\frac{4}{3}\pi R^3}$

we get

$$\frac{P_C}{R} \sim 2\frac{GM}{R^2} \cdot \frac{M}{\frac{4}{3}\pi R^3} = \frac{3}{2\pi}\frac{GM^2}{R^5}$$

$$\Rightarrow \boxed{P_C \sim \frac{3}{2\pi}\frac{GM^2}{R^4}}$$

For the sun, we get $P_C \sim 5.4 \cdot 10^{15} \frac{g}{s^2 cm}$

But more importantly: $P_C \propto M^2$, $P_C \propto R^{-4}$

Temperature

Using the law for perfect gasses:

$$PV = NkT$$

$$\rightarrow P = \frac{N}{V} kT = nkT = \frac{\rho}{\mu m_u} kT$$

with k = Boltzmann constant

μ = mean molecular weight in units of m_u

m_u = atomic mass weight

This gives us

$$T = \frac{\mu m_u P}{k \rho}$$

using $\bar{P} \sim \frac{1}{2} P_c \sim \frac{1}{2} \frac{3}{2\pi} \frac{GM^2}{R^4}$

$\bar{\rho} \sim \frac{4}{3} \pi M / R^3$

$$\rightarrow \bar{T} \sim \frac{\mu m_u}{k} \frac{3}{4\pi} \frac{GM^2}{R^4} \cdot \frac{3R^3}{4\pi M}$$

$$\sim \frac{\mu m_u}{k} \frac{GM}{R}$$

$$\Rightarrow \bar{T} \propto M, \quad \bar{T} \propto R^{-1}$$

Dynamical timescales

The dynamical timescale characterises departures from mechanical equilibrium.
Suppose the initial pressure gradient in a gravitationally bound configuration becomes negligible (e.g. collapse), then

$$\ddot{r} = -\frac{G M_r}{r^2}$$

we approximate again:

$$\ddot{r} \sim \frac{R_f - R_S}{\tau_{dyn}^2} = \frac{-R}{\tau_{dyn}^2}$$

$$M_r \sim M$$

$$r \sim R$$

$$\rightarrow \quad \frac{R}{\tau_{dyn}^2} \sim \frac{GM}{R^2}$$

$$\Rightarrow \boxed{\tau_{dyn} \sim \sqrt{\frac{R^3}{GM}} \sim \frac{1}{\sqrt{G\varrho}}}$$

If the internal pressure is negligible, the star collapses under its own gravity and the dynamical timescale τ_{dyn} is essentially the free-fall timescale.

The dynamical timescale is very short for stars:
~ a few seconds for white dwarfs, ~ 30 min for the sun and ~ 40 days for red giants.

This implies that the mechanical equilibrium is always very quickly and closely adjusted due to the fast mechanical response of the star.

We can arrive at the same result if we assume that the gravity becomes negligible compared to the pressure gradient:

$$\ddot{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{GM_r}{r^2} \approx -\frac{1}{\rho}\frac{\partial P}{\partial r}$$

$$\rightsquigarrow \frac{R}{\tau_{dyn}^2} \sim \frac{1}{2\bar{\rho}}\frac{P}{R} = -\frac{1}{\bar{\rho}}\frac{\frac{1}{2}(P_c + P_S)}{R_c - R_S}$$

$$\rightarrow \frac{R^2}{\tau_{dyn}^2} \sim \frac{P_c}{2\bar{\rho}} \sim 2\frac{1}{2}\frac{P_c}{M/R^3\frac{4\pi}{3}} \sim \frac{\frac{3}{4\pi}\frac{GM^2}{R^4}}{\frac{M}{\frac{4}{3}\pi R^3}} = \frac{GM}{R}$$

$$\Rightarrow \tau_{dyn} \sim \sqrt{\frac{GM}{R^3}}^{-1} \approx \frac{1}{\sqrt{G\bar{\rho}}}$$

The sound speed in gas is given by

$$c_s = \sqrt{\frac{\Gamma_1 P}{\rho}}$$

for a perfect gas, $\Gamma_1 = \gamma = \frac{5}{3} = \frac{c_p}{c_v}$

Since we had (factors omitted)

$$\frac{R}{\tau_{dyn}} \sim \sqrt{\frac{P}{\rho}} \sim c_s$$

$$\Rightarrow \tau_{dyn} \sim R/c_s$$

This means that the mean dynamical timescale of a star is of the order necessary for the sound speed to cross the stellar radius.

This is not surprising since the sound velocity characterizes the pressure adjustments.

The Potential Energy

Mechanical equilibrium of a star implies that the gravitational energy, i.e. the potential energy, is of the same order as the thermal energy, which supports the star against gravitation.

Consider a non-rotating spherical star in the process of formation by addition of new mass elements. Let M_r be the mass already collected at the interior of radius r. The work dW provided by the gravitational force $\vec{F}$ when it brings a new mass element δM_r from radius $r+dr$ to r is

$$dW = \vec{F} \cdot d\vec{r} = \frac{G M_r \, \delta M_r}{r^2} \, dr$$

Since $\vec{F}$ and $d\vec{r}$ have the same direction, this work is positive.

The work δW to bring δM_r from infinity to the radius r is then

$$\delta W = G M_r \, \delta M_r \int_{\infty}^{r} \frac{dr}{r^2} = -\frac{GM}{r} \delta M_r .$$

One defines the potential energy for a mass element as $\delta\Omega = \delta W$. The formation of an entire star of mass M represents a potential energy

$$\boxed{\Omega = -G \int_0^M \frac{M_r \, dM_r}{r}}$$

This is the energy lost by the reservoir of gravitational energy during the formation of a star. The energy lost by the initial cloud is gained, for example, by the thermal energy of the gas.

Often one writes the potential energy of a star in the simplified form:

$$\Omega = -q \frac{GM^2}{R}$$

where q is a numerical factor which depends on the internal energy distribution.

$$q = \frac{5}{3} \quad \text{for constant density}$$

$$q = \frac{3}{2} \quad \text{for MS stars.}$$

We can also express the gravitational energy via the potential:

$$\vec{g} = -\vec{\nabla}\phi \quad , \quad g = \frac{\partial \phi}{\partial r} \quad \text{in spherical c.}$$

Starting from the Poisson equation:

$$\vec{\nabla}^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 4\pi G \rho$$

$$r^2 \frac{\partial \phi}{\partial r} = \int 4\pi G \rho r^2 dr = G M_r$$

$$\phi(r) = \int_0^r \frac{G M_r}{r^2} dr + \text{const}$$

the const is chosen such that $\phi(r=\infty) = 0$.

We can write:

$$-\Omega = G \int_0^M \frac{M_r \, dM_r}{r} = G \int_0^M \frac{1}{2} \frac{dM_r^2}{r}$$

We can also express the potential energy as a function of pressure:

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \qquad \frac{dM_r}{dr} = 4\pi\rho r^2$$

$$\Rightarrow \frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \cdot 1 = -\rho \frac{GM_r}{r^2} \frac{dM_r}{dr} \frac{1}{4\pi\rho r^2}$$

$$= \frac{-GM_r \, dM_r}{4\pi r^4 \, dr}$$

$$\Rightarrow dP = \frac{-GM_r}{4\pi r^4} dM_r \qquad \Rightarrow \quad -\frac{GM_r}{r} dM_r = 4\pi r^3 dP$$

Then

$$\Omega = -G \int_0^M \frac{M_r}{r} dM_r = + \int_0^R dP \, 4\pi r^3$$

integrate by parts:

$$u = 4\pi r^3 \qquad v = P$$

$$u' = 12\pi r^2 dr \qquad v' = dP$$

$$= 4\pi r^3 P \Big|_0^R - \int_0^R 12\pi r^2 P \, dr \qquad \Big| dV = 4\pi r^2 dr$$

$$\boxed{= 4\pi R^3 P(R) - 3 \int P dV}$$

at the surface, we can write $P(R) \approx 0$

$$\hookrightarrow \Omega = -3 \int P dV$$

We can once again estimate the temperature of the star via

$$P = \frac{k}{\mu m_u} \varrho T \quad , \quad \Omega = q \frac{GM^2}{R}$$

$$\Rightarrow \Omega = -3 \int P \, dV = -3 \frac{k}{\mu m_u} \int \varrho T \, dV$$

$$= -3 \frac{k}{\mu m_u} \int_0^M T \, dM_r = -3 \frac{k}{\mu m_u} \overline{T} M$$

$$= q \frac{GM^2}{R}$$

$$\Rightarrow \overline{T} = \frac{-\mu m_u}{k} q \frac{GM}{R}$$

again, we get

$$\overline{T} \propto M \quad , \quad \overline{T} \propto R^{-1}$$

where $\overline{T}$ is the internal temperature averaged over the stellar mass.

The Virial Theorem for Stars

Let u be the density of the translational kinetic energy of a gas.

We have

$$P = \frac{2}{3} u \quad \text{for non-relativistic gas}$$

$$P = \frac{1}{3} u \quad \text{for relativistic gas.}$$

We have that

$$\frac{1}{3} \leq \frac{P}{u} \leq \frac{2}{3}.$$

For a perfect gas, we have

$$P = \frac{k}{\mu m_H} \rho T$$

$$u = \frac{3}{2} N k T = \frac{3}{2} \frac{\rho}{\mu m_H} k T$$

Using again our expression for the potential energy

$$\Omega = -3\int P\,dV = -3\int \frac{2}{3} u\,dV = -2E_{kin}$$

for non-relativistic gas

$$\Rightarrow \boxed{2E_{kin} + \Omega = 0}$$

The total energy of a star is given by

$$E_{tot} = E_{kin} + \Omega = -\frac{\Omega}{2} + \Omega = \frac{\Omega}{2} = -E_{kin} < 0$$

On the other hand, we have for relativistic particles:

$$\Omega = -3\int P\,dV = -3\int \frac{1}{3} u\,dV = -E_{kin}$$

$$\Rightarrow E_{tot} = E_{kin} + \Omega = 0$$

$\Rightarrow$ A star of relativistic particles is not stable. Any negligible energy would spread it out.

Let us first consider the virial equilibrium for a star of perfect gas.

For every average particle, we have

$$\overline{E}_{kin,\,per\,part} = \frac{1}{2}\mu m_u \overline{v^2} = \frac{3}{2} k \overline{T}$$

$$U = C_v \mu m_u \overline{T} \qquad \text{internal energy}$$

And for the entire star, it is

$$E_{kin} = \frac{3}{2} N k T$$

$$U = C_v N \mu m_u T$$

C_v is the specific heat at constant volume, and C_p is the specific heat at constant pressure. We always have $C_p > C_v$, since for C_p the volume isn't constant, and you therefore need to insert extra energy to increase the volume.

The internal energy U contains all forms of energy: thermal, radiation, atomic excitation, ionization, electron degeneracy, etc.

We can relate U and E_{kin} via temperature:

$$NT = \frac{2}{3} \frac{E_{kin}}{k}$$

$$\Rightarrow U = c_v T N \mu m_u = c_v \frac{2}{3} \frac{E_{kin}}{\frac{k}{\mu m_u}}$$

$$= \frac{2}{3} \frac{c_v}{c_p - c_v} E_{kin} = \frac{2}{3} \frac{1}{\gamma - 1} E_{kin}$$

where we used $c_p - c_v = \frac{k}{\mu m_u}$, which we will show later, and

$$\gamma = \frac{c_p}{c_v} > 1 \quad \text{for ideal gases.}$$

For a perfect gas, we have only kinetic energy as internal energy, i.e. $U = E_{kin}$

In that case:

$$E_{kin} = \frac{2}{3} \frac{1}{\gamma - 1} E_{kin}$$

$$\hookrightarrow \gamma - 1 = \frac{2}{3} \quad \Rightarrow \quad \underline{\gamma = \frac{5}{3}}$$

The virial theorem for a star of perfect gas becomes

$$\boxed{3(\gamma - 1)U + \Omega = 0}$$

Computing the total energy

$$E_{tot} = \Omega + U = \Omega + \frac{2 E_{kin}}{3(\gamma-1)}$$

$$= \Omega + \frac{2}{3(\gamma-1)}\left(-\frac{1}{2}\Omega\right) = \Omega\left(1 - \frac{1}{3(\gamma-1)}\right)$$

$$= \Omega\left(\frac{3\gamma - 3 - 1}{3\gamma - 3}\right) = \Omega \, \frac{3\gamma - 4}{3\gamma - 3}$$

In order to have a bound system, E_{tot} should be negative. Since $\Omega \leq 0$, we have the condition that

$$\frac{3\gamma - 4}{3\gamma - 3} > 0$$

Since $\gamma = c_p / c_v$ and $c_p > c_v$, we can't have a singularity at $\gamma = 1$.
However, there is no stable / bound solution for $\gamma < 4/3$.
$\Longrightarrow$ The thermodynamical properties of a gas are immensely important to the stability of a star.

This condition is satisfied for the perfect gas with $\gamma = 5/3$.

What happens when the star is perturbed, i.e. changes the radius?
We assume it doesn't lose mass. Since the dynamical timescale is very short, we can also consider this to be an adiabatic perturbation.

We have from the first law of thermodynamics:

$$dQ = dU + P\,dV$$

for specific quantities: $q = \frac{Q}{M}$, $u = \frac{U}{M}$,

$$\frac{dV}{M} = d\left(\frac{1}{\varrho}\right) = -\frac{1}{\varrho^2}\,d\varrho \quad \text{for } M = \text{const.}$$

we get

$$dq = du - \frac{P}{\varrho^2}\,d\varrho$$

We may approximate:

$$\Delta q = \Delta u - \frac{P}{\varrho^2}\,\Delta\varrho = 0$$ for adiabatic processes.

We can also introduce a generalized adiabatic exponent

$$\Gamma_1 \equiv \left.\frac{\partial \ln P}{\partial \ln \rho}\right|_{ad} \longrightarrow \frac{\Delta P}{P} - \Gamma_1 \frac{\Delta \rho}{\rho} = 0$$

Then using

$$\Omega = -3\int P\,dV = -3\int_0^M P\,\frac{dM_r}{\rho}$$

we can express the change in potential energy as

$$\Delta\Omega = -3\Delta\int_0^M \frac{P}{\rho}\,dM_r =$$

$$= -3\int\left(\frac{\Delta P}{\rho} - \frac{P}{\rho^2}\Delta\rho\right) dM_r$$

$$\frac{\Delta P}{P} = \Gamma_1 \frac{\Delta \rho}{\rho}$$

$$\Delta u = \frac{P}{\rho^2}\Delta\rho$$

$$\hookrightarrow \frac{\Delta P}{\rho} = \Gamma_1 \frac{\Delta\rho}{\rho^2}\cdot P = \Gamma_1 \Delta u$$

$$= -3\int(\Gamma_1 \Delta u - \Delta u)\,dM_r$$

$$= -3(\overline{\Gamma_1}\Delta U - \Delta U)$$ with $\overline{\Gamma_1}$ mass-averaged Γ_1

After integrating over the change & assuming the integration constant is zero, we can write

$$\Omega = -3U(\bar{\Gamma}_1 - 1)$$

$$\Rightarrow \Omega + 3U(\bar{\Gamma}_1 - 1) = 0$$

This applies for a star in equilibrium for a general EoS with $\Gamma_1 \equiv \frac{d \ln P}{d \ln \rho}$.

If $\Gamma_1 < 4/3$, we again have the total energy $=$ [illegible] and instabilities may appear. Many unstable or explosive events in stellar evolution occur because Γ_1: stellar pulsations, pair instability supernovae, core collapse, neutron star formation, black hole formation, etc.

Γ_1 appears as a most fundamental physical parameter in stars.

Kelvin-Helmholtz timescale: Slow Contraction

Consider a slow contraction: timescale of the contraction is much larger than dynamical timescale $\tau_{dyn} \sim \frac{1}{\sqrt{G\bar{\varsigma}}}$. Since the dynamical timescale characterizes the typical times for departures from mechanical equilibrium, and the contraction takes much more time, we can assume that the virial theorem is satisfied throughout.

This implies:

$$R \rightarrow R - \Delta R$$

$$\Omega \rightarrow \Omega - \Delta\Omega \,, \Delta\Omega < 0$$

$$U \rightarrow U + \Delta U \,, \Delta U > 0$$

Once again we can relate the internal energy

$$U = \frac{2}{3} \frac{1}{\gamma - 1} E_{kin}$$

to the kinetic energy and insert it into the virial theorem:

$$\Omega + 2 E_{kin} = \Omega + 3(\gamma - 1) U = 0$$

The virial theorem relates the changes ΔU and $\Delta \Omega$ accordingly:

$$\Delta \Omega + 3(\gamma - 1)\Delta U = 0$$

$$\rightarrow \Delta U = -\frac{\Delta \Omega}{3(\gamma - 1)}$$

Clearly the gravitational energy change will not be entirely converted into internal energy, the factor $[3(\gamma-1)]^{-1}$ is present.

The difference between the energy produced by gravitation and what is converted into internal energy will be radiated away:

$$\Delta E_{rad} = (-\Delta \Omega) - \Delta U = -\Delta \Omega + \frac{\Delta \Omega}{3(\gamma - 1)}$$

$$= (-\Delta \Omega) \frac{3(\gamma-1) - 1}{3(\gamma - 1)} = (-\Delta \Omega) \frac{3\gamma - 4}{3\gamma - 3}$$

For monoatomic gas, we have $\gamma = 5/3$ and $\Delta E_{rad} = (-\Delta \Omega) \frac{5-4}{5-3} = \frac{-\Delta \Omega}{2}$

so half the energy liberated by the decrease of potential energy is radiated away.

Expressing $\Omega = -q \frac{GM^2}{R}$ and $E_{rad} = \bar{L} t$,

we get

$$E_{rad} = \bar{L} t = \frac{3\gamma - 4}{3\gamma - 3} \, q \, \frac{GM^2}{R}$$

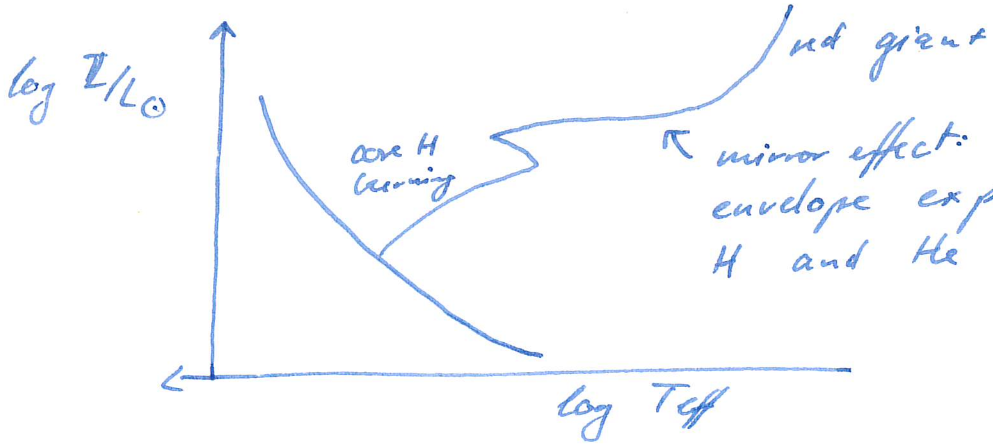
From which we can define a typical timescale,

$$\tau_{KH} \sim \frac{GM^2}{R \bar{L}}$$

the Kelvin–Helmholtz timescale. It is the timescale during which a star can produce an average luminosity $\bar{L}$ at the expense of the gravitational energy. For the sun,

$$\tau_{KH} \sim 3 \cdot 10^7 \text{ yrs.}$$

It is not a good estimate for the age of the star because it neglects the internal heat sources. Nevertheless, there are stages in the lifetime of a star that have τ_{KH} as the typical timescale, e.g. pre-main sequence phase of evolution and contraction phases which separate the phases of nuclear burning.



↖ mirror effect: core contracts, envelope expands between H and He burning.

Again using the virial theorem, we can show that if the core collapses the envelope will expand.

We assume that the contraction is once again slow, so the virial theorem holds, and that the radiated energy is negligibly small.

Then:

$$\Omega + 2E_{kin} = 0$$

$$E_{tot} = \Omega + E_{kin} = \text{const} = \frac{1}{2}\Omega$$

$\Rightarrow$ both Ω and E_{kin} are const.

Since $\Omega = \text{const} \Rightarrow \frac{d\Omega}{dt} = 0$

We can separate Ω into two terms for the envelope and the core:

$$\Omega = \Omega_c + \Omega_{env}$$

$$= -\frac{GM_c^2}{R_c} - \frac{GM_c M_{env}}{R}$$

with $\frac{d\Omega}{dt} = 0$:

$$\frac{d\Omega}{dt} = + \frac{G M_c^2}{R_c^2} \frac{dR_c}{dt} + \frac{G M_c M_{env}}{R^2} \frac{dR}{dt} = 0$$

$\Rightarrow$ $\frac{dR}{dt}$ will have opposite sign to $\frac{dR_c}{dt}$

in this situation.

$\Rightarrow$ An expanding core will lead to a contracting envelope and vice versa.

The Energetic Equilibrium of Stars

The energetic equilibrium, i.e. the balance between the energy production and its transport out of the star, is another essential piece of the general stellar equilibrium. The energy production "needs" to adjust its rate to match radiative losses at the surface.

If it for example overproduced energy in the core, energy would accumulate at the core and the core would expand. This inflation would decrease the inner pressure and temperature and as a consequence slow down the nuclear reactor.

There are four main processes to move energy:

- radiative transfer
- convection (macroscopic movement)
- conduction (microscopic movement)
- neutrinos

Radiative Transfer

Radiative transfer is the most important process of energy transport in stars.

The radiative flux is given by

$$F_\nu = -\frac{4acT^3}{3\kappa_\nu \rho}\frac{dT}{dr}$$ for frequency ν

$$F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$ bolometric with κ = Rosseland Mean opacity

- the flux depends on the temperature gradient
- It depends on the opacity κ; The higher the opacity, the lower the flux.
- The mean free path of a photon is given by $l \approx \frac{1}{\kappa\rho}$
- For spherical systems, $F_\nu = \frac{L_\nu}{4\pi r^2}$

We can already make a <u>mass-luminosity relation</u> estimate:

$$F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} = \frac{L}{4\pi R^2}$$

approximate $\frac{dT}{dr} \sim \frac{T}{R}$, and use

$$T \propto \frac{\mu M}{R}, \quad \rho \propto \frac{M}{R^3}$$

Then

$$\frac{L}{4\pi R^2} \propto \frac{T^3}{\kappa\rho}\frac{T}{R}$$

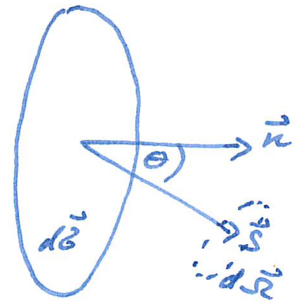
$$\rightarrow L \propto \frac{RT^4}{\kappa\rho} \propto \frac{R\,\frac{\mu^4 M^4}{R^4}}{\kappa\,\frac{M}{R^3}} = \frac{\mu^4 M^3}{\kappa}$$

$$\Rightarrow L \propto \mu^4, \quad L \propto M^3, \quad L \propto \kappa^{-1}$$

$\Rightarrow$ more transparent matter radiates more

Let us go through some basic definitions of radiative transfer.

Let I_ν be the specific intensity of a radiation beam.



Consider a medium with a radiation beam at frequency ν in a given direction $\vec{s}$. The energy dU_ν transmitted by a surface element $d\vec{\sigma}$ in a direction making an angle θ with the normal $\vec{n}$ to $d\vec{\sigma}$ over the length ds centered on a solid angle $d\Omega$, during the time dt and in the frequency interval $d\nu$

is

$$dU_\nu \equiv I_\nu \, d\sigma \cos\theta \, d\nu \, d\Omega \, dt$$

this defines the specific intensity I_ν.

Its units are

$$[I_\nu] = \frac{\text{ergs}}{\text{cm}^2 \text{ strad } s^{-1} \text{ Hz}}$$

You can describe dU_ν as the energy going through the surface $d\sigma$ in the solid angle $d\Omega$ during the time dt along the direction $\vec{s}$.

The <u>density of radiation energy</u> is the ratio

$$\frac{dU_\nu}{dV} = \frac{dU_\nu}{d\sigma \cos\Theta\, c\, dt}$$

For monochromatic energy, we have

$$du_\nu = \frac{dU_\nu}{dV\, d\nu} = \frac{dU_\nu}{d\sigma \cos\Theta\, c\, dt\, d\nu} = \frac{I_\nu}{c}\, d\Omega$$

By integrating over all directions, we get

$$\boxed{u_\nu = \int_\Omega \frac{I_\nu}{c}\, d\Omega = \int_0^\pi \frac{I_\nu}{c}\, 2\pi \sin\Theta\, d\Theta = \frac{4\pi}{c} I_\nu}$$

If we only consider one direction, the total radiation density is

$$u = \frac{1}{c} \int_\nu I_\nu\, d\nu$$

The <u>flux</u> is defined as the energy (passing through a surface) per unit surface per unit time.

For a monochromatic flux, we have

$$dF_\nu = \frac{dU_\nu}{dA\, dt\, d\nu} = I_\nu \cos\theta \, d\Omega$$

$$\Rightarrow \boxed{F_\nu = \int_\Omega I_\nu \cos\theta \, d\Omega}$$

In case of isotropy: $I_\nu = I_{\nu,0}$, independent of direction.

Then

$$F_\nu = \int_\Omega I_\nu \cos\theta \, d\Omega = \int_\Omega I_\nu \cos\theta \, 2\pi \sin\theta \, d\theta =$$

$$= 2\pi I_{\nu,0} \int_0^\pi \cos\theta \sin\theta \, d\theta = 2\pi I_{\nu,0} \int_0 \frac{\cos^2(\theta)}{2}$$

$$= 2\pi I_{\nu,0} \, \frac{1}{2} \sin^2\theta \Big|_0^\pi = 0$$

However, we may only consider the outgoing flux F_ν^+, where we integrate $\theta \in [0, \pi/2]$ only.

We then get

$$F_\nu = F_\nu^+ + F_\nu^- = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \, I_{\nu,0} \cos\theta \sin\theta \quad + \int_{\pi/2}^{\pi} [\ldots]$$

$$= 2\pi I_{\nu,0} \cdot \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} + [\ldots]_{\pi/2}^{\pi}$$

$$= \pi I_{\nu,0} \quad + (-\pi I_{\nu,0})$$

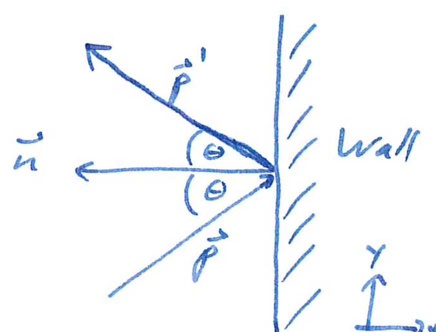
The monochromatic luminosity L_ν is the energy emmited by a star over all directions by units of time and frequency:

$$dL_\nu = \frac{dU_\nu}{dt\, d\nu} = I_\nu \cos\theta \, d\Omega \, d\sigma$$

$$L = \int_\Sigma \int_\Omega I_\nu \cos\theta \, d\Omega \, d\sigma = \int_\Sigma F_\nu^+ \, d\sigma = 4\pi R^2 F_\nu^+$$

where Σ is the total stellar surface.

To derive an expression for radiation pressure, consider particles hitting on a wall with incoming angle θ w.r.t. the surface normal.



The change in momentum of the particle is $2p\cos\theta$:

$$p_y' = p_y \,, \quad p_x' = -p_x = -p\cos\theta$$

$$\hookrightarrow |\vec{p} - \vec{p}\,'| = |(0,\ 2p_x)| = 2p\cos\theta$$

Now let $n(p, \theta)$ be the number of particles per unit volume having the momentum p and moving along the direction given by θ.

This gives us

$$dP(p,\theta) = \underbrace{n(p,\theta)\overbrace{\cos\theta\, v(p)}^{\text{velocity in the right direction}}}_{\substack{\text{Flux of particles in}\\ \theta\text{-direction;}\\ \text{Number density} \cdot \text{velocity}}} \underbrace{2p\cos\theta\, d\theta\, dp}_{\text{momentum change}}$$

$$\Rightarrow P = \iint_{p\,\Omega} 2pv(p)\, n(p,\theta) \cos^2 d\Omega$$

can as well do $d\Omega$ instead just $d\theta$

Assuming the distribution of the directions are isotropic, we can write

$$n(p, \theta, \varphi)\, dp\, d\Omega = n(p)\, dp\, \frac{d\Omega}{4\pi}$$

This gives us

$$P_{iso} = \int_p \int_\Omega 2 p v(p)\, n(p,\theta) \cos^2\theta \, d\Omega$$

$$= \int_p \int_{\Omega^+} 2 p v(p)\, n(p) \cos^2\theta \, \frac{d\Omega}{4\pi}$$

$$= \int_p dp \int_{2\pi} d\varphi \int_0^{\pi/2} d\theta \, \frac{2p}{4\pi} v(p)\, n(p) \cos^2\theta \sin\theta \, d\theta$$

$$\boxed{= \frac{1}{3} \int_p n(p)\, v(p)\, p \, dp}$$

For non-relativistic particles:

$$p = mv, \quad v = \frac{p}{m}$$

$$P = \frac{1}{3} \int n(p) \frac{p^2}{m} dp$$

$$= \frac{1}{3} \int_p n(p)\, 2 E_{kin} \, dp$$

$$= \frac{2}{3} \int_p n(p)\, E_{kin} \, dp = \frac{2}{3} u_{kin}$$

where u_{kin} is the kinetic energy density.

For completely relativistic particles, we have

$$v = c, \quad p = \frac{E}{c}$$

$$P = \frac{1}{3}\int n(p)\, p v\, dp = \frac{1}{3}\int n(p) \frac{E}{c} c\, dp = \frac{1}{3} u_{kin}$$

This is also the expression for radiation pressure. To relate the radiation pressure to the specific intensity I_ν, we express the number density of the particles as

$$n(\theta, p)\, d\theta\, dp = \frac{dU_\nu}{dV\, E_\nu} = \frac{dU_\nu}{h\nu\, d\sigma\, c\, dt} =$$

$$= \frac{I_\nu\, d\sigma\, dt\, d\nu\, d\Omega}{c h\nu\, d\sigma\, dt} = \frac{I_\nu}{c h\nu}\, d\nu\, d\Omega$$

giving us

$$P_{rad} = \iint_{\Omega^+ p} 2 p\, v(p)\, n(p, \theta) \cos^2\theta\, d\Omega$$

$v = c$
$p = h\nu$
$dp \to d\nu$

$$= \iint_{\Omega^+ \nu} 2 \frac{h\nu}{c} c \frac{I_\nu}{c h\nu}\, d\nu\, d\Omega \cos^2\theta$$

$$= \frac{2}{c}\int_\nu \int_{\Omega^+} I_\nu \cos^2\theta\, d\Omega$$

$$= \frac{1}{c}\iint_{\Omega\,\nu} I_\nu \cos^2\theta\, d\Omega = \frac{1}{3}\int_\nu u_\nu\, d\nu = \frac{1}{3} u$$

with $u = \frac{1}{c}\int_\nu I_\nu\, d\nu$

For isotropic radiation I_0, the radiation pressure becomes

$$P_{rad,\,iso} = \frac{1}{c}\int_0^{\pi} I_0 \cos^2 d\Omega = \frac{4\pi}{3}\frac{I_0}{c}$$

Stars have the unique property that they are sustained both by radiation pressure and gas pressure.

For a medium in equilibrium at constant temperature T in an isolated box, the intensity of the radiation is that of a <u>black-body</u>, given by Planck's law

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT}-1}$$

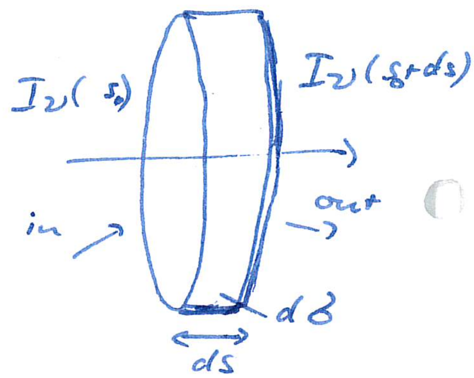
At thermal equilibrium, we have

- Intensity $\quad I = B(T) = \frac{\sigma}{\pi}T^4$
- Flux $\quad F^+ = \pi I = \sigma T^4$
- Energy Density $\quad u = \frac{4\pi I}{c} = \frac{4\sigma}{c}T^4 = aT^4$
- radiation pressure $\quad P_{rad} = \frac{1}{3}u = \frac{aT^4}{3}$

This also gives the luminosity

$$L = 4\pi R^2 F^+ = 4\pi R^2 \sigma T^4$$

To derive the equation of radiative transfer, we consider the radiation beam going through a surface $d\sigma$.



The incoming and outgoing energies are given by

$$dU_{in} = I_\nu(s_0)\, d\sigma\, d\Omega\, d\nu\, dt$$

$$dU_{out} = I_\nu(s_0+ds)\, d\sigma\, d\Omega\, d\nu\, dt$$

$$\Rightarrow dU_{out} - dU_{in} = dI\, d\sigma\, d\Omega\, d\nu\, dt$$

Within the material, we may have sources and sinks.

We define:

j_ν : emissivity per unit mass of the material.

κ_ν : opacity of the material.

With these, we can express

- The emitted energy of the material is given by

$$dU_{em} = j_\nu \underbrace{d\sigma\, \rho\, ds}_{dm} \; d\nu\, d\Omega\, dt$$

- The absorbed energy of the material is given by

$$dU_{abs} = \kappa_\nu I_\nu \underbrace{d\sigma\, ds\, \rho}_{dm} \; d\nu\, d\Omega\, dt$$

$$\Rightarrow dU_{in} - dU_{out} = dU_{em} - dU_{abs}$$

$$= (j_\nu - \kappa_\nu I_\nu)\, d\sigma\, \rho\, ds\, d\nu\, d\Omega\, dt$$

$$= dI_\nu\, d\sigma\, d\Omega\, d\nu\, dt$$

$$\Rightarrow \boxed{\frac{dI_\nu}{ds} = \rho(j_\nu - \kappa_\nu I_\nu)}$$

In a medium where $j_\nu = 0$, we have

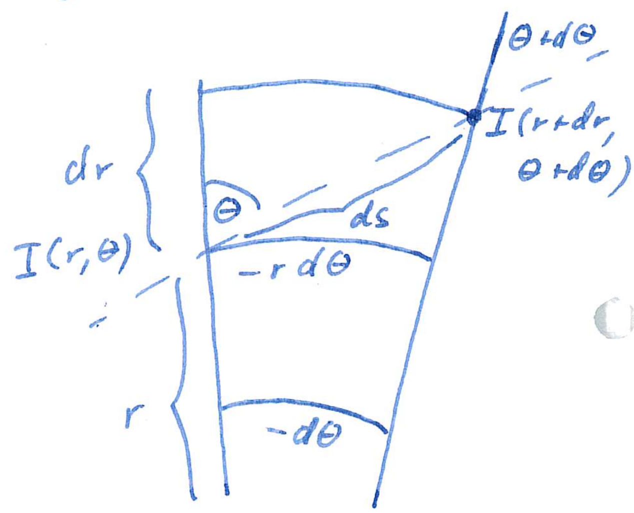
$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho I_\nu$$

$$\Rightarrow I_\nu(s) = I_\nu(s_0)\, e^{-(s-s_0)\kappa_\nu \rho}$$

$$= I_\nu(s_0)\, e^{-\frac{s-s_0}{\ell_\nu}} \qquad \text{with } \ell_\nu = \frac{1}{\kappa_\nu \rho}$$

so the intensity decreases exponentially.

For a spherical star, we can write the equation of radiative transfer as a function of (r, θ).



$$\cos\theta = \frac{dr}{ds}$$

$$\sin\theta = \frac{-r\,d\theta}{ds}$$

Then

$$\frac{d}{ds} = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{\partial s}\frac{\partial}{\partial \theta}$$

$$= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

This gives us the equations of radiative transfer in spherical coordinates:

$$\boxed{\begin{aligned} \frac{dI_\nu}{ds} &= \cos\theta \frac{dI_\nu}{dr} - \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta} = \rho j_\nu - \kappa_\nu \rho I_\nu \\ \frac{\partial P_{rad,\nu}}{\partial r} &+ \frac{1}{r}\left(3 P_{rad,\nu} - u_\nu\right) + \frac{\rho \kappa_\nu F_\nu}{c} = 0 \end{aligned}}$$

In stellar interiors, the radiation field is assumed to be that of a blackbody, and thus characterized by the local temperature.

This situation is called "Local Thermodynamic equilibrium" (LTE).

The justification can be shown for this assumption by comparing the temperature gradient with the mean free path of the photons:

The relative variation of the temperature as typically encounted by a photon is

$$\frac{\Delta T}{T} \approx \frac{\ell \left|\frac{dT}{dr}\right|}{T} \approx \frac{\frac{1}{K\rho}\frac{T_c}{R_\odot}}{T} \sim 10^{-11}$$

so for photons, the temperature appears $\approx$ constant.

In outer regions, the LTE is less valid.

Some attention must be brought to the definition of the mean opacity K.

If we assume LTE, then the radiation is blackbody radiation, and

$$P_{rad,\nu} = \frac{1}{3} u_\nu$$

Then our equation of radiative transfer for the pressure reduces to

$$\frac{\partial P_{rad,\nu}}{\partial r} + \frac{1}{r}\left(3 P_{rad,\nu} - u_\nu\right) + \frac{\rho K_\nu F_\nu}{c} = 0$$

$$= \frac{\partial P_{rad}}{\partial r} + \frac{\rho K_\nu F_\nu}{c} = 0$$

To get bolometric quantities from specific ones, we integrate over all frequencies.

It must hold that

$$P_{rad} = \int_0^\infty P_{rad,\nu}\, d\nu$$

$$F_{rad} = \int_0^\infty F_{rad,\nu}\, d\nu$$

So if we want to write the equation for the bolometric quantities, we have

$$\frac{\partial P_{rad}}{\partial r} + \frac{\rho \kappa F}{c} = 0$$

but in this form, $\kappa = \int_0^\infty \kappa_\nu\, d\nu$ can't be true!

So we need to derive the correct expression for κ, which is called the <u>Rosseland Mean Opacity</u>

We are assuming LTE: Then

$$P_{rad} = \frac{4\pi}{3}\frac{I_0}{c} = \frac{1}{3}u_{rad} \quad \left(= \frac{1}{3}\int_\Omega \frac{I_\nu}{c} d\Omega \right)$$

$$I_0 = B(T)$$

Then our equation(s) become(s)

$$\frac{\partial P_{rad}}{\partial r} = -\frac{\kappa \rho F}{c} \quad , \quad \frac{\partial P_{rad,\nu}}{\partial r} = -\frac{\kappa_\nu \rho F_\nu}{c}$$

$$= \frac{4\pi}{3c}\frac{\partial B}{\partial r} \qquad\qquad = \frac{4\pi}{3c}\frac{\partial B_\nu}{\partial r}$$

We then equate

$$F = \int_\nu F_\nu \, d\nu$$

$$-\frac{c}{\kappa\rho}\frac{\partial P_{rad}}{\partial r} = \int_\nu -\frac{c}{\kappa_\nu \rho}\frac{\partial P_{rad,\nu}}{\partial r} d\nu$$

$$= \frac{-c}{\kappa\rho}\frac{4\pi}{3c}\frac{\partial B}{\partial r} = \int_\nu -\frac{c}{\kappa_\nu \rho}\frac{4\pi}{3c}\frac{\partial B_\nu}{\partial r} d\nu$$

$$\Rightarrow \boxed{\frac{1}{\kappa} = \frac{1}{\frac{\partial B}{\partial r}} \int_\nu \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial r} d\nu}$$

Both B and B_ν are functions of r through the temperature distribution $T(r)$, i.e.

$$\frac{dB}{dr} = \frac{dB}{dT}\frac{dT}{dr} = \frac{d}{dT}\left(\frac{\sigma T^4}{\pi}\right)\frac{dT}{dr}$$

$$= \frac{4\sigma T^3}{\pi}\frac{dT}{dr}$$

Then we can write the Rosseland Mean Opacity as

$$\frac{1}{\kappa} = \frac{1}{\frac{dB}{dT}\frac{\partial T}{\partial r}}\int_\nu \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial r}\,d\nu$$

$$\boxed{\frac{1}{\kappa} = \frac{1}{\frac{dB}{dT}}\int_\nu \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}$$

Furthermore, we can get an expression for the bolometric radiative flux from

$$\frac{\partial P_{rad}}{\partial r} + \frac{\kappa \rho F}{c} = 0$$

$$\Rightarrow F = -\frac{c}{\kappa\rho}\frac{\partial P_{rad}}{\partial r} = -\frac{c}{\kappa\rho}\frac{4\pi}{3c}\frac{\partial B}{\partial r} =$$

$$= -\frac{4\pi}{3\kappa\rho}\frac{\partial B}{\partial T}\frac{\partial T}{\partial r} = -\frac{4\pi}{3\kappa\rho}\frac{4\sigma T^3}{\pi}\frac{dT}{dr} \quad \Big| \; a = \frac{4\sigma}{c}$$

$$\boxed{F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}}$$

Energetic Equilibrium: Energy Conservation

Energy conservation is another fundamental property. It brings a closure to our current equations of stellar structure:

$$\frac{\partial P}{\partial r} = -\rho \frac{G M_r}{r^2}$$

$$\frac{\partial M_r}{\partial r} = 4\pi \rho r^2$$

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho}{4acT^3} F = -\frac{3\kappa\rho}{4acT^3} \frac{L}{4\pi r^2}$$

We have 3 equations, but 4 independent variables (P, M_r or r, T, F or L)

→ we need an additional equation.

Let us consider a volume element.
Let ε be the energy produced per unit mass and unit time, q the energy provided to the system, and F the outward flux from the surface of the volume element.

Energy conservation implies:

$$\int_0^M \frac{dq}{dt}\, dm = \underbrace{\int_0^M \varepsilon\, dm}_{\text{energy produced per time}} - \underbrace{\oint_\Sigma \vec{F} \cdot d\vec{\sigma}}_{\text{outwards energy flux}}$$

ε is also called the "energy generation rate".

We can use $dm = \rho\, dV$ and the gauss theorem:

$$\int_0^V \frac{dq}{dt} \rho\, dV = \int_0^V \varepsilon \rho\, dV - \int_0^V \operatorname{div} F\, dV$$

$$\Rightarrow \quad \frac{dq}{dt} = \varepsilon - \frac{1}{\rho} \operatorname{div} F$$

$\frac{dq}{dt}$ is the energy provided to the system during a change of structure. If it is zero, the outward flux directly balances the produced energy, so the system will remain in equilibrium.

The change of structure is typically an expansion or contraction. We call

$$\frac{dq}{dt} \equiv -\varepsilon_{grav}$$

if $\varepsilon_{grav} > 0$: energy is provided by the star, i.e. $\frac{dq}{dt} < 0 \Rightarrow$ flux $> \varepsilon \Rightarrow$ star's energy is being eaten up to balance the process. This is typically a contraction.

if $\varepsilon_{grav} < 0$: Some extra energy is produced by the system, and absorbed by the star, e.g. in stellar expansion situations.

With this definition, we have

$$\frac{1}{\rho} \operatorname{div} F = \varepsilon + \varepsilon_{grav}$$

Turning our attention to ε:

The nuclear energy production rate consists generally of

$$\varepsilon_{nucl} = \varepsilon_{nucl,\gamma} + \varepsilon_{nucl,\nu}$$

a photon and neutrino contribution. The neutrino energy in general must not be counted in the radiative transfer. The neutrinos in stars are both the neutrinos emitted by nuclear reactions $\varepsilon_{nucl,\nu}$ and by various processes (photo-neutrinos, pair-neutrinos, plasma-neutrinos etc) which remove a lot of energy for central temperatures above 6×10^8 K.

Thus we have

$$\varepsilon = \underbrace{\varepsilon_{nucl,\gamma} + \varepsilon_{nucl,\nu}}_{\text{production}} - \underbrace{(\varepsilon_{nucl,\nu} + \varepsilon_{pair,photo,plasma,\nu})}_{\text{energy escape}}$$

Until the end of the He burning phase, we can neglect the neutrino production, so we can use $\varepsilon = \varepsilon_{nucl,\gamma}$

For a spherical star, we can rewrite

$$\frac{1}{\rho} \operatorname{div} F = \varepsilon + \varepsilon_{grav}$$

by using

$$\operatorname{div} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2$$

$$F = \frac{L}{4\pi r^2}$$

giving

$$\frac{1}{\rho} \operatorname{div} F = \frac{1}{\rho} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{L}{4\pi r^2} \right)$$

$$= \frac{1}{4\pi \rho r^2} \frac{\partial}{\partial r} L$$

$$= \boxed{\frac{\partial L}{\partial M_r} = \varepsilon + \varepsilon_{grav}}$$

Energy Generation Rate from Gravitational Contraction

Using the first principle of thermodynamics:

$$dq = du + p\,dv = du - \frac{p}{\rho^2}d\rho = -\varepsilon_{grav}\,dt$$

$$\Rightarrow \varepsilon_{grav} = -\frac{du}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt}$$

$$= -\left(\left.\frac{\partial u}{\partial \rho}\right|_T \frac{d\rho}{dt} + \left.\frac{\partial u}{\partial T}\right|_\rho \frac{dT}{dt}\right) + \frac{P}{\rho^2}\frac{d\rho}{dt}$$

$$= -\left.\frac{\partial u}{\partial T}\right|_\rho \frac{dT}{dt} + \left(\frac{P}{\rho^2} - \left.\frac{\partial u}{\partial \rho}\right|_T\right)\frac{d\rho}{dt}$$

$$= -\frac{dq}{dt} = \frac{-T\,ds}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{T}\left.\frac{\partial u}{\partial T}\right|_\rho \frac{dT}{dt} - \frac{1}{T}\left(\frac{P}{\rho^2} - \left.\frac{\partial u}{\partial \rho}\right|_T\right)\frac{d\rho}{dt}$$

$$\stackrel{\text{definition}}{=} \left.\frac{\partial s}{\partial T}\right|_\rho \frac{dT}{dt} + \left.\frac{\partial s}{\partial \rho}\right|_T \frac{d\rho}{dt}$$

$$\Rightarrow \left.\frac{\partial s}{\partial T}\right|_\rho = \frac{1}{T}\left.\frac{\partial u}{\partial T}\right|_\rho$$

$$\left.\frac{\partial s}{\partial \rho}\right|_T = \frac{1}{T}\left.\frac{\partial u}{\partial \rho}\right|_T - \frac{1}{T}\frac{P}{\rho^2}$$

The entropy differential ds is an exact differential

$$\Rightarrow \frac{\partial}{\partial T}\left(\frac{\partial s}{\partial \rho}\right) = \frac{\partial}{\partial \rho}\left(\frac{\partial s}{\partial T}\right)$$

$$\Rightarrow \frac{\partial}{\partial \rho}\left(\frac{1}{T}\frac{\partial u}{\partial T}\Big|_s\right) = \frac{\partial}{\partial T}\left(\frac{1}{T}\frac{\partial u}{\partial \rho}\Big|_T - \frac{P}{T\rho^2}\right)$$

$$\frac{1}{T}\frac{\partial^2 u}{\partial T \partial \rho} = \frac{1}{T}\frac{\partial^2 u}{\partial \rho \partial T} - \frac{1}{T^2}\frac{\partial u}{\partial \rho} + \frac{1}{T^2}\frac{P}{\rho^2} - \frac{1}{T\rho^2}\frac{\partial P}{\partial T}\Big|_s$$

$$\Rightarrow \frac{1}{\rho^2 T}\frac{\partial P}{\partial T}\Big|_s = \frac{1}{T^2}\left(\frac{P}{\rho^2} - \frac{\partial u}{\partial \rho}\Big|_T\right)$$

$$\Rightarrow \frac{T}{\rho^2}\frac{\partial P}{\partial T}\Big|_s = \frac{P}{\rho^2} - \frac{\partial u}{\partial \rho}\Big|_T$$

We insert this expression back into our original equation:

$$\varepsilon_{grav} = -\frac{du}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt}$$

$$= -\frac{\partial u}{\partial T}\Big|_s \frac{dT}{dt} + \left(\frac{P}{\rho^2} - \frac{\partial u}{\partial \rho}\Big|_T\right)\frac{d\rho}{dt}$$

$$= -\frac{\partial u}{\partial T}\Big|_s \frac{dT}{dt} + \frac{T}{\rho^2}\frac{\partial P}{\partial T}\Big|_s \frac{d\rho}{dt}$$

by definition

$$= -c_V \frac{dT}{dt} + \frac{T}{\rho^2}\frac{\partial P}{\partial T}\Big|_s \frac{d\rho}{dt}$$

Now let us choose a general equation of state of the form

$$d \ln \rho = \alpha \, d \ln P - \delta \ln T + \varphi \, d \ln \mu$$

where
$$\alpha = \left.\frac{d \ln \rho}{d \ln P}\right|_{T,\mu}$$

$$\delta = \left.\frac{d \ln \rho}{d \ln T}\right|_{P,\mu}$$

$$\varphi = \left.\frac{d \ln \rho}{d \ln \mu}\right|_{P,T}$$

if we assume that $\mu \approx$ const:

$$\frac{d\rho}{\rho} \approx \alpha \frac{dP}{P} - \delta \frac{dT}{T}$$

$$\Rightarrow \frac{d\rho}{dt} = \frac{\alpha \rho}{P} \frac{dP}{dt} - \frac{\delta \rho}{T} \frac{dT}{dt}$$

We insert $\frac{d\rho}{dt}$ as such in our previous equation:

$$\varepsilon_{grav} = -c_V \frac{dT}{dt} + \frac{T}{\rho^2} \frac{\partial P}{\partial T}\Big|_s \frac{d\rho}{dt}$$

$$= -c_V \frac{dT}{dt} + \frac{T}{\rho^2} \frac{\partial P}{\partial T}\Big|_s \left(\frac{\alpha \rho}{P} \frac{dP}{dt} - \frac{\delta \rho}{T} \frac{dT}{dt} \right)$$

$$= \underbrace{\left(-c_V - \frac{\delta \rho}{T} \frac{T}{\rho^2} \frac{\partial P}{\partial T}\Big|_s \right)}_{\equiv -c_P} \frac{dT}{dt} + \frac{T}{\rho^2} \frac{\partial P}{\partial T}\Big|_s \frac{\alpha \rho}{P} \frac{dP}{dt}$$

$$= -c_P \frac{dT}{dt} + \alpha \frac{T}{P\rho} \frac{\partial P}{\partial T} \frac{dP}{dt}$$

$$= -c_P \frac{dT}{dt} + \frac{\alpha}{\rho} \frac{\partial \ln P}{\partial \ln T} \frac{dP}{dt}$$

To find an expression for $\frac{\partial \ln P}{\partial \ln T}$, we use the Maxwell identities:

$$d\ln P = \frac{\partial \ln P}{\partial \ln \rho}\Big|_T d\ln \rho + \frac{\partial \ln P}{\partial \ln T}\Big|_s d\ln T$$

$$d\ln \rho = \frac{\partial \ln \rho}{\partial \ln P}\Big|_T d\ln P + \frac{\partial \ln \rho}{\partial \ln T}\Big|_P d\ln T$$

Combining these two we get

$$d\ln P = \left.\frac{\partial \ln P}{\partial \ln \rho}\right|_T \left(\left.\frac{\partial \ln \rho}{\partial \ln P}\right|_T d\ln P + \left.\frac{\partial \ln \rho}{\partial \ln T}\right|_P d\ln T \right) + \frac{\partial \ln P}{\partial \ln T} d\ln T$$

$$= \underbrace{\frac{\partial \ln P}{\partial \ln \rho} \frac{\partial \ln \rho}{\partial \ln P}}_{=1} d\ln P + \frac{\partial \ln P}{\partial \ln \rho} \frac{\partial \ln \rho}{\partial \ln T} d\ln T + \frac{\partial \ln P}{\partial \ln T} d\ln T$$

$$= d\ln P + \left(\frac{\partial \ln P}{\partial \ln \rho} \frac{\partial \ln \rho}{\partial \ln T} + \frac{\partial \ln P}{\partial \ln T} \right) d\ln T$$

$$\Rightarrow \frac{\partial \ln P}{\partial \ln \rho} \frac{\partial \ln \rho}{\partial \ln T} + \frac{\partial \ln P}{\partial \ln T} = 0$$

$$\Rightarrow \frac{\partial \ln P}{\partial \ln T} = - \frac{\partial \ln P}{\partial \ln \rho} \frac{\partial \ln \rho}{\partial \ln T} = \frac{\delta}{\alpha}$$

Finally, this gives us

$$\varepsilon_{grav} = - c_p \frac{dT}{dt} + \frac{\alpha}{\rho} \left.\frac{\partial \ln P}{\partial \ln T}\right|_\rho \frac{dP}{dt}$$

$$= \boxed{- c_p \frac{dT}{dt} + \frac{\delta}{\rho} \frac{dP}{dt}}$$

We have seen that

$$c_p = c_v + \frac{T}{\varrho^2}\frac{\delta\varrho}{T}\left.\frac{\partial P}{\partial T}\right|_s$$

$$\Rightarrow c_p - c_v = \frac{T}{\varrho^2}\frac{\delta\varrho}{T}\left.\frac{\partial P}{\partial T}\right|_s = \frac{\delta}{\varrho}\left.\frac{\partial P}{\partial T}\right|_s = \frac{\delta P}{\varrho T}\frac{\partial \ln P}{\partial \ln T}$$

$$= \frac{\delta P}{\varrho T}\frac{\delta}{\alpha} = \frac{P\delta^2}{\varrho T\alpha}$$

for a perfect gas EoS, we have

$$\alpha = \delta = \varphi = 1.$$

$$\Rightarrow \boxed{c_p - c_v = \frac{P}{\varrho T} = \frac{k}{\mu m_u}}$$

Furthermore, for a perfect gas:

$$u = E_{kin} = \frac{3}{2}\frac{kT}{\mu m_u}$$

$$\rightarrow c_v = \left.\frac{\partial u}{\partial T}\right|_v = \frac{3}{2}\frac{k}{\mu m_u}$$

$$c_p = c_v + \frac{k}{\mu m_u} = \frac{5}{2}\frac{k}{\mu m_u}$$

$$\gamma = \frac{c_p}{c_v} = \frac{5}{3}$$

If we have an adiabatic contraction / transformation, we have

$$dq = 0 \quad \Rightarrow \quad \frac{dq}{dt} = 0 = \varepsilon_{grav}$$

$$\Rightarrow \quad \varepsilon_{grav} = -c_p \frac{dT}{dt} + \frac{\delta}{\rho} \frac{dP}{dt} = 0$$

$$\Rightarrow \boxed{c_p \frac{dT}{dt} = \frac{\delta}{\rho} \frac{dP}{dt}}$$

$$\Rightarrow \quad c_p \, dT = \frac{\delta}{\rho} dP$$

$$c_p T \, d\ln T = \frac{\delta}{\rho} P \, d\ln P$$

$$\Rightarrow \quad \frac{\partial \ln T}{\partial \ln P} = \frac{\delta P}{\rho T c_p} \overset{\text{for perfect gas}}{=} \frac{1 \cdot \frac{k}{\mu m_u}}{\frac{5}{2} \frac{k}{\mu m_u}} = \frac{2}{5} = 0.4$$

$\frac{\partial \ln T}{\partial \ln P}$ is called the adiabatic gradient.

This relation tells us that

$$\frac{\Delta T}{T} = 0.4 \frac{\Delta P}{P}$$

$\Rightarrow$ a 10% temperature increase results in a 4% pressure increase.

If a star is contracting by how much does the central temperature increase if the density increases by 1%?

We know that

$$P \propto \frac{M^2}{R^4} \quad \Rightarrow \quad \frac{\Delta P}{P} = -4\frac{\Delta R}{R} \left(= \frac{-4\frac{M^2}{R^5}\Delta R}{\frac{M^2}{R^4}} \right)$$

$$\rho \propto \frac{M}{R^3} \quad \Rightarrow \quad \frac{\Delta \rho}{\rho} = -3\frac{\Delta R}{R}$$

$$\Rightarrow \boxed{\frac{\Delta P}{P} = \frac{4}{3}\frac{\Delta \rho}{\rho}}$$

Using the general equation of state again:

$$d\ln\rho = \alpha\, d\ln P - \delta\, d\ln T + \underbrace{\varphi\, d\ln\mu}_{\approx 0}$$

$$\Rightarrow \frac{\Delta\rho}{\rho} = \alpha\frac{\Delta P}{P} - \delta\frac{\Delta T}{T}$$

$$\Rightarrow \frac{\Delta P}{P} = \frac{1}{\alpha}\frac{\Delta\rho}{\rho} + \frac{\delta}{\alpha}\frac{\Delta T}{T} = \frac{4}{3}\frac{\Delta\rho}{\rho}$$

$$\Rightarrow \frac{\Delta T}{T} = \frac{\alpha}{\delta}\frac{\Delta\rho}{\rho}\left(\frac{4}{3} - \frac{1}{\alpha}\right) = \left(\frac{4\alpha-3}{3\alpha}\right)\frac{\alpha}{\delta}\frac{\Delta\rho}{\rho}$$

$$= \frac{4\alpha-3}{3\delta}\frac{\Delta\rho}{\rho}$$

$$\overset{\text{perfect gas}}{=} \frac{1}{3}\frac{\Delta\rho}{\rho}$$

Note that this expression can analogously be derived with differentials:

$$d \ln T = \frac{1}{3} d \ln \rho$$

and both these equations are valid for non-adiabatic contraction, as it e.g. occurs in the pre-main sequence phase or between H and He-burning phases.

For an adiabatic process for an ideal gas, we have

$$p V^\gamma = \text{const}$$

$$p V = NkT = \frac{\rho V}{m_u \mu} kT \Rightarrow p = \frac{\rho}{m_u \mu} kT$$

$$V = \frac{m_u \mu}{\rho}$$

$$\Rightarrow p V^\gamma = \frac{\rho}{m_u \mu} kT \left(\frac{m_u \mu}{\rho} \right)^\gamma = \text{const}$$

assuming μ is const:

$$\rho^{1-\gamma} T = \frac{\text{const}}{(m_u \mu)^{\gamma - 1}}$$

$$\Rightarrow d(\rho^{1-\gamma} T) = 0 = (1-\gamma) \rho^{-\gamma} T d\rho + \rho^{1-\gamma} dT$$

$$\Rightarrow (1-\gamma) \frac{d\rho}{\rho} + \frac{dT}{T} = 0$$

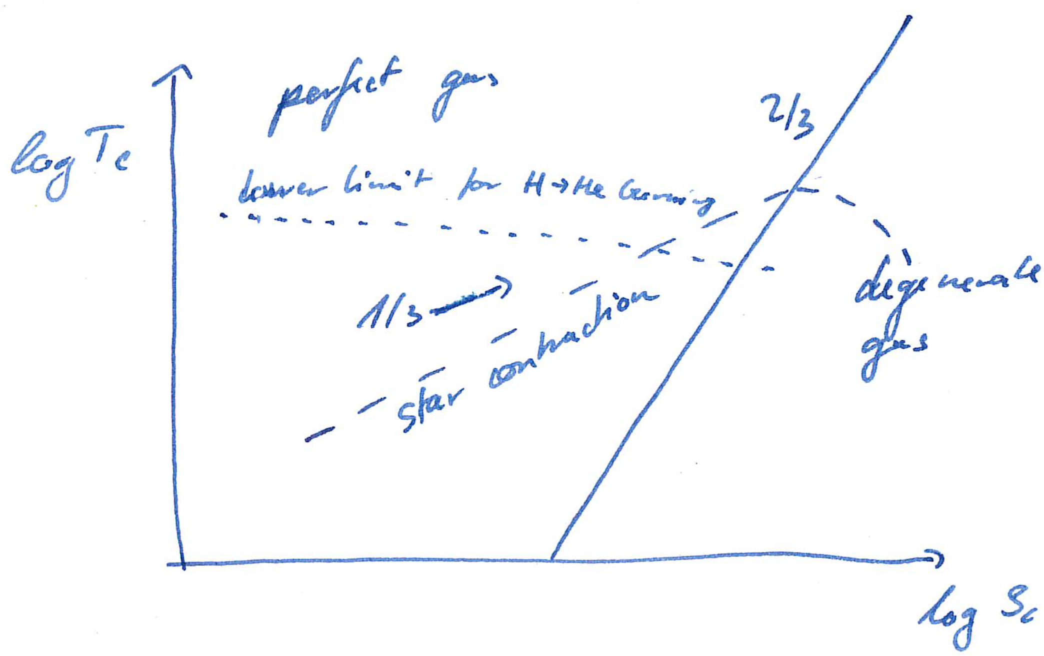
$$\Rightarrow d \ln T = (\gamma - 1) \frac{d\rho}{\rho} = \frac{2}{3} d \ln \rho$$

So we have

$$d \ln T = \frac{1}{3} d \ln \rho \qquad \text{non-adiabatic}$$

$$d \ln T = \frac{2}{3} d \ln \rho \qquad \text{adiabatic}$$

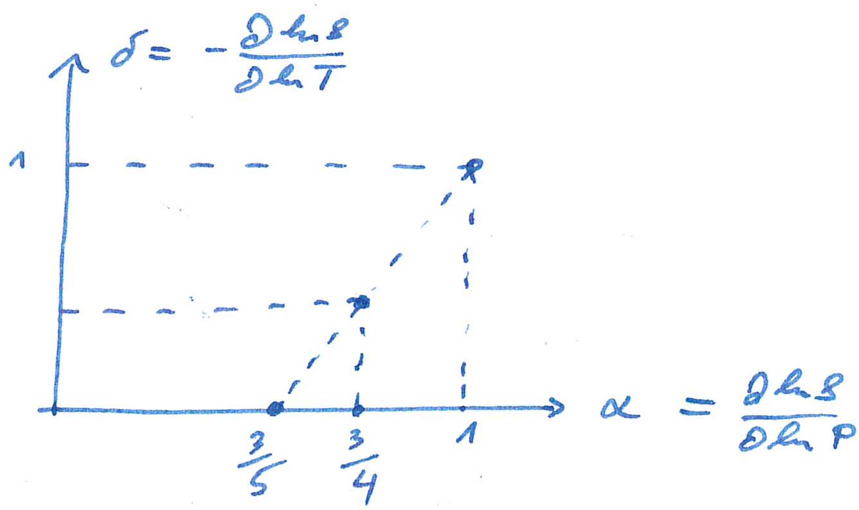
So a temperature increase leads to twice the density increase in the adiabatic case than in a non-adiabatic one. This is understandable, because half the energy gets radiated away. The slope of $\sim 1/3$ applies when the evolution proceeds with the Kelvin-Helmholtz timescale.



In the $\log T - \log \varrho$ plane, the separation between the domain of the perfect and degenerate gas has a slope $2/3$.

The evolution of stellar centers proceeds with a flatter slope, implying that during evolution a star unavoidably moves towards the domain of degenerate gas.

$\Rightarrow$ Stars evolve to degenerate end points.



When a star enters the domain of degeneracy, we get

$$\alpha \to \frac{3}{5}, \quad \delta \to 0$$

$\Rightarrow$ At some point during evolution, $\alpha < 3/4$ while $\delta \neq 0$

The problem with that is that in general, we have

$$d\ln T = \frac{4\alpha - 3}{3\delta} d\ln \rho$$

$$\rightarrow \frac{4\alpha-3}{3\delta} < 0$$

$\Rightarrow$ for a star entering the degenerate domain, contraction doesn't produce an increase in temperature.

If we have a situation when all electrons are degenerate, by increasing the density the electrons are forced to the highest energy levels. The required energy to get to these levels is taken from ions, which effectively reduces the temperature.

These diagrams also give the lower limit for brown dwarfs. If a star enters the degenerate domain with a temperature below the limit for hydrogen burning at $T \sim 6 \times 10^6$ K, its further contraction does not produce an increase of T and H ignition will never occur.

The lower mass limit for H burning is $\sim 0.08\,M_\odot$ at solar metallicity. The objects between $\sim 0.01 - 0.08\,M_\odot$ resulting from the contraction-fragmentation process are the brown dwarfs.

Secular Stability of Nuclear Burning

We have

$$\frac{\partial L_\nu}{\partial M_r} = \varepsilon_{nucl} + \varepsilon_{grav} = \varepsilon_{nucl} - c_p \frac{dT}{dt} + \frac{\delta}{\rho}\frac{dP}{dt}$$

Let us suppose that for a non-perturbed situation, we have

$$\frac{\partial L_{\nu,0}}{\partial M_r} = \varepsilon_{nucl,0}$$

Now let us suppose a small temperature perturbation T_1 around the equilibrium value T_0, such that

$$T = T_0 + T_1$$

We also suppose that the timescale is so short for this temperature increase that $L_\nu = L_{\nu,0}$, i.e. that the temperature increase is not immediately radiated away.

Since we have

$$\frac{\partial L_{\nu,0}}{\partial M_r} = \varepsilon_{nucl,0}$$

$$= \varepsilon_{nucl,0} + \left(-c_p \dot{T}_0 + \frac{\delta}{\varrho} \dot{P}_0\right)$$

it follows that

$$\dot{T}_0 = \dot{P}_0 = 0$$

in the equilibrium state.

Then in the perturbed state, we have for an adiabatic perturbation:

$$\frac{\partial L_\nu}{\partial M_r} = \frac{\partial L_\nu}{\partial M_r} = 0 = \varepsilon_{nucl,0} + \left(-c_p \dot{T}_0 + \frac{\delta}{\varrho}\dot{P}_0\right) + \xi = 0$$

$$+ \varepsilon_{nucl,1} + \left(-c_p \dot{T}_1 + \frac{\delta}{\varrho} \dot{P}_1\right)$$

$$\Rightarrow \varepsilon_{nucl,1} = c_p \dot{T}_1 - \frac{\delta}{\varrho} \dot{P}_1$$

We can express the $\varepsilon_{nucl,1}$ via a Taylor expansion from a simplified form

$$\varepsilon_{nucl} = \varepsilon_0 \varrho T^\nu$$

Then we expand:

$$\varepsilon_{nucl} = E_0 \rho T^{\nu} = E_0 \rho_0 T_0^{\nu} + E_0 \rho_0 T_0^{\nu-1} \nu T_1 + \mathcal{O}(T_1^2)$$

$$\approx E_0 \rho_0 T_0^{\nu} \left(1 + \frac{\nu T_1}{T_0}\right)$$

$$= \varepsilon_{nucl,0} \left(1 + \frac{\nu T_1}{T_0}\right) = \varepsilon_{nucl,0} + \varepsilon_{nucl,1}$$

$$\Rightarrow \varepsilon_{nucl,1} = \varepsilon_{nucl,0} \frac{\nu T_1}{T_0}$$

$$= c_p \dot{T}_1 - \frac{\delta}{\rho} \dot{P}_1$$

$$= c_p T_0 \left(\frac{\dot{T}_1}{T_0} - \underbrace{\frac{\delta P_0}{\rho c_p T_0}}_{\nabla_{ad}} \frac{\dot{P}_1}{P_0} \right)$$

$$= c_p T_0 \left(\frac{\dot{T}_1}{T_0} - \nabla_{ad} \frac{\dot{P}_1}{P_0} \right)$$

We again use the approximations and EOS:

$$\frac{\Delta P}{P} = \frac{4}{3} \frac{\Delta \rho}{\rho} \quad ; \quad \frac{\Delta \rho}{\rho} = \alpha \frac{\Delta P}{P} - \delta \frac{\Delta T}{T}$$

$$\Rightarrow \frac{\Delta P}{P} = \frac{4}{3} \frac{\Delta \rho}{\rho} = \frac{4}{3}\left(\alpha \frac{\Delta P}{P} - \delta \frac{\Delta T}{T} \right)$$

$$\frac{\Delta P}{P}\left(1 - \frac{4}{3}\alpha\right) = \left(\frac{-4\alpha + 3}{3}\right) \frac{\Delta P}{P} = -\frac{4}{3} \delta \frac{\Delta T}{T}$$

$$\Rightarrow \frac{\Delta P}{P} = -\frac{4\delta}{3} \frac{3}{3 - 4\alpha} \frac{\Delta T}{T} = \frac{4\delta}{-3 + 4\alpha} \frac{\Delta T}{T} \propto \frac{\dot{P}_1}{P}$$

This gives us

$$\frac{T_1}{T_0}\varepsilon_{nucl,0}\,\nu = c_p T_0\left(\frac{\dot{T}_1}{T_0} - \nabla_{ad,0}\frac{\dot{P}_1}{P_0}\right)$$

$$= c_p T_0\left(\frac{\dot{T}_1}{T_0} - \nabla_{ad,0}\frac{4\delta}{4\alpha-3}\frac{\dot{T}_1}{T_0}\right)$$

$$\Rightarrow \boxed{\frac{\dot{T}_1}{T_0} = \frac{\varepsilon_{nucl,0}\,\nu}{c_p T_0\left(1-\nabla_{ad,0}\frac{4\delta}{4\alpha-3}\right)}\frac{T_1}{T_0}}$$

In shorthand: $\frac{\dot{T}_1}{T_0} = A\frac{T_1}{T_0}$

$A > 0$: Unstable: The perturbation will grow.

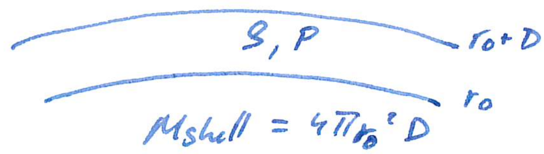
$A < 0$: Stable: Increase of temperature produces cooling.

The sign of A is determined by the adiabatic gradient in the factor

$$1 - \nabla_{ad,0}\frac{4\delta}{4\alpha-3}$$

For perfect gas, we have $\nabla_{ad} = 0.4$, $\delta = 1 = \alpha$
$\Rightarrow A < 0$. A nuclear perturbation of perfect gas is stable. Degenerate gas however isn't. $[\delta \to 0]$

Another source of instabilities may be <u>shell burning instabilities</u>. In asymptotic giant branch stars, nuclear burning occurs in shells around the core. In such stars, the burning in the burning shells is unstable and produces thermal pulses.



Let us consider the case of a thin shell burning at a distance r_0 from the center in a very thin shell of thickness $D \ll r_0$.

The mass in the shell is $M = 4\pi r_0^2 \rho D$.

We suppose that ρ can only change with D through expansion.

If the shell extends, due to some perturbation keeping M and $r_0 \approx$ const, the relative change in density is

$$\frac{d\rho}{\rho} = \frac{d\left(\frac{M}{4\pi r_0^2 D}\right)}{\frac{M}{4\pi r_0^2 D}} = \frac{-\frac{1}{D^2}\, dD}{1/D} = \frac{-dD}{D}$$

we also can expand:

$$\frac{dD}{D} = \frac{dD}{r_0}\,\frac{r_0}{D} = \frac{(r+dr-r)}{r_0}\,\frac{r_0}{D} = \frac{dr}{r_0}\,\frac{r_0}{D}$$

giving us

$$\frac{d\rho}{\rho} = -\frac{dD}{D} = -\frac{r_0}{D}\frac{dr}{r_0}$$

In the earlier derivation for nuclear instabilities we used the spherical case with

$$P \propto R^{-4} \Rightarrow \frac{\Delta P}{P} = -4\frac{\Delta R}{R}$$

$$\rho \propto R^{-3} \Rightarrow \frac{\Delta \rho}{\rho} = -3\frac{\Delta R}{R}$$

$$\Rightarrow \frac{\Delta P}{P} = \frac{4}{3}\frac{\Delta \rho}{\rho}$$

In this case, we have

$$\frac{\Delta \rho}{\rho} = -\frac{\Delta R}{R}\frac{r_0}{D}$$

$$\Rightarrow \frac{\Delta P}{P} = 4\frac{\Delta \rho}{\rho}\frac{r_0}{D} = 4\frac{r_0}{D}\left(\alpha\frac{\Delta P}{P} - \delta\frac{\Delta T}{T}\right)$$

$$\Rightarrow \frac{\Delta P}{P}\left(1 - 4\alpha\frac{r_0}{D}\right) = -4\delta\frac{\Delta T}{T}$$

$$\frac{\Delta P}{P} = \frac{4\delta}{1-4\alpha\frac{r_0}{D}}\frac{\Delta T}{T}$$

We insert this in our perturbed equation for $\varepsilon_{nucl,1}$ again:

$$\frac{T_1}{T_0}\varepsilon_{nucl,0}\,\nu = c_p T_0\left(\frac{\dot{T}_1}{T_0} - \nabla_{ad,0}\frac{\dot{P}_1}{P_0}\right)$$

$$= c_p T_0\left(\frac{\dot{T}_1}{T_0} - \nabla_{ad}\frac{4\delta}{1-4\alpha\frac{r_0}{D}}\frac{\dot{T}_1}{T_0}\right)$$

For perfect gas, we have $\alpha = \delta = 1$ and hence

$$\frac{\dot{T}_1}{T_0} = \frac{\varepsilon_{nucl,0}\,\nu}{c_p T_0\left(1 + \nabla_{ad}\frac{4}{1-4\frac{r_0}{D}}\right)}\frac{T_1}{T_0}$$

$$= \frac{\varepsilon_{nucl,0}\,\nu}{c_p T_0\left(1 + \frac{8}{5(1-4\frac{r_0}{D})}\right)}\frac{T_1}{T_0}$$

where we used $\nabla_{ad,\text{perfect gas}} = \frac{2}{5} = 0.4$ and $r_0 \gg D$

Here, the situation is always unstable for $\frac{\dot{T}_1}{T_0} > 0$. This instability is responsible for the thermal pulses in the advanced AGB phases.

The Role of Radiation Pressure in Stars

Radiation has two main impact in stars:

1) Energy transport: $F_{rad} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$

2) Pressure: $P_{rad} = \frac{1}{3}aT^4$ (Blackbody)

The outward flowing radiation may be compared to a wind blowing through the star and helping to distend it against gravity.

We introduce the parameter $\beta \equiv \frac{P_{gas}}{P}$

$$P = P_{tot} = P_{gas} + P_{rad}, \qquad \frac{P_{rad}}{P} = (1-\beta)$$

$$P_{gas} = \beta P = \frac{k}{\mu m_u}\rho T$$

$$P_{rad} = (1-\beta)P = \frac{1}{3}aT^4$$

$$\Rightarrow \quad T = \frac{\beta P \mu m_u}{\rho k}$$

$$P = \frac{\frac{1}{3}aT^4}{1-\beta} = \frac{1}{3}\frac{a}{1-\beta}\frac{\beta^4 P^4}{\rho^4 k^4}(\mu m_u)^4$$

$$\Rightarrow \quad P = \rho^{4/3}\left[\frac{3}{a}\left(\frac{k}{\mu m_u}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3}$$

If β was constant inside the star, we would have $P \propto \rho^{4/3}$. This is a problem:

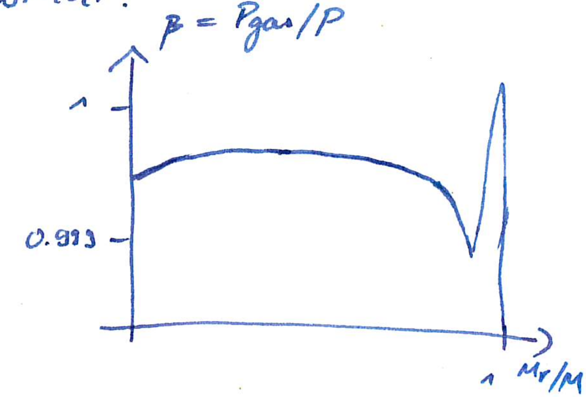
The virial theorem gives an expression for the total energy via internal energy:

$$3(\gamma - 1) U + \Omega = 0$$

$$E = U + \Omega = \frac{-\Omega}{3(\gamma-1)} + \Omega = \frac{3\gamma-4}{3\gamma-3}\Omega$$

So if $\gamma < 4/3$, $E > 0$, and the star is unstable. Since we generally have $P \propto \rho^{\gamma}$, and for constant $\beta \Rightarrow \gamma = 4/3$, this would be unstable.

Therefore, β cannot be constant.



However, models show that β is nearly constant throughout the star. Around the atmosphere, β goes down to zero, since at the top of the atmosphere the gas pressure is zero and there is only the radiation pressure left.

How does the radiative pressure change as a function of the star's mass?

We have

$$\frac{P_{rad}}{P_{gas}} = \frac{1/3\, a T^4}{\frac{k}{m_u \mu} \rho T} = \frac{1}{3} a \frac{k}{m_u \mu} \frac{T^3}{\rho}$$

we approximate:

$$\rho \rightarrow \bar{\rho} = \frac{M}{4/3 \pi R^3}$$

$$T \rightarrow \bar{T} = \frac{\mu m_u}{3k} \, 9 \, \frac{GM}{R} \beta$$

which we get from the virial theorem:

$$\Omega = -3 \int \frac{P}{\beta} dV = -3 \int \frac{k}{\mu m_u \beta} \rho T \, dV$$

$$= -3 \frac{k}{\mu m_u \beta} M \bar{T} = -9 \frac{GM^2}{R}$$

$$\Rightarrow \bar{T} = \frac{1}{3} \frac{\mu m_u}{k} \, 9 \, \frac{GM}{R} \beta$$

where we assumed that $P \sim P_{gas} = \frac{P}{\beta}$

Inserting these approximation gives

$$\frac{P_{rad}}{P_{gas}} = \frac{1}{3} a \frac{\mu m_u}{k} \frac{T^3}{S}$$

$$= \frac{1}{3} a \frac{\mu m_u}{k} \left(\frac{\mu m_u}{3k} q \frac{GM}{R} \beta \right)^3 \frac{4/3 \pi R^3}{M}$$

$$= \frac{4a\pi}{3^5} \left(\frac{\mu m_u}{k} \right)^4 q^3 G^3 \beta^3 M^2$$

$$= \frac{(1-\beta)P}{\beta P} = \frac{1-\beta}{\beta}$$

$$\Rightarrow \frac{1-\beta}{\beta^4} = \frac{4\pi}{3^5} a \left(\frac{\mu m_u}{k} \right)^4 q^3 G^3 M^2$$

Separating the variables from constants, we have

$$\frac{1-\beta}{\beta^4} M^{-2} \sim \frac{4\pi}{3^5} a \left(\frac{\mu m_u}{k} \right)^4 q^3 G^3$$

$\Rightarrow$ For constant μ, we have

$$\frac{1-\beta}{\beta^4} \sim M^2$$

$\Rightarrow$ For $\beta \to 0$, $\quad \beta^{-4} \sim M^2 \Rightarrow \beta \sim \frac{1}{\sqrt{M}}$

$\Rightarrow$ β decreases with increasing M

As radiation pressure opposes its force to gravity and becomes larger for more luminous stars, there is a luminosity limit, when the surface layers are no longer bound. This is called the Eddington Luminosity.

We have

$$\vec{g} = -\frac{GM_r}{r^2}\frac{\vec{r}}{r} \quad \text{and} \quad \vec{g}_{rad} = -\frac{1}{\rho}\frac{dP_{rad}}{dr}\frac{\vec{r}}{r}$$

with $\frac{dP_{rad}}{dr} < 0$.

and $$\frac{dP_{rad}}{dr} = -\frac{\kappa\rho F}{c} = -\frac{\kappa\rho}{c}\frac{L_r}{4\pi r^2}$$

$$\Rightarrow \vec{g} + \vec{g}_{rad} = -\left(\frac{GM_r}{r^2} - \frac{1}{\rho}\frac{\kappa\rho}{c}\frac{L_r}{4\pi r^2}\right)$$

$$= -\frac{GM}{R^2}\left(1 - \frac{\kappa L}{4\pi c GM}\right) \quad \text{for } r = R$$

One defines the Eddington Luminosity

$$L_{EDD} = \frac{4\pi c GM}{\kappa}$$

Numerically, we have

$$\frac{L_{EDD}}{L_\odot} = 1.3 \cdot 10^4 \frac{1}{\kappa} \frac{M}{M_\odot}$$

Also one commonly defines

$$\Gamma_{edd} = \frac{L}{L_{EDD}} \quad \text{Eddington factor}$$

If the Eddington luminosity is reached at the surface of a star, then the upper layers of the star are no longer bound and heavy mass losses are expected. So for a given mass M, the Eddington luminosity is an upper limit of the luminosity.

Already at luminosities well below L_{EDD} massive stars lose mass by stellar winds driven by radiation pressure. The reason is that some ions have at certain frequencies very large opacities, sufficient to absorb enough momentum to accelerate them outward. By collision coupling, global stellar winds are generated.

The closer to the Eddington limit, the higher the stellar winds.

The Eddington luminosity is generally defined with κ the electron scattering opacity

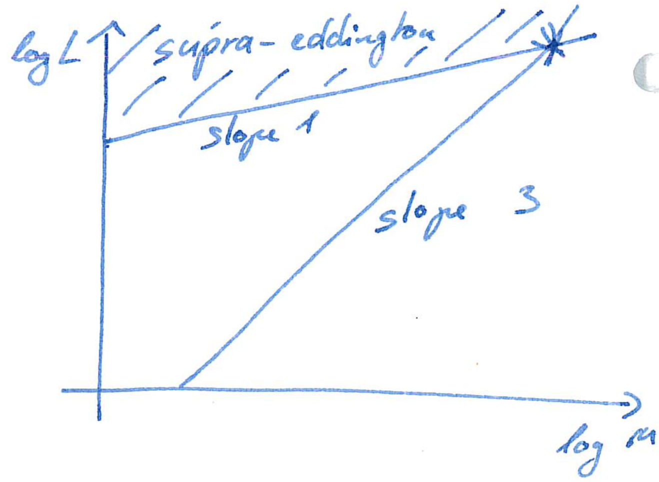
$$\kappa_{es} = 0.2\,(1+X)\,cm^2 g^{-1} \qquad X: \text{H mass fraction.}$$

The condition $L < L_{EDD}$ can be translated in a condition on T_{eff}:

$$L = 4\pi R^2 \sigma T_{eff}^4 \overset{!}{<} \frac{4\pi c G M}{\kappa}$$

$$\Rightarrow T_{eff}^4 < \frac{c g}{\sigma \kappa}$$

where g is the surface gravity.



The Eddington Luminosity puts an upper threshold on stellar masses:

$$L \propto M^3$$

$$L_{EDD} = \frac{4\pi G c M}{\kappa} \propto M$$

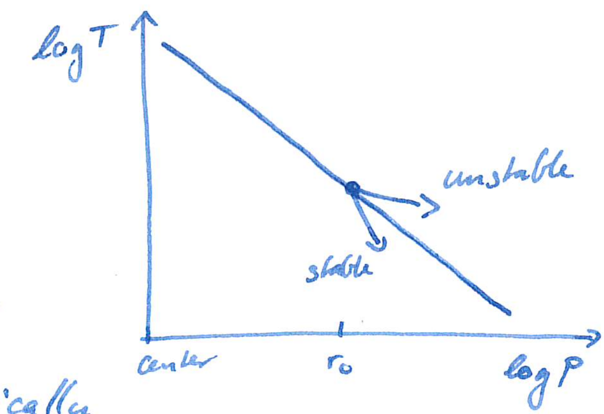
The upper mass is found through $\frac{L_{EDD}}{L_\odot} \sim 38200 \frac{M}{M_\odot}$ and $\frac{L}{L_{EDD}} \propto M^2$, giving $\frac{M_{max}}{M_\odot} \sim 195$

Stellar Convection

In stars, a heat excess (w.r.t. what radiation can transfer) drives turbulent chaotic convective motions. Convection, i.e. the turbulent turnover of matter in a medium heated from below, is a basic mechanism of energy transport in stars together with radiative transfer. In addition, it produces fast mixing of the chemical elements, generally leading to the chemical homogeneity of the convective regions. Convection also transports angular momentum, thus influencing the evolution of rotation. It also imposes a particular density structure characterized by a relatively low-density contrast and generates sonic waves observable in helioseismology and asteroseismology.

Gravity Waves and Brunt-Väisälä Frequency

Let us consider a fluid element in a star at some level r_0 in equilibrium with the surrounding medium.



If the cell is moved vertically adiabatically and neglecting viscous effects, its equation of motion is

$$\rho_i \frac{d^2 r}{dt^2} = -\rho_i g + \rho_e g = g(\rho_e - \rho_i)$$

index i : interior of the cell

e : exterior of the cell

For a small displacements, we can expand

$$\rho_i(r) = \rho_i(r_0) + \left.\frac{d\rho_i}{dr}\right|_{r_0}(r-r_0)$$

$$\rho_e(r) = \rho_e(r_0) + \left.\frac{d\rho_e}{dr}\right|_{r_0}(r-r_0)$$

giving, assuming $\rho_i(r_0) = \rho_e(r_0)$

$$\rho_i \frac{d^2 r}{dt^2} = g\left(\left.\frac{d\rho_e}{dr}\right|_{r_0} - \left.\frac{d\rho_i}{dr}\right|_{r_0}\right)(r-r_0)$$

This equation is the equation of harmonic motions without damping. The solution is of the form

$$(r - r_0) = A \exp iNt \, ,$$

which gives

$$-\rho_i A N^2 e^{iNt} + g \left(\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr} \right) A e^{iNt} = 0$$

N is the Brunt-Väisälä frequency:

$$N^2 = \frac{g}{\rho} \left(\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr} \right) = \frac{g}{\rho} \frac{d\Delta\rho}{dr}$$

These oscillations are known as gravity waves, since gravity is the restoring force. ρ is the density evaluated at the equilibrium position, where $\rho_i = \rho_e$, so no indices are required.

What is the physical meaning of N?

- $N^2 > 0 \Rightarrow N$ is real $\Rightarrow (r - r_0)$ is periodic, the motion is oscillatory. The fluid element experiences a negligible displacement from the equilibrium level r_0.
  This is the case for $\frac{d\rho_i}{dr} > \frac{d\rho_e}{dr}$, e.g. when the cell is displaced upwards.

- $N^2 < 0 \Rightarrow (r - r_0)$ has an exponential solution

$$r - r_0 = A \exp\left[\pm\sqrt{\frac{d\rho_e}{dr} - \frac{d\rho_i}{dr}}\, t\right]$$

  The fluid elements experiences a motion that removes it exponentially from the equilibrium level. The medium is unstable with respect to convection.
  This is the case for $\frac{d\rho_e}{dr} > \frac{d\rho_i}{dr}$

In summary:

$\frac{d\Delta\rho}{dr} > 0 \Rightarrow$ stable; $\frac{d\Delta\rho}{dr} < 0 \Rightarrow$ unstable

We can express this criterion as a function of the temperature gradient.

We use the equation of state in the general form

$$\frac{d \ln \rho}{dr} = \alpha \frac{d \ln P}{dr} - \delta \frac{d \ln T}{dr} + \varphi \frac{d \ln \mu}{dr}$$

We assume that

1) The composition of the cell in motion doesn't change, i.e. $d\mu_i = 0$

2) $v_{cell} \ll v_{sound}$; This implies that the pressure equilibrium has time to be realised at every point along the movement, i.e. $P_e = P_i$.

We get the following equations:

$$\frac{d \ln \rho_i}{dr} = \alpha \frac{d \ln P_i}{dr} - \delta \frac{d \ln T_i}{dr}$$

$$\frac{d \ln \rho_e}{dr} = \alpha \frac{d \ln P_e}{dr} - \delta \frac{d \ln T_e}{dr} + \varphi \frac{d \ln \mu_e}{dr}$$

We assume $\alpha_i = \alpha_e$, $\delta_i = \delta_e$.

For a stable convection (unstable: flip inequality)

we have the condition

$$\frac{d\varrho_i}{dr} - \frac{d\varrho_r}{dr} > 0$$

$$= \alpha \underbrace{\left(\frac{d\ln P_i}{dr} - \frac{d\ln P_e}{dr} \right)}_{=0:\ P_e = P_i} - \delta \left(\frac{d\ln T_i}{dr} - \frac{d\ln T_e}{dr} \right) - \varphi \frac{d\ln\mu_e}{dr}$$

$$= -\delta \frac{d\ln T_i}{dr} + \delta \frac{d\ln T_e}{dr} - \varphi \frac{d\ln\mu_e}{dr} > 0$$

$$\Rightarrow \frac{d\ln T_e}{dr} > \frac{d\ln T_i}{dr} + \frac{\varphi}{\delta} \frac{d\ln\mu_e}{dr}$$

$$\Rightarrow \underbrace{\frac{d\ln T_e}{d\ln P}}_{\equiv \nabla_e} \frac{d\ln P}{dr} > \underbrace{\frac{d\ln T_i}{d\ln P}}_{\equiv \nabla_i} \frac{d\ln P}{dr} + \frac{\varphi}{\delta} \underbrace{\frac{d\ln\mu_e}{d\ln P}}_{\equiv \nabla_\mu} \frac{d\ln P}{dr}$$

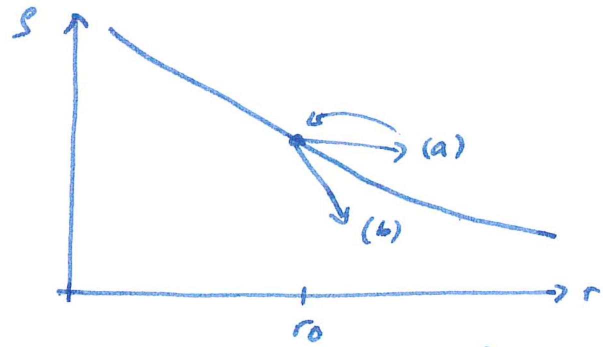
$$\Rightarrow \nabla_e < \nabla_i + \frac{\varphi}{\delta} \nabla_\mu$$

Note that the sign flips here because $\frac{d\ln P}{dr} < 0$!

So the stability condition can be written in terms of temperature gradients

$$\boxed{\nabla_e < \nabla_i + \frac{\varphi}{\delta} \nabla_\mu}$$

In summary, there are three important things about convection:

1) 

a perturbed cell can be pushed into

(a) a denser region. Then it will be pushed back. This is a stable condition. (Denser region usually is downwards.)

(b) a less dense region: unstable solution. Grows exponentially.

2) The equation of motion is a harmonic oscillation with the general solution

$$r = A e^{iNt}, \quad N^2 = \frac{g}{\rho} \frac{d}{dr}(\rho_i - \rho_e)$$

From this Brunt-Väisälä frequency N, we get the stability condition

$N^2 > 0$: Stable, for $\frac{d}{dr}(\varrho_i - \varrho_e) > 0$

$N^2 < 0$: Unstable, for $\frac{d}{dr}(\varrho_i - \varrho_e) < 0$

3) The stability condition can be expressed via temperature gradients

$$\nabla_e < \nabla_i + \frac{\varphi}{\delta} \nabla_\mu$$

The Brunt-Väsälä frequency N can also be expressed via the entropy gradient.

We have:

$$d\varrho = \left.\frac{\partial \varrho}{\partial s}\right|_P ds + \left.\frac{\partial \varrho}{\partial P}\right|_s dP$$

$$\Rightarrow \frac{d\varrho_i}{dr} - \frac{d\varrho_e}{dr} = \left.\frac{\partial \varrho_i}{\partial s_i}\right|_P \frac{ds_i}{r} + \left.\frac{\partial \varrho_i}{\partial P_i}\right|_s dP_i -$$

$$- \left.\frac{\partial \varrho_e}{\partial s_e}\right|_P \frac{ds_e}{dr} - \left.\frac{\partial \varrho_e}{\partial P_e}\right|_s dP_e$$

We assume adiabatic convection of the perturbed cell, i.e. $ds_i = 0$

Also we again assume $v_{cell} \ll v_{sound}$, thus

$P_i = P_e \approx \text{const} \quad \Rightarrow dP_i = dP_e \approx 0$

$$\Rightarrow \frac{d\rho_i}{dr} - \frac{d\rho_e}{dr} = - \left.\frac{\partial \rho_e}{\partial s}\right|_p \frac{ds_e}{dr}$$

We can express the density and entropy differentials as

$$d\rho = \left.\frac{\partial \rho}{\partial s}\right|_p ds + \left.\frac{\partial \rho}{\partial P}\right|_s dP$$

$$ds = \left.\frac{\partial s}{\partial p}\right|_T dP + \left.\frac{\partial s}{\partial T}\right|_p dT$$

And in case of constant pressure, we have from the first law of thermodynamics:

$$T ds = dq = du + \underbrace{p}_{=0} dV = \left.\frac{\partial u}{\partial T}\right|_p dT + \underbrace{\left.\frac{\partial u}{\partial p}\right|_T dp}_{=0}$$

$$= c_p \, dT$$

$$\Rightarrow T ds = T \left.\frac{\partial s}{\partial T}\right|_p dT = c_p \, dT \quad \Rightarrow \left.\frac{\partial s}{\partial T}\right|_p = \frac{c_p}{T}$$

Then

$$d\rho = \frac{\partial\rho}{\partial s}\Big|_p ds + \frac{\partial\rho}{\partial P}\Big|_s dP$$

$$= \frac{\partial\rho}{\partial s}\Big|_p \left[\frac{\partial s}{\partial P}\Big|_T dP + \frac{\partial s}{\partial T}\Big|_p dT \right] + \frac{\partial\rho}{\partial P}\Big|_s dP$$

$$= \frac{\partial\rho}{\partial s}\Big|_p \left[\frac{\partial s}{\partial P}\Big|_T dP + \frac{c_p}{T} dT \right] + \frac{\partial\rho}{\partial P}\Big|_s dP$$

$$= \frac{\partial\rho}{\partial s}\Big|_p \frac{c_p}{T} dT + \left[\frac{\partial\rho}{\partial s}\Big|_p \frac{\partial s}{\partial P}\Big|_T + \frac{\partial\rho}{\partial P}\Big|_s \right] dP$$

$$= \frac{\partial\rho}{\partial T}\Big|_p dT + \frac{\partial\rho}{\partial P}\Big|_T dP$$

$$\Rightarrow \frac{\partial\rho}{\partial T}\Big|_p = \frac{\partial\rho}{\partial s}\Big|_p \frac{c_p}{T} \quad \Rightarrow \frac{\partial\rho}{\partial s}\Big|_p = \frac{\partial\rho}{\partial T}\Big|_p \frac{T}{c_p}$$

This gives us

$$\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr} = - \left.\frac{\partial \rho_e}{\partial s}\right|_p \frac{ds_e}{dr}$$

$$= - \left.\frac{\partial \rho}{\partial T}\right|_p \frac{T}{c_p} \frac{ds_e}{dr}$$

And for the Brunt-Väisälä frequency

$$N^2 = \frac{g}{\rho} \frac{d}{dr} (\rho_i - \rho_e)$$

$$= - \frac{g}{\rho} \left.\frac{\partial \rho}{\partial T}\right|_p \frac{T}{c_p} \frac{ds_e}{dr} = - \frac{g}{c_p} \underbrace{\frac{\partial \ln \rho}{\partial \ln T}}_{= -\delta} \frac{ds_e}{dr}$$

$$= \frac{g\delta}{c_p} \frac{ds_e}{dr}$$

So the condition for stability is

$$\frac{ds_e}{dr} > 0$$

$\Rightarrow$ Every time in a star if the entropy decreases toward the exterior, we will have convection

We have so far introduced 4 gradients:

1) $\nabla = \nabla_e = \frac{\partial \ln T_e}{\partial \ln P}$

2) $\nabla_i = \frac{\partial \ln T_i}{\partial \ln P}$

3) $\nabla_{ad} = \frac{\partial \ln T}{\partial \ln P}\Big|_{ad} = \frac{P\delta}{C_p \rho T}$

4) $\nabla_{rad} = \frac{3}{16\pi a c G} \frac{\kappa L_r P}{M_r T^4}$ (to be derived)

- ∇, ∇_i and ∇_{ad} are all positive quantities, since P and T grow simultaneously.

- $\nabla_{int} > \nabla_{ad}$, i.e. the adiabatic gradient is flatter, because the temperature in an upward moving cell decreases faster due to heat losses than if the cell is adiabatic.

To derive an expression for ∇_{rad}, we start with the total flux in a convective region:

$$F_{total} = F_{conv} + F_{rad}$$

$$F_{rad} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

$$\frac{dT}{dr} = \frac{T d\ln T}{dr} = \frac{T d\ln T}{d\ln P}\frac{d\ln P}{dr} = \frac{T\nabla}{-H_p}$$

with $\frac{d\ln P}{dr} \equiv -\frac{1}{H_p}$, H_p = pressure scale height

The pressure scale height is the characteristic scale for the pressure decrease over height:

$$P = P_0 e^{-r/H_p}$$

$$\ln P = \ln P_0 - r/H_p$$

$$\frac{d\ln P}{dr} = -\frac{1}{H_p}$$

Then we can write the radiation flux as

$$F_{rad} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} = \frac{4acT^3}{3\kappa\rho}\frac{T\nabla}{H_p} = \frac{4acT^4}{3\kappa\rho}\frac{\nabla}{H_p}$$

The radiative gradient ∇_{rad} is defined as the thermal gradient which would be necessary to carry the sum

$$\boxed{F_{tot} = F_{conv} + F_{rad} = \frac{4acT^4}{3\kappa\rho}\frac{\nabla_{rad}}{H_p}}$$

Furthermore, we can find a better expression.
First, for the H_p:

$$\frac{d\ln P}{dr} = \frac{1}{P}\frac{dP}{dr} = \frac{1}{P}(-\rho g) = -\frac{\rho}{P}\frac{GM_r}{r^2} = -\frac{1}{H_p}$$

$$\Rightarrow \boxed{H_p = \frac{Pr^2}{\rho GM_r}}$$

Then, if we express F_{tot} as

$$F_{tot} = \frac{L_r}{4\pi r^2}$$

$$= \frac{4acT^4}{3\kappa\rho}\frac{\nabla_{rad}}{H_p}$$

$$\Rightarrow \nabla_{rad} = \frac{3\kappa\rho H_p}{4acT^4}\frac{L_r}{4\pi r^2}$$

$$= \frac{3\kappa\rho}{4acT^4}\frac{L_r}{4\pi r^2}\frac{Pr^2}{\rho GM_r}$$

$$= \frac{3\kappa L_r P}{16acT^4GM_r}$$

In a convective zone, the following relations of order apply:

$$\nabla_{rad} > \nabla_{ext} > \nabla_{int} > \nabla_{ad}$$

The radiative gradient ∇_{rad} is defined as the thermal gradient which would be necessary to carry the sum $F_{tot} = F_{conv} + F_{rad}$ of the radiative and convective fluxes by radiation only. This is a fictitious, but calculable, gradient. In a convective zone, it is evidently larger than the other gradients, since in order to carry both the convective and the radiative energies by radiation only, one would need a steeper T gradient.

Why is $\nabla_{ext} > \nabla_{int}$?

We had the stability criterion

$$\nabla_{ext} < \nabla_{int} + \frac{\varphi}{\delta} \nabla_{\mu} = \nabla_{int}$$ for a chemically homogeneous medium.

But for convection, we need an unstable situation: $\Rightarrow \nabla_{ext} > \nabla_{int}$

Lastly, $\nabla_i > \nabla_{ad}$ because ∇_i can lose energy (temperature) faster than an adiabatic (by being adiabatic, thus not losing heat) gradient can.

In outer regions of stars, you have to take into account non-adiabatic convection.

Mixing Length Theory

Convection is essentially a non-local process. At a given level in the star the convective flux and other properties are determined by what happens in the surrounding levels. Fluid elements have a certain spectrum of sizes, velocities and temperatures.

"Mixing length theory" is a phenomenological theory, not based on physical principles, but describing the phenomenon without understanding details.

The basic hypothesis of MLT is that the fluid elements can be represented by an average cell, which moves over an average vertical distance l called the mixing length, over which the cell keeps its identity before dissolving in the ambient medium and deliver its energy excess.

Usually the mixing length l is taken to be of the order of the pressure scale height H_p

$$l = \alpha H_p$$

with a coefficient α of the order of unity.

It is unlikely that a convective cell keeps its identity over several scale heights. On the other side, if $\alpha \ll 1$, there would be little significant fluid motions.

We start by finding an expression for the convective flux. At a level r, the average flux results from the motions of the cells which have an average velocity and an average temperature excess ΔT. Both the hotter upward moving fluid elements and the cooler downward moving elements contribute to the outward transport of energy. We model for the convective flux of energy.

is

$$F_{conv} = C_p \, \rho \, \bar{v} \, \overline{\Delta T}$$

C_p is the specific heat per unit mass at constant pressure. We again consider subsonic flows, which implies that the pressure is in equilibrium with the cell.

$C_p \Delta T$ is an energy per unit mass, multiplied by the density ρ and vertical velocity v, giving an energy flux.

For a motion over the distance ℓ, the average velocity $\bar{v}$ and temperature excess are estimated at $\ell/2$, for example

$$\overline{\Delta T} = \overline{T_{int} - T_{ext}} = T_{int}(r_0) + \left.\frac{\partial T_{int}}{\partial r}\right|_{r_0} \frac{\ell}{2} - T_{ext}(r_0) - \left.\frac{\partial T_{ext}}{\partial r}\right|_{r_0} \ell/2$$

$$= \left(\left.\frac{\partial T_{int}}{\partial r}\right|_{r_0} - \left.\frac{\partial T_{ext}}{\partial r}\right|_{r_0} \right) \ell/2$$

Now using

$$\frac{dT}{dr} = T\frac{d\ln T}{dr} = T\frac{d\ln T}{d\ln P}\frac{d\ln P}{dr} = T\nabla\cdot\left(\frac{-1}{H_p}\right)$$

gives the expression

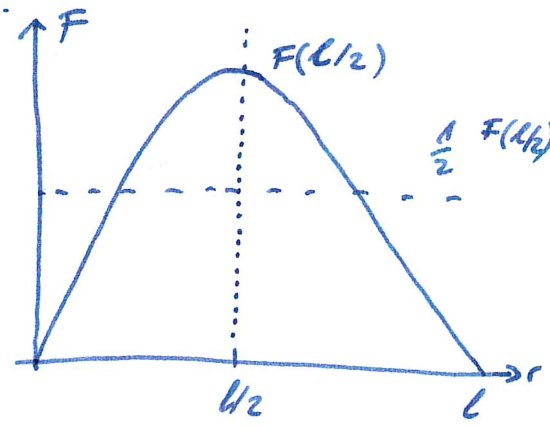
$$\overline{\Delta T} = \left(\frac{\partial T_{int}}{\partial r}\bigg|_{r_0} - \frac{\partial T_{ext}}{\partial r}\bigg|_{r_0}\right)\frac{\ell}{2} =$$

$$= \left(\frac{-T\nabla_i}{H_p} + \frac{T\nabla_e}{H_p}\right)\frac{\ell}{2} = (\nabla_e - \nabla_i)\frac{T\ell}{2H_p}$$

Now for the velocity: We start off by expressing the force per unit volume as

$$\mathcal{F} = -\rho_i g + \rho_e g = -g(\rho_i - \rho_e)$$

The mean value of the force is taken as half the value at the middle of ℓ



The force increases with the temperature gradient, but at some point, the cell cools down, and the gradient decreases again.

The work up to the distance $\ell/2$ is $\bar{\mathcal{F}}\cdot\ell/2$.
In MLT, it is assumed that half of this work is converted into the kinetic energy of the fluid element, the rest is assumed to be dissipated on the way.

For the half of the work converted to kinetic energy, we have

$$\frac{1}{2}\bar{\mathcal{F}}\cdot\ell/2 = \frac{1}{2}\rho\bar{v}^2$$

$$= \frac{1}{2}\left(\frac{1}{2}\mathcal{F}(\ell/2)\right)\cdot\frac{\ell}{2}$$

$$\Rightarrow \quad \bar{v}^2 = \frac{1}{\rho}\,\frac{\ell}{4}\,\mathcal{F}(\ell/2)$$

We expand the expression for $\mathcal{F}(\ell/2)$:

$$\mathcal{F}(\ell/2) = -g(\rho_i - \rho_e)$$

$$\approx -g\left(\rho_i(r_0) + \left.\frac{d\rho_i}{dr}\right|_{r_0}\frac{\ell}{2} - \rho_e(r_0) - \left.\frac{d\rho_e}{dr}\right|_{r_0}\frac{\ell}{2}\right)$$

$$= -g\left(\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr}\right)_{r_0}\frac{\ell}{2}$$

Once again we use

$$d\ln\rho = \alpha\, d\ln P - \delta\, d\ln T$$

$$\Rightarrow \quad \frac{d\rho_i}{dr} - \frac{d\rho_e}{dr} = \frac{\alpha\rho}{P}\frac{dP_i}{dr} - \frac{\delta\rho}{T}\frac{dT_i}{dr} - \frac{\alpha\rho}{P}\frac{dP_e}{dr} + \frac{\delta\rho}{T}\frac{dT_e}{dr}$$

with $\rho_i(r_0) = \rho_e(r_0) = \rho$, $dP = 0$ for const. P

$$= \frac{-\delta\rho}{T}\left(\frac{dT_i}{dr} - \frac{dT_e}{dr}\right)_{r_0}$$

$$= -\delta\rho\left(\frac{d\ln T_i}{d\ln P}\frac{d\ln P}{dr} - \frac{d\ln T_e}{d\ln P}\frac{d\ln P}{dr}\right)$$

$$= -\delta\rho\left(\frac{-\nabla_i}{H_p} + \frac{\nabla_e}{H_p}\right)$$

This gives us

$$\mathcal{F}(\ell/2) \approx -g\left(\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr}\right)\frac{\ell}{2}$$

$$\approx -g\frac{\ell}{2}\cdot\frac{(-\delta\rho)}{H_p}\left(\nabla_e - \nabla_i\right)$$

$$= \frac{\rho g \ell\delta}{2H_p}\left(\nabla_e - \nabla_i\right)$$

and finally

$$\bar{v}^2 = \frac{1}{\rho}\frac{\ell}{4}\mathcal{F}(\ell/2) = \frac{\ell}{\rho 4}\frac{\rho g \ell\delta}{2H_p}\left(\nabla_e - \nabla_i\right)$$

$$\boxed{\bar{v}^2 = \frac{g\ell^2\delta}{8H_p}\left(\nabla_e - \nabla_i\right)}$$

Once again we have the condition

$$\nabla_e > \nabla_i$$

which is the condition for an unstable situation coming from the Brunt–Väisälä frequency.

With the expressions for $\overline{\Delta T}$ and $\bar{v}$, we get for the convective flux

$$F_{conv} = c_p \rho \bar{v} \overline{\Delta T}$$

$$= c_p \rho \left[\frac{g \ell^2 \delta}{8 H_p} (\nabla_e - \nabla_i) \right] (\nabla_e - \nabla_i) \frac{T \ell}{2 H_p}$$

$$\boxed{F_{conv} = \frac{c_p \rho T \sqrt{g \delta}}{4 \sqrt{2}} \frac{(\nabla_e - \nabla_i)^{3/2} \ell^2}{H_p^{3/2}}}$$

With these expressions, we can make some orders of magnitude estimates.

Let us consider a point at $r = R_\odot/2$ and $M_r = M_\odot/2$ in the interior of the sun as well as a point at the solar envelope at $r = R_\odot$ and $M_r = M_\odot$.

Let us assume that the whole flux is carried by convection and search the temperature gradient difference $\left(\frac{dT_i}{dr} - \frac{dT_e}{dr}\right)$ which would be necessary to carry the whole flux by convection.

We have

$$\frac{dT}{dr} = \frac{T\, d\ln T}{d\ln P} \frac{d\ln P}{dr} = \frac{-T\nabla}{H_p}$$

$$\Rightarrow \frac{\nabla}{H_p} = -\frac{1}{T}\frac{dT}{dr}$$

So from the convection flux, we have

$$F_{conv} = \frac{C_p\, \varsigma T \sqrt{g\delta}}{4\sqrt{2}} \left(\frac{\nabla_e - \nabla_i}{H_p}\right)^{3/2}$$

$$\Rightarrow \frac{\nabla_e - \nabla_i}{H_p} = \left[\frac{4\sqrt{2}}{C_p\, \varsigma T \sqrt{g\delta}} F_{conv}\right]^{2/3} = \frac{1}{T}\left(\frac{dT_i}{dr} - \frac{dT_e}{dr}\right)$$

$$\Rightarrow \frac{dT_i}{dr} - \frac{dT_e}{dr} = \left[\frac{F_{conv} \, 4\sqrt{2}}{c_p \, \rho \left(\frac{g\delta}{T}\right)^{1/2} \ell^2} \right]^{2/3}$$

We approximate $F_{conv} = \frac{L}{4\pi R^2}$, and obtain

$$\overline{\Delta T} \approx \left(\frac{dT_{int}}{dr} - \frac{dT_{ext}}{dr} \right) \frac{\ell}{2} \sim 1.8 \cdot 10^{-2} \, K$$

for $\ell \sim 0.1 R_\odot$ in the stellar interior,

and $\frac{dT_{int}}{dr} - \frac{dT_{ext}}{dr} \sim 5 \cdot 10^{-12} \frac{K}{cm}$

So convective cells with an excess ΔT of only $\sim 10^{-2}$ K are able to carry the whole stellar flux in an ambient medium of $T \approx 10^7$ K.

$\Rightarrow$ The outgoing flux is a very small fraction of the thermal energy content of the star.

$\Rightarrow$ the very small $\overline{\Delta T} = \overline{(T_{int} - T_{ext})}$ implies that convection is essentially adiabatic in stellar interiors.

We can also estimate the "turnover" time, i.e. the travel time of the cell over the distance l is

$$t_{turnover} = \frac{l}{v} \approx 2 \times 10^{2} \text{ days for the sun}$$

This timescale is very short with respect to the MS lifetime.

$\Rightarrow$ During most of evolution, the convective mixing can be considered instantaneous and that the convective zones are chemically homogeneous.

The "interior" of a star can in general be taken as the regions where H and He are completely ionised.

In the outer layers of the stars, we get

$$\frac{dT_{int}}{dr} - \frac{dT_{ext}}{dr} \sim 10^{-4} \frac{K}{cm}$$

which is $\sim 10^8$ times bigger than in the interior.

The average temperature difference is

$$\overline{\delta T} \sim 1000K$$

which is $\sim 10^5$ times bigger than in the interior.

Clearly, convection isn't adiabatic in the outer regions of stars. Non-adiabatic convection is less effective: For maximum efficiency the cell would keep all its energy up until the point where it dissolves.

Convection in Stellar Interiors

We have seen that in stellar interiors, convections are essentially adiabatic.
A convenient rule to define the "stellar interior" is the layers when H and He are fully ionized.

Since the convection is adiabatic, we can write

$$\nabla_i = \nabla_{ad}$$

$$\frac{d\ln T_i}{d\ln P} = \frac{d\ln T}{d\ln P}\bigg|_{ad} = \frac{P\delta}{c_p \rho T}$$

Furthermore, using the same order of magnitude estimate as before, we have that

$$\frac{\nabla_e - \nabla_i}{\nabla_e} \sim 10^{-8}$$

So we can also set

$$\nabla_e \approx \nabla_i = \nabla_{ad}$$

So in a convective zone, the inequalities

$$\nabla_{rad} > \nabla_e > \nabla_i > \nabla_{ad}$$

reduce to

$$\nabla_{rad} > \nabla_{ad} \qquad \text{(Schwarzschild criterion)}$$

from the original expression of the condition

$$\nabla_e > \nabla_i$$

similarly, we get the Ledoux criterion

$$\nabla_{rad} > \nabla_{ad} + \frac{\varphi}{\delta} \nabla_\mu$$

which is easier to derive by considering the conditions for a radiative zone.

In a radiative zone, we just have

$$\nabla_e = \nabla_{rad}$$

since the total flux, which depends on ∇_{rad}, is equal to the radiative flux, which depends on ∇. We also have

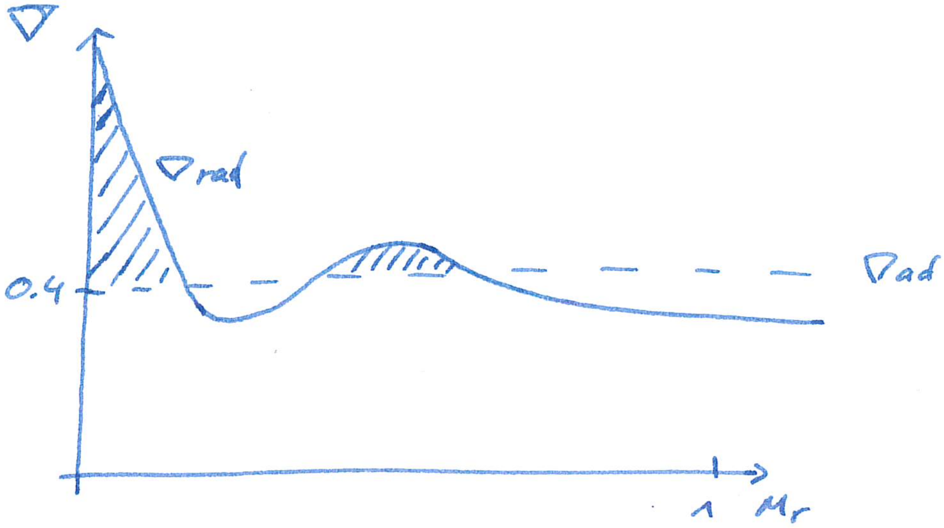
$$\nabla_i = \nabla_{ad}$$

since the fluid motions are adiabatic.

The radiative region needs to be stable w.r.t. convective instabilities, thus

$$\nabla_{rad} < \nabla_{ad}$$

$$\nabla_{rad} < \nabla_{ad} + \frac{\varphi}{\delta} \nabla_\mu$$



Convection occurs in the marked areas, where the condition

$$\nabla_{rad} = \nabla_e > \nabla_i = \nabla_{ad}$$

is satisfied.

The equation expressing the temperature gradient in a convection zone is

$$\frac{d\ln T}{d\ln P} = \frac{P}{T}\frac{dT}{dP} = \nabla \quad \Rightarrow \frac{dT}{dr} = \frac{T}{P}\frac{dP}{dr}\nabla$$

$$\Rightarrow \frac{dT}{dr} = \frac{T}{P}\nabla\frac{dP}{dr} = \frac{T}{P}\nabla\frac{dP}{dM_r}\frac{dM_r}{dr}$$

$$= \frac{T}{P}\nabla\frac{1}{4\pi r^2\rho}\frac{dP}{dM_r} = \frac{dT}{dM_r}\frac{dM_r}{dr} = \frac{1}{4\pi r^2\rho}\frac{dT}{dM_r}$$

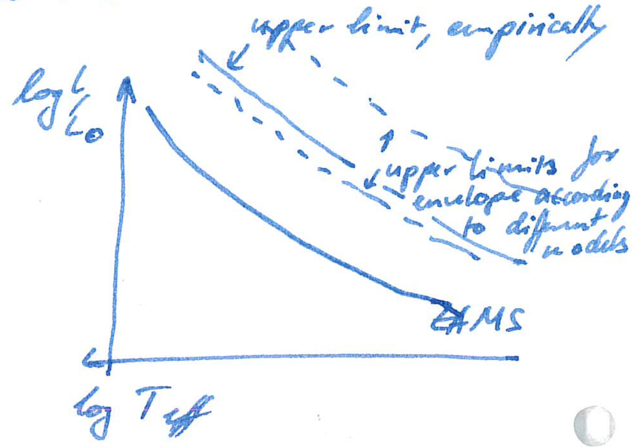
$$\Rightarrow \boxed{\frac{dT}{dM_r} = \frac{T}{P}\frac{dP}{dM_r}\nabla = -\frac{T}{P}\frac{GM_r}{4\pi r^4}\nabla}$$

In this equation, $\nabla = \nabla_{ad}$ in a convective zone and $\nabla = \nabla_{rad}$ in a radiative zone.

A problem is how to define the edge of the convective zone. Only because the force is

$$\rho_i \ddot{r} = g(\rho_e - \rho_i) = 0$$

at some point, it doesn't mean that the movement of the convected cell stops; it still has momentum. This problem is known as "overshooting".



A possibility to deal with this problem is to parametrize (the effects of) overshooting and fit the parameters such that the observed upper envelopes on the HR-diagram are reproduced.

In case of mono-atomic perfect gas with $\nabla_{ad} = \frac{2}{5}$, one has the following relation between structural variables in internal convective regions:

$$\nabla_{ad} = \frac{2}{5} = \frac{d\ln T}{d\ln P} \rightarrow d\ln T = \frac{2}{5} d\ln P$$

$$\Rightarrow \boxed{T \sim P^{2/5}, \quad P \sim T^{5/2}}$$

from $pV = NkT = \frac{V\rho}{m_a \mu} kT$: $p \sim \rho T \sim T^{5/2}$

$$\Rightarrow \boxed{\rho \sim T^{3/2}}$$

$$\Rightarrow p \sim T^{5/2} = (T^{3/2})^{5/3} \sim \rho^{5/3}$$

$$\boxed{p \sim \rho^{5/3}}$$

In the adiabatic interior, the Brunt-Väisälä frequency becomes

$$N^2 = \frac{g}{\rho}\left(\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr}\right)$$

and using the general EoS

$$d\ln \rho = \alpha\, d\ln P - \delta\, d\ln T + \varphi\, d\ln \mu$$

we get for $P_i = P_e$, $dP_{i,e} = 0$, $d\mu_i = 0$:

$$\frac{d S_i}{dr} - \frac{d S_r}{dr} = -\delta \frac{d\ln T_i}{dr} + \delta \frac{d\ln T_e}{dr} - \varphi \frac{d\ln \mu_e}{dr}$$

$$= \delta \left(\frac{d\ln T_e}{dr} - \frac{d\ln T_i}{dr} \right) - \varphi \frac{d\ln \mu_e}{dr}$$

$$= \delta \left(\frac{d\ln T_e}{d\ln P} \frac{d\ln P}{dr} - \frac{d\ln T_i}{d\ln P} \frac{d\ln P}{dr} \right) - \varphi \frac{d\ln \mu_e}{d\ln P} \frac{d\ln P}{dr}$$

$$= -\frac{1}{H_p} \left[\delta (\nabla_e - \nabla_i) - \varphi \nabla_\mu \right]$$

giving us

$$N^2 = \frac{g}{H_p} \left[(\nabla_i - \nabla_e) \delta + \varphi \nabla_\mu \right]$$

in the adiabatic case, we have shown that $\nabla_i = \nabla_{ad}$

$$\Rightarrow N^2_{ad} = \frac{g}{H_p} \left[\delta (\nabla_{ad} - \nabla_e) + \varphi \nabla_\mu \right]$$

$$= \frac{g\delta}{H_p} \left[\nabla_{ad} - \nabla_e \right] + \frac{g\varphi}{H_p} \nabla_\mu$$

$$= N^2_{T,ad} + N^2_\mu$$

We can also derive a different expression directly:

Using $\Gamma_1 = \left.\frac{d\ln P}{d\ln \rho}\right|_{ad}$

Then

$$N^2 = \frac{g}{\rho}\frac{d}{dr}(\Delta\rho) = \frac{g}{\rho}\left(\frac{d\rho_i}{dr} - \frac{d\rho_e}{dr}\right)$$

$$= g\left(\frac{d\ln\rho_i}{dr} - \frac{1}{\rho}\frac{d\rho_e}{dr}\right)$$

$$= g\left(\frac{d\ln\rho_i}{d\ln P}\frac{d\ln P}{dr} - \frac{1}{\rho}\frac{d\rho_e}{dr}\right)$$

$$= g\left(\frac{1}{\Gamma_1 P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho_e}{dr}\right)$$

Non-Adiabatic Convection

In convective stellar envelopes, the excess $\overline{\Delta T}$ is not negligible w.r.t. the local temperature ($\overline{\Delta T} \sim 1000\,K$) and convection is no longer adiabatic. In practice, one generally considers that a non-adiabatic treatment is required in regions where the ionizations of H and He are incomplete.

Let us first quantify the amount of radiative losses of the convected cell.

The energy lost by a turbulent eddy of diameter ℓ, temperature excess $\overline{\Delta T}$, surface Σ during its travel time $\Delta t = \ell/\bar{v}$ is

$$\Delta U_{lost} = |F_{rad}| \cdot \Sigma \cdot \Delta t$$

$$= \left| -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr} \right| \Sigma \Delta t$$

$$\approx \frac{4acT^3}{3\kappa\rho}$$

We assume that the cell is optically thick, such that the energy inside the cell is not lost instantaneously. We take the size of the cell to be $= \ell$ to avoid introducing a new scale length. Furthermore, we take the expression for $\overline{\Delta T}$ from mixing length theory:

$$\overline{\Delta T} \approx \left(\frac{dT_{int}}{dr} - \frac{dT_{ext}}{dr} \right) \frac{\ell}{2}$$

(which comes from expanding $T_i - T_{ext}$ to 1st order)

$$= \left(\frac{T \, d\ln T}{d\ln P} \frac{d\ln P}{dr} - \frac{T \, d\ln T_{ext}}{d\ln P} \frac{d\ln P}{dr} \right) \frac{\ell}{2}$$

$$= (\nabla_e - \nabla_i) \frac{T\ell}{2H_p}$$

Then

$$\Delta U_{lost} = \frac{4acT^3}{3\kappa\rho} (\nabla_e - \nabla_i) \frac{T\ell}{2H_p} \cdot \frac{1}{\ell/2} \cdot \Sigma \cdot \Delta t$$

$$= \frac{4acT^3}{3\kappa\rho} (\nabla_e - \nabla_i) \frac{T}{H_p} \cdot \frac{\ell}{v} \Sigma$$

and per unit volume, for a spherical cell, we have $\frac{\Sigma}{V} = \frac{4\pi (\ell/2)^2}{4/3 \, \pi (\ell/2)^3} = \frac{3\ell^2/4}{\ell^3/8} = 6/\ell$

and $$\frac{\Delta U_{lost}}{V} = \frac{4acT^3}{3\kappa\rho} (\nabla_e - \nabla_i) \frac{T}{H_p} \frac{6}{v}$$

This loss of energy is made at the expense of heat. The first law of thermodynamics:

$$\delta q = C_p \, dT - \frac{\delta}{\rho} dP$$

is when translating from specific (per mass) quantities to energies per unit volume:

$$\frac{m}{V} \delta q = \rho \, dq = \rho C_p dT - \delta dP$$

(We have computed this form of the first law of thermodynamics when we were looking at the energetic equilibrium of stars. Cornerstone ideas ~~da~~ are to use the entropy, that ds is an exact differential, and Maxwell identities.)

We then set:

$$\nabla_{ad} = \frac{P\delta}{C_p \rho T} \quad \Rightarrow \quad \delta = \frac{C_p \rho T}{P} \nabla_{ad}$$

$$\rightarrow \delta dP = \delta \frac{dP}{dr} dr = \frac{C_p \rho T}{P} \nabla_{ad} \frac{dP}{dr} dr$$

$$= - \frac{C_p \rho T}{H_P} \nabla_{ad} \, dr$$

and

$$\rho C_p dT = \rho C_p T \frac{d\ln T}{d\ln P} \frac{d\ln P}{dr} dr$$

$$= - \frac{\rho C_p T \nabla_{int}}{H_P} dr$$

In summary, this gives

$$\rho\delta q = \rho C_p \, dT - \delta \, dP$$

$$= -\frac{\rho C_p T \nabla_{int}}{H_p} dr + \frac{C_p \rho T}{H_p} \nabla_{ad} \, dr$$

$$= \frac{\rho C_p T}{H_p} (\nabla_{ad} - \nabla_{int}) \, dr$$

$$\Rightarrow \rho \Delta q \approx \frac{\rho C_p T}{H_p} (\nabla_{ad} - \nabla_{int}) \underbrace{\ell}_{\approx \Delta r}$$

We now can equate the energy loss per unit volume with the heat:

$$\rho \Delta q = -\frac{U_{loss}}{V}$$

$$\frac{\rho C_p T}{H_p} (\nabla_{ad} - \nabla_{int}) \ell = -\frac{4acT^3}{3\kappa\rho} (\nabla_{ext} - \nabla_{int}) \frac{T}{H_p} \frac{6}{\bar{v}}$$

$$\Rightarrow \frac{\nabla_{ext} - \nabla_{int}}{\nabla_{int} - \nabla_{ad}} = \frac{\rho C_p T}{H_p} \frac{\ell}{6} \frac{\bar{v} H_p}{T} \frac{3\kappa\rho}{4acT^3}$$

$$= \frac{\kappa \rho^2 C_p \, \ell \bar{v}}{8acT^3}$$

$$\boxed{= \frac{3\rho^2 C_p \kappa}{4ac \, T^3} \frac{\bar{v}\ell}{6} \equiv \Gamma}$$

The ratio Γ is called the "convection efficiency", and is the ratio

$$\Gamma = \frac{\text{thermal energy transported by fluid element}}{\text{energy lost by radiation during transport}}$$

We can re-write Γ as

$$\frac{\nabla_e - \nabla_i}{\nabla_i - \nabla_{ad}} = \frac{3\rho^2 c_p \kappa}{4acT^3} \frac{\bar{v}\ell}{6} \equiv \frac{\bar{v}\ell}{6K}$$

$$\text{with} \quad K \equiv \frac{4acT^3}{3\rho^2 c_p \kappa}$$

for which we can find physical meaning.

Let us consider a small volume in the envelope of a star, where $\varepsilon_{nuc} = 0$.

Energy conservation dictates:

$$\rho \frac{dq}{dt} = \varepsilon + \operatorname{div} \vec{F}$$

If we write the flux as the radiative flux

$$F = F_{rad} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

we get

$$\rho\frac{dq}{dt} = \varepsilon + \mathrm{div}\,F = 0 + \mathrm{div}\left(\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}\right)$$

$$= \mathrm{div}\left(-C_{rad}\vec{\nabla}T\right)$$ with $C_{rad} = \frac{4acT^3}{3\kappa\rho}$ radiative conductivity, assume = const

$$= \underbrace{\rho \cdot c_p \frac{dT}{dt}}_{\frac{dq}{dt}}$$

$$\Rightarrow \frac{dT}{dt} = -\frac{C_{rad}}{\rho c_p}\nabla^2 T$$ diffusion equation of temperature

$$= -\frac{4acT^3}{3\kappa\rho^2 c_p}\nabla^2 T$$

$$= -K\nabla^2 T$$

$\Rightarrow$ K is the diffusion coefficient in the thermal diffusion equation, and is called "thermal diffusivity" or "heat conductivity".

Using the diffusion equation, it is straightforward to show that the thermal adjustment timescale is given by

$$\tau_{therm} \sim l^2/K :$$

$$\frac{dT}{dt} = K\nabla^2 T \quad \rightarrow \quad \frac{T}{\tau_{therm}} \sim K\frac{T}{l^2}$$

$$\Rightarrow \tau_{therm} \sim \frac{l^2}{K}$$

however, we can go even further:

using $K = \dfrac{4acT^3}{3\rho^2 c_p \kappa}$ and $F_{rad} = \dfrac{4acT^3}{3\kappa\rho}$,

we can write

$$\tau_{therm} \sim \frac{l^2}{K} \sim R^2 \frac{3\rho^2 c_p \kappa}{4acT^3} = R^2 \frac{\rho c_p}{F_{rad}} \left|\frac{dT}{dr}\right|$$

$$= R^2 \frac{\rho c_p 4\pi R^2}{L} \left|\frac{dT}{dr}\right| \sim \frac{\rho c_p 4\pi R^4}{L} \frac{\overline{T}}{R}$$

now using $\rho = \dfrac{M}{4/3\,\pi R^3}$, $\overline{T} = \dfrac{1}{3}\dfrac{\mu m_u}{k} q \dfrac{GM}{R}$

we get

$$\tau_{therm} \sim \frac{\rho c_p 4\pi R^3}{L}\overline{T} = \frac{3M\, 4\pi R^3}{L\, 4\pi R^3} \frac{1}{3} \frac{\mu m_u}{k} q \frac{GM}{R}$$

$$= \frac{\mu m_u}{k} q \frac{GM^2}{RL} \sim \frac{GM^2}{RL} \sim \tau_{KH}$$

So the thermal adjustment timescale is of the same order as the Kelvin–Helmholtz timescale!
τ_{KH} is the typical lifetime of a star that produces its luminosity exclusively from gravitational contraction.

Why are they of the same order?
The virial theorem tells us that in mechanical equilibrium, $U \sim \Omega$, and we have

$$\tau_{KH} \sim \frac{\Omega}{L} \quad \text{and} \quad \tau_{therm} \sim \frac{U}{L}$$

Let us now look at some limiting cases for

$$\Gamma = \frac{\nabla_e - \nabla_i}{\nabla_i - \nabla_{ad}}$$

For $\Gamma \to \infty$: $\nabla_i - \nabla_{ad} \to 0$

$\Rightarrow \nabla_i \simeq \nabla_{ad}$, we're in the limit of adiabatic convection. Radiative losses are negligible.

For $\Gamma \to 0$: $\nabla_e \simeq \nabla_i$, and (for undiscussed reasons) $\nabla_e \approx \nabla_{rad}$. Convection transports negligible energy, we're in a radiative zone.

Solutions of the Equations and Simple Models

Hydrostatic and Hydrodynamic Models

Hydrostatic models apply when evolution is slow w.r.t. the dynamical time $t_{dyn} \sim \frac{1}{\sqrt{G\bar{\rho}}}$

Let us write again the four equations of stellar structure:

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}$$

$$\frac{dr}{dM_r} = \frac{1}{4\pi \rho r^2}$$

$$\frac{dL_r}{dM_r} = \varepsilon_{nucl} + \varepsilon_{grav} - \varepsilon_\nu$$

$$\frac{dT}{dM_r} = -\frac{GM_r}{4\pi r^4}\frac{T}{P}\nabla$$

$\nabla \to \nabla_{rad}$ in radiative zone

$\nabla \to \nabla_{ad}$ in convective zone in interior

In the interior, convection is present if $\nabla_{rad} > \nabla_{ad}$, otherwise the energy transfer is radiative.

M_r is the independent variable in general, the unknowns are P, T, L and r.

The EoS, opacities and nuclear reactions are expressed by the functions

$$\rho = \rho(P, T, X_i), \quad \kappa = \kappa(P, T, X_i), \quad \varepsilon = \varepsilon(P, T, X_i)$$

generally given by numerical tables for different abundances X_i.

Note that $\varepsilon_{grav} = -\frac{dq}{dt}$ is the only part of these equations where time appears explicitly. At a second point, it appears implicitly in the chemical composition.

Usual abundances are

$$\boxed{X = 0.72, \quad Y = 0.266, \quad Z = 0.014}$$

The four equations need boundary conditions.
Given a mass M, we set

	r	P	T	L_r
surface	R	0	0	L
center	0	P_c	T_c	0

However, integrating numerically from a given value of R to the center does not automatically guarantee that all boundary conditions are satisfied at the center. Typically, the calculation needs to be done iteratively. The root of this problem is that some boundary conditions are given at the center, while others are at the surface.

This leads to the Vogt-Russell theorem:
The properties of a star of a given composition and in equilibrium are entirely determined by its mass.
Or: M and X, Y, Z specify a unique set of P, L, R, T.

The Vogt-Russel theorem implies the existence of relations such as

$$L = L(M, X_i), \quad R = R(M, X_i).$$

Let us consider the Lagrangian form of hydrostatic equations. At the center, some equations diverge, e.g. the hydrostatic equilibrium.

Near the center, we can expand the density

$$\rho(r) = \rho_c + \frac{d\rho_c}{dr}\Big|_r \approx \rho_c$$

because $\frac{d\rho_c}{dr} \approx 0$ at the center because of spherical symmetry. In that small region around the center, we have

$$M_r = \frac{4}{3}\pi r^3 \rho_c \quad \Rightarrow r = \left(\frac{3M_r}{4\pi\rho_c}\right)^{1/3}$$

and for the hydrostatic equilibrium:

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} = -\frac{GM_r}{4\pi}\left(\frac{4\pi\rho_c}{3M_r}\right)^{4/3}$$

$$= -\frac{G}{4\pi}\left(\frac{4\pi\rho_c}{3}\right) M^{-1/3}$$

We can integrate this:

$$\int dP = \int dM_r \frac{-G}{4\pi} \left(\frac{4\pi \rho_c}{3} \right)^{4/3} M_r^{-1/3}$$

$$P - P_c = -\frac{G}{4\pi} \left(\frac{4\pi \rho_c}{3} \right)^{4/3} \cdot \frac{3}{2} M_r^{2/3}$$

$$= -\frac{3G}{8\pi} \left(\frac{4\pi \rho_c}{3} \right)^{4/3} M_r^{2/3}$$

By using

$$L = 4\pi R^2 F = 4\pi R^2 \sigma T_{eff}^4$$

we can similarly find an expression for

$T^4 - T_c^4$.

A good method to deal with boundary conditions is to start from the surface with the three parameters M, L, and R, and integrate towards the center.

Starting from outside, we have in the numerical model a first region called 'the atmosphere', and then a second region called the envelope before we reach the interior of a star.

In the atmosphere, we can assume that M, L, and R are $\approx$ const. We introduce the optical depth τ as the independent variable in the atmosphere:

$$d\tau = -\kappa \rho \, dr$$

The hydrostatic equilibrium equation becomes

$$\frac{dP}{dr} = -\rho g = -\rho \underbrace{\frac{GM}{R^2}}_{\text{const}} = -\rho g$$

$$= -\frac{\rho g \kappa}{\kappa}$$

$$\Rightarrow \frac{dP}{-\rho \kappa dr} = = \frac{g}{\kappa} = \frac{dP}{d\tau}$$

If we treat κ as a constant:

$$\frac{dP}{d\tau} = \frac{g}{\kappa} \rightarrow P(\tau) = P(\tau=0) \frac{g\tau}{\kappa}$$

In the atmosphere, $P(\tau=0) = P_{rad} = \frac{1}{3} a T^4$, there is only radiation pressure present.

However, we need another equation to solve for $T(\tau)$, the only unknown left.
This will be the radiative transfer equation.

$$\frac{dP_{rad}}{dr} = -\frac{\kappa \rho F}{c} \Rightarrow \frac{dP_{rad}}{d\tau} = \frac{F}{c}$$

Let us assume that $F = \frac{L}{4\pi R^2} \approx \text{const.}$
In particular, $F = F_{surface} = \sigma T_{eff}^4$.

$$\Rightarrow \frac{dP_{rad}}{d\tau} = \frac{\sigma T_{eff}^4}{c}$$

$$\Rightarrow P_{rad}(\tau) = P_{rad}(\tau=0) + \frac{\sigma T_{eff}^4}{c}\tau$$

$$\frac{1}{3} a T^4(\tau) = \frac{1}{3} a T^4(\tau=0) + \frac{\sigma T_{eff}^4}{c}\tau$$

$$\Rightarrow T^4(\tau) = T^4(\tau=0) + 3\frac{\sigma}{ac} T_{eff}^4 \tau$$

$$= T^4(\tau=0) + 3\frac{ac}{4ac} T_{eff}^4 \tau \qquad \Big| \; \sigma = \frac{ac}{4}$$

$$= \frac{3}{4} T_{eff}^4 (\tau + q)$$

$$\text{with } q \equiv \frac{4}{3}\frac{T^4(\tau=0)}{T_{eff}^4}$$

It can be shown that if the outgoing specific intensity is constant and if there is no ingoing specific intensity, then

$$q = 2/3$$

$$\Rightarrow T^4(\tau) = \frac{3}{4} T_{eff} (\tau + 2/3)$$

This allows us to start integrating from outwards until $\tau = 2/3$, where we have $T(\tau = \frac{2}{3}) = T_{eff}$, at which point it is considered that the atmosphere ends and the envelope begins.

In the envelope, the mass does not vary a lot. It is therefore better to use a quantity that exhibits great change in the envelope as the independent variable instead of the mass. Furthermore, partial ionization and non-adiabatic convection are allowed in the envelope.

Below the envelope, we have the interior of the star, where convection is adiabatic and where the gas is completely ionized.

We use:

$$\frac{dP}{dr} = -\rho \frac{GM}{r^2} \qquad \text{and}$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3} \frac{L_r}{4\pi r^2}$$

to get

$$\frac{dP}{dT} = \frac{dP}{dr}\frac{dr}{dT} = \rho \frac{GM}{r^2} \frac{4acT^3 \, 4\pi r^2}{3\kappa\rho L_r}$$

$$= \frac{16\pi GM acT^3}{3\kappa L_r}$$

we also use $\rho = \dfrac{P}{\frac{k}{\mu m_u} T}$

and $\kappa = \kappa_0 \rho T^{-3.5}$

(Kramer's opacity law)

to obtain

$$\frac{dP}{dt} = \frac{16\pi GM acT^3}{3\kappa L_r} = \frac{16\pi GM acT^3}{3\kappa_0 \frac{P}{k/\mu m_u} T^{-1} T^{-3.5} L_r} = \frac{16\pi GM ac\, k\, T^{7.5}}{3\kappa_0 \mu m_u P L_r}$$

or re-written:

$$\frac{dP}{dT} = \left(\frac{16\pi}{3} \frac{GM}{\kappa_0 L_r} \frac{k}{\mu m_u} \right) \frac{T^{7.5}}{P}$$

where we consider all the terms in the parentheses to be constant. Then we can integrate:

$$\frac{dP}{dT} = C \frac{T^{7.5}}{P} \Rightarrow P\,dP = C T^{7.5} dT$$

$$\Rightarrow \frac{1}{2} P^2 = \frac{1}{8.5} C T^{8.5}$$

$$\Rightarrow P = \left(\frac{2}{8.5} C \right)^{1/2} T^{4.25} = \left(\frac{20}{85} C \right)^{1/2} T^{4.25}$$

$$= \left(\frac{4}{17} \cdot \frac{16\pi}{3} \frac{GM}{\kappa_0 L_r} \frac{k}{\mu m_u} \right)^{1/2} T^{4.25}$$

$$= \left(\frac{64}{51}\pi \frac{GM}{\kappa_0 L_r} \frac{k}{\mu m_u} \right)^{1/2} T^{4.25}$$

The additive constant is zero: We have $T=0$ for $P=0$.

Or, in other words, we can express this as

$$d \ln P \propto 4.25\, d \ln T$$

What happens if we have adiabatic convection in the envelope?

$$\nabla_{ad} = \frac{2}{5} = 0.4 = \left.\frac{d\ln T}{d\ln P}\right|_{ad}$$

$\Rightarrow$ Then we have $P \propto T^{5/2} = T^{2.5}$

$\Rightarrow$ In a radiative zone, temperature increases more slowly ($d\ln T \propto \frac{1}{4.25} d\ln P$) with pressure than in a convective zone ($d\ln T \propto 0.4\, d\ln P$)

We can also write the $\frac{dP}{dT}$ relation as a $\frac{dP}{dr}$ relation:

$$P = C\,T^{4.25}$$

$$\Rightarrow \frac{dP}{dr} = 4.25\, C\, T^{3.25} \frac{dT}{dr}$$

and find an expression for $\frac{dT}{dr}$:

$$\frac{dP}{dr} = 4.25\, C\, T^{3.25} \frac{dT}{dr} = -\rho \frac{GM}{r^2} = -\frac{P}{\frac{k}{m_u \mu} T} \frac{GM}{r^2}$$

$$\Rightarrow \frac{dT}{dr} = -\frac{m_u \mu}{k} \frac{P}{T} \frac{GM}{r^2} \frac{1}{4.25\, C\, T^{3.25}}$$

$$= -\frac{m_u \mu}{k} \underbrace{\frac{P}{T^{4.25}}}_{\substack{P = CT^{4.25} \\ \Rightarrow \frac{P}{T^{4.25}} = C}} \frac{1}{4.25\, C} \frac{GM}{r^2}$$

$$= -\frac{1}{4.25} \frac{m_u \mu}{k} \frac{GM}{r^2}$$

$$\Rightarrow T(r) - T_{eff} = \frac{1}{4.25} \frac{m_u \mu}{k} GM \left(\frac{1}{r} - \frac{1}{R} \right)$$

$\Rightarrow$ the temperature in the outer layers varies as $1/r$

This is also true for a convective envelope. T_{eff} appears as an additive constant for an envelope of a star. Thus, when at some depth T reaches $\sim 10^6$ K, the fact that at the surface $T_{eff} \sim 6000$K makes no significant difference for $T(r)$ in the interior.

This shows a general property of stellar structure: the outer boundary conditions only have very little effect on the internal structure. For a star of given mass and composition, different surface conditions rapidly converge toward the same values in the stellar interior.

The Henyey Method

The spherical star is divided in m concentric mass shells from the surface to the center.

The idea is to discretize the 4 equations of stellar structure between each of the two shells, e.g.

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho} \rightarrow \frac{r_i - r_{i+1}}{M_{r\,i} - M_{r\,i+1}} = \frac{1}{4\pi r_{i+1/2}^2 \rho_{i+1/2}}$$

this gives us

$$4 \cdot (m-1) + \underset{\text{surface conditions}}{2} = 4m - 2$$ equations to solve.

Luckily, we also have

$4 \cdot m - 2$ unknowns
known approximations at the center/boundary

Thus, the system is solvable.

If we have an approximate solution, we can linearize the problem, e.g.

$$P^{i+1} = P^{i} + \delta P$$

and solve the system of equations iteratively.

However, this method requires the knowledge of an initial approximate solution, thus we need approximate models first. If you want to follow the evolution of a star over time, you can use the previous timestep as an initial guess.

Homology Transformations

Homology transformations are scaling relations between nearby models, useful for obtaining order of magnitude estimates of what happens when the mass of the star is changed.

Let us consider stars of nearby masses M and M'. We furthermore assume that the homology relations are also valid in the interior, i.e.

$$M' = C_M M \, , \qquad R' = C_R R$$

with $C_M, C_R \approx 1$

Then in interior:

$$M'_r = C_M M \, , \qquad r' = C_R r$$

$$P' = C_P P \, , \qquad T' = C_T T \, , \qquad L'_r = C_L L_r$$

In general, homology relations are applied to homogeneous models.

We assume simple gas models with

$$\rho = \frac{\mu m_u}{kT} P \quad , \quad \kappa = \kappa_0 \rho T^{-3.5} \quad , \quad \varepsilon = \varepsilon_0 \rho T^{\nu}$$

and the relations

$$\mu' = C_\mu \mu, \quad \kappa_0' = C_\kappa \kappa_0, \quad \varepsilon_0' = C_\varepsilon \varepsilon_0$$

The two models M and M' must satisfy the basic equilibrium relations.

Using hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \quad \text{and} \quad \frac{dP'}{dr'} = -\frac{GM'}{r'^2}\rho' =$$

$$= \frac{C_P}{C_R}\frac{dP}{dr} = -\frac{G C_M M}{C_R^2 r^2}\frac{C_M}{C_R^3}\rho$$

$$\Rightarrow \frac{dP}{dr} = -\frac{GM}{r^2}\rho = -\frac{C_M^2 C_R}{C_R^5 C_P}\frac{GM}{r^2}\rho$$

$$\Rightarrow \boxed{C_P = \frac{C_M^2}{C_R^4}}$$

From the eqation of state:

$$\rho = \frac{\mu m_u}{kT} P \quad \Rightarrow C_\rho = \frac{C_M}{C_R{}^3} = \frac{C_\mu}{C_T} C_P = \frac{C_\mu}{C_T} \frac{C_M{}^2}{C_R{}^4}$$

$$\Rightarrow \boxed{C_T = \frac{C_M}{C_R} C_\mu}$$

From energetic equilibrium:

$$\frac{\partial L}{\partial M_r} = \varepsilon = \varepsilon_0 \rho T^\nu \Rightarrow \frac{C_L}{C_M} = C_\varepsilon \frac{C_M}{C_R{}^3} C_T{}^\nu$$

$$\Rightarrow C_L = C_\varepsilon \frac{C_M{}^2}{C_R{}^3} C_T{}^\nu = C_\varepsilon \frac{C_M{}^2}{C_R{}^3} \frac{C_M{}^\nu}{C_R{}^\nu} C_\mu{}^\nu$$

$$\Rightarrow \boxed{C_L = C_\varepsilon C_M{}^{2+\nu} C_R{}^{-3-\nu} C_\mu{}^\nu}$$

For a radiative star, we get from radiative transfer

$$F_{rad} = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr} = \frac{L_r}{4\pi r^2} = -\frac{4ac\,T^3}{3\kappa_0 \rho^2 T^{-3.5}}$$

$$\Rightarrow \frac{C_L}{C_R{}^2} = \frac{C_T{}^3}{C_\kappa \frac{C_M{}^2}{C_R{}^6} C_T{}^{-3.5}} \frac{C_T}{C_R}$$

$$= \frac{C_T{}^{7.5} C_R{}^5}{C_\kappa C_M{}^2}$$

$$\Rightarrow C_L = C_\kappa{}^{-1} C_M{}^{-2} C_R{}^7 C_T{}^{7.5}$$

$$= C_E^{-1} C_R^{7} C_M^{-2} \frac{C_\mu^{7.5} C_M^{7.5}}{C_R^{7.5}}$$

$$\Rightarrow \boxed{C_L = C_E^{-1} C_R^{-1/2} C_\mu^{7.5} C_M^{5.5}}$$

Using these two expressions for C_L:

$$C_L = C_\varepsilon C_M^{v+2} C_\mu^{v} C_R^{-3-v}$$

$$\Rightarrow C_R = \left(C_L C_\varepsilon^{-1} C_\mu^{-v} C_M^{-v-2} \right)^{\frac{-1}{3+v}}$$

$$= C_L^{\frac{1}{-3-v}} C_\varepsilon^{\frac{1}{3+v}} C_\mu^{\frac{v}{v+3}} C_M^{\frac{v+2}{v+3}}$$

$$\Rightarrow C_L = C_R^{-1/2} C_E^{-1} C_\mu^{7.5} C_M^{5.5} =$$

$$= C_L^{+\frac{1}{6+2v}} C_\varepsilon^{\frac{-1}{6+2v}} C_\mu^{-\frac{v}{2v+6}} C_M^{-\frac{v+2}{2v+6}} C_E^{-1} C_\mu^{7.5} C_M^{5.5}$$

$$= C_L^{\frac{1}{6+2v}} C_\varepsilon^{-\frac{1}{6+2v}} C_\mu^{\frac{15v+45-v}{2v+6}} C_E^{-1} C_M^{5.5-\frac{v+2}{2v+6}}$$

$$= C_L^{\frac{1}{6+2v}} C_\varepsilon^{\frac{-1}{6+2v}} C_\mu^{\frac{14v+45}{2v+6}} C_E^{-1} C_M^{\frac{10v-31}{2v+6}}$$

$$\Rightarrow C_L^{1-\frac{1}{6+2v}} = C_L^{\frac{6+2v-1}{6+2v}} = C_L^{\frac{5+2v}{6+2v}} = C_\varepsilon^{()} C_\mu^{()} C_E^{-1} C_M$$

$$\Rightarrow C_L = (\ )^{\frac{6+2v}{5+2v}} = C_\varepsilon^{-\frac{1}{5+2v}} C_\mu^{\frac{14v+45}{2v+5}} C_E^{-\frac{6+2v}{5+2v}} C_M^{\frac{10v+3}{2v+5}}$$

So we finally got

$$\boxed{C_L = C_\varepsilon^{\frac{-1}{5+2\nu}} \; C_\mu^{\frac{14\nu+45}{2\nu+5}} \; C_\kappa^{\frac{-6+2\nu}{5+2\nu}} \; C_M^{\frac{10\nu+31}{2\nu+5}}}$$

ν depends on the type of nuclear reaction dominating the nuclear burning.

For pp-chains: $\nu = 6$: $L \sim \varepsilon_0^{-0.08} \kappa_0^{-1.07} \mu^{7.8} M^{5.5}$

For CNO-chains: $\nu = 17$: $L \sim \varepsilon^{-0.02} \kappa_0^{-1.03} \mu^{7.3} M^{5.2}$

Conclusions:

1) L increases fast with mass.
(exponent 5 for $\sim 1 M_\odot$ stars ~ 3 for massive stars)

2) L varies strongly with μ
He stars are much more luminous than H stars.

3) $L \sim 1/\kappa$

4) L varies very weakly with ε_0. The increase of energy production will change the star's structure, but not increase its luminosity.

We can do the same exercise for C_R and C_T for the two cases for ν and get

pp-chains: $\nu = 6$

$$R \sim (\kappa_0 \varepsilon_0)^{0.15} \mu^{-0.54} M^{0.08}$$

$$T_{\rm eff} \sim \kappa_0^{-0.35} \varepsilon_0^{-0.10} \mu^{2.2} M^{1.33}$$

HCNO-cycles: $\nu = 17$

$$R \sim (\kappa_0 \varepsilon_0)^{0.51} \mu^{0.49} M^{0.69}$$

$$T_{\rm eff} \sim \kappa_0^{-0.28} \varepsilon_0^{-0.03} \mu^{1.6} M^{0.3}$$

- The radius has little sensitivity to the mass of solar-type stars. (Remember, we used no convection here.)
- The radius has weak dependence on μ.
- For a given mass, stars with higher μ are hotter and much brighter.

How do these relations change if we instead use $\kappa = \kappa_{\rm electron\ scattering} = \kappa_0$?

For $\kappa = \kappa_0 = \text{const}$, we repeat the procedure:

$$\rho = \frac{\mu m_u}{kT} P \quad , \qquad \kappa = \kappa_0 , \qquad \varepsilon = \varepsilon_0 \rho T^{\nu}$$

$$\frac{dP}{dr} = - \frac{GM}{r^2} \rho \quad \Rightarrow \quad \frac{C_P}{C_R} = \frac{C_M}{C_R^2} \frac{C_M}{C_R^3}$$

$$\Rightarrow \boxed{C_P = \frac{C_M^2}{C_R^4}}$$

$$\rho = \frac{\mu m_u}{kT} P \quad \Rightarrow \quad \frac{C_M}{C_R^3} = \frac{C_\mu}{C_T} C_P = \frac{C_\mu}{C_T} \frac{C_M^2}{C_R^4}$$

$$\Rightarrow \boxed{C_T = \frac{C_M}{C_R} C_\mu}$$

$$\frac{\partial L}{\partial M_r} = \varepsilon = \varepsilon_0 \rho T^{\nu} \Rightarrow \frac{C_L}{C_M} = C_\varepsilon \frac{C_M}{C_R^3} C_T^{\nu}$$

$$\Rightarrow C_L = C_\varepsilon C_M^2 C_R^{-3} C_T^{\nu} =$$

$$= C_\varepsilon C_M^2 C_R^{-3} \left(\frac{C_M}{C_R} C_\mu \right)^{\nu}$$

$$\boxed{= C_\varepsilon C_M^{2+\nu} C_R^{-3-\nu} C_\mu^{\nu}}$$

(so far, everything is the same)

$$F_{rad} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} = \frac{L_r}{4\pi r^2} = -\frac{4acT}{3\kappa_0\,\rho}\,\frac{dT}{dr}$$

$$\Rightarrow \frac{C_L}{C_R^2} = \frac{C_T^3}{C_\kappa\, C_M/C_R^3}\,\frac{C_T}{C_R} = \frac{C_T^4\, C_R^2}{C_\kappa\, C_M}$$

$$\Rightarrow C_L = \boxed{C_\kappa^{-1}\, C_M^{-1}\, C_T^4\, C_R^4}$$

$$= C_\kappa^{-1}\, C_M^{-1}\left(\frac{C_M}{C_R}\, C_\mu\right)^4 C_R^4$$

$$= \boxed{C_\kappa^{-1}\, C_M^3\, C_\mu^4}$$

$$\Rightarrow L \propto \frac{T^4 R^4}{\kappa_0 M}$$

$$L \propto \frac{M^3 \mu^4}{\kappa_0}$$

How does this situation change with radiation pressure in the picture?

We again use

$$P_{tot} = P = P_{gas} + P_{rad}$$

with $P_{gas} = \beta P$, $P_{rad} = (1-\beta)P$

Then the equation of state reads

$$\rho = \frac{\mu m_u}{kT} P_{gas} = \frac{\mu m_u}{kT} \beta P$$

So we only need to replace $P \to \beta P$, i.e.

$$C_T = C_\mu C_R^3 C_M^{-1} C_P \to C_\mu C_R^3 C_M^{-1} C_\beta C_P$$

$$= C_\mu \frac{C_R^3}{C_M} C_\beta \frac{C_M^2}{C_R^4} = C_\mu C_M C_R^{-1} C_\beta$$

giving lastly

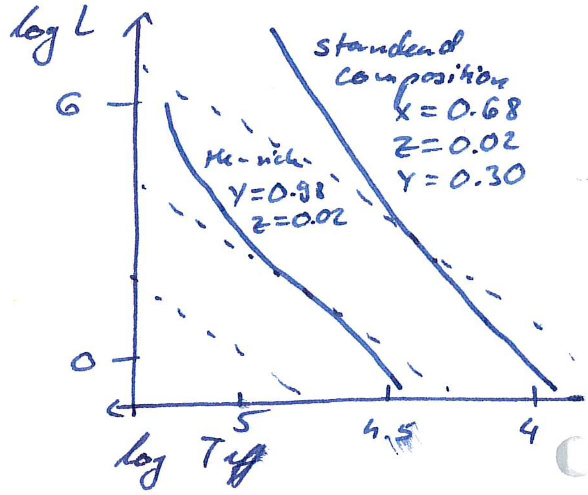
$$L \sim \frac{\mu^4 \beta^4 M^3}{\kappa_0}$$

We have seen that as β increases, the mass decreases. For very high radiation pressure, we have $\beta \propto M^{-1/2}$. So for this extreme case, we get

$$L \sim \frac{\mu^4 M}{\kappa_0}$$

The Helium and Generalized Main Sequences

The zero-age main sequence is the location in the HR diagram of chemically homogeneous stars at the beginning of MS evolution. As stars evolve, they become inhomogeneous, forming a He core. The core size represents various mass fractions depending on masses, ages, mass loss and mixing.



For helium-rich stars of a given mass are much brighter and hotter, while their radii are smaller.

The He-burning reactions are much more sensitive with $\nu \approx 30$ ($\varepsilon = \varepsilon_0 \rho T^{\nu}$) compared to typical MS stars.

The mass fractions of the convective cores in He stars are larger, $0.27 M_\odot$ in a $1 M_\odot$ star, while for a sun it is zero. The central and average densities are much bigger.

$\frac{P_{rad}}{P} = (1-\beta)$ is also larger in He-rich stars;

1.5% in He-rich stars compared to 5×10^{-4} in normal cores of stars with $M = 1 M_\odot$

A consequence is the larger Eddington ratio

$$\Gamma = \frac{L}{L_{edd}} = \frac{L}{\frac{4\pi c G M}{\kappa}}$$

and this sets the maximum stellar mass of He-stars at $16 M_\odot$, compared to $\sim 150 M_\odot$ for normal stars.

The generalized main sequences are a useful concept and are formed by models consisting of a He core of mass fraction q and an envelope of mass fraction $1-q$ with a standard composition.

There is a discontinuity of composition at the edge of the core. The models are supposed in equilibrium, He is burning in the core and H is burning in a shell at the base of the envelope. The models are described by the two parameters M and q.

For smaller values of q, the H-rich envelope produces larger opacities and thus stellar inflation. The star is shifted to the right of the He-sequence which we obtain for $q = 1$.

Polytropic Models

Useful simplified models. A polytropic model has an EoS of the type

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$$

$$\gamma = 1 + \frac{1}{n} \quad ; \quad n = \frac{1}{\gamma - 1} = \text{polytropic index.}$$

K is taken to be a constant.

Together with

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

we have the unknowns P, M_r and ρ: 3 unknowns for 3 equations.

Let us introduce a dimensionless variable θ so that

$$\rho \equiv \rho_c \, \theta^n$$

$$\Rightarrow \quad P = K \rho^{1+\frac{1}{n}} = \underbrace{K \rho_c^{1+\frac{1}{n}}}_{P_c} \theta^{n(1+\frac{1}{n})} \equiv P_c \, \theta^{n+1}$$

From hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho \frac{G M_r}{r^2}$$

$$\Rightarrow \quad M_r = -\frac{1}{\rho G} r^2 \frac{dP}{dr}$$

$$\Rightarrow \quad \frac{dM_r}{dr} = -\frac{d}{dr}\left(\frac{1}{\rho G} r^2 \frac{dP}{dr}\right) = 4\pi \cdot \rho r^2$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{1}{\rho G} r^2 \frac{dP}{dr}\right) + 4\pi \cdot \rho r^2 = 0$$

now insert $\rho = \rho_c \theta^n$, $P = P_c \theta^{n+1}$

$$\Rightarrow \frac{1}{G}\frac{d}{dr}\left(\frac{1}{\rho_c \theta^n} r^2 \frac{d}{dr}\left(P_c \theta^{n+1}\right)\right) + 4\pi \cdot \rho_c \theta^n r^2 = 0$$

$$= \frac{1}{G}\frac{d}{dr}\left(\frac{1}{\rho_c \theta^n} r^2 \frac{d}{dr}\left(K \rho_c^{1+\frac{1}{n}} \theta^{n+1}\right)\right) + 4\pi \cdot \rho_c \theta^n r^2$$

$$\Rightarrow \frac{1}{G}\frac{d}{dr}\left(\frac{1}{\rho_c\Theta^n} r^2 P_c (n+1)\Theta^n \frac{d\Theta}{dr}\right) + 4\pi G\rho_c\Theta^n r^2 = 0 \quad | \cdot G$$

$$\Rightarrow \quad \frac{P_c(n+1)}{\rho_c}\frac{d}{dr}\left(\frac{1}{\Theta^n} r^2 \Theta^n \frac{d\Theta}{dr}\right) + 4\pi G\rho_c\Theta^n r^2 = 0$$

$$= \quad \frac{K\rho_c^{1+\frac{1}{n}}}{\rho_c}(n+1)\frac{d}{dr}\left(r^2\frac{d\Theta}{dr}\right) + 4\pi G\rho_c\Theta^n r^2 = 0$$

$$= \quad K\rho_c^{1/n}(n+1)\frac{d}{dr}\left(r^2\frac{d\Theta}{dr}\right) + 4\pi G\rho_c\Theta^n r^2 = 0 \quad | : r^2$$

$$K\rho_c^{1/n}(n+1)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Theta}{dr}\right) + 4\pi G\rho_c\Theta^n = 0$$

$$\Rightarrow \frac{K\rho_c^{1/n}(n+1)}{4\pi G\rho_c}\quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Theta}{dr}\right) + \Theta^n = 0$$

Let $\quad \alpha^2 = \frac{4\pi G\rho_c}{K\rho_c^{1/n}(n+1)} = \frac{4\pi G}{K\rho_c^{1/n-1}(n+1)}$

and let

$$\xi = \alpha r$$

then

$$d\xi = \alpha\, dr \quad \Rightarrow \quad dr = \frac{1}{\alpha}d\xi$$

$$r = 1/\alpha\, \xi$$

For our derivatives in the equation, we get

$$\frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = \frac{d}{\frac{1}{\alpha} d\xi}\left(\frac{1}{\alpha^2}\, \xi^2 \frac{d}{\frac{1}{\alpha} d\xi}\, \theta\right)$$

$$= \frac{d}{d\xi}\left(\xi^2 \frac{d}{d\xi}(\theta)\right)$$

And our equation simplifies to

$$\frac{1}{\alpha^2}\, \frac{1}{r^2}\, \frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) + \theta^n = 0$$

$$= \boxed{\frac{1}{\xi^2}\, \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0}$$

This is called the Lane-Emden equation.

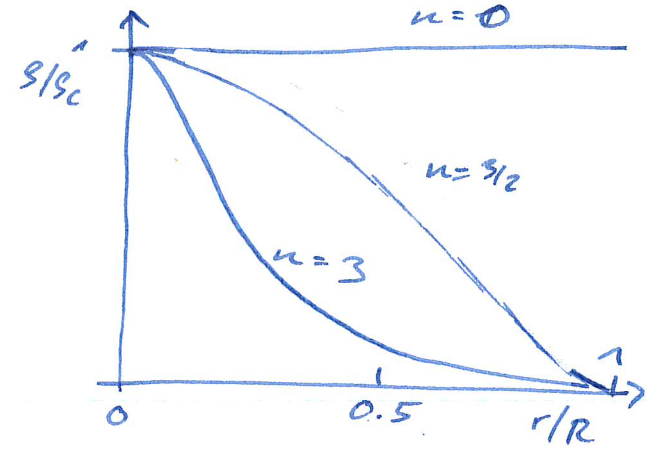
We have the boundary conditions

center: $\xi = 0$, $\theta = 1$, $\frac{d\theta}{d\xi} = 0$ $[r=0, P=P_c, \rho=\rho_c]$

surface: $\xi = \xi_1$, $\theta(\xi_1) = 0$ [some radius pressure and density $= 0$]

There are analytical solutions for $n = 0$, 1 and 5.
At $n = 5$, the radius becomes infinite.

When n increases from 0 to 5, we obtain a sequence of increasing central concentration.



We can consider some special cases we've seen:

- When the EoS is $P = P_{gas} + P_{rad}$, i.e. radiation pressure is not negligible, then $\gamma \approx 4/3 = 1 + \frac{1}{n} \Rightarrow n = 3$

- In an adiabatic convective zone, $\gamma = 5/3 \Rightarrow n = \frac{3}{2}$
  So the adiabatic case will have a less concentrated core, thus have smaller density contrast, compared to radiative zones.
  Fully degenerate gases (i.e. in white dwarfs) also have $\gamma = 5/3$.

For $\gamma = 1$, we can obtain an isothermal sphere. A finite self-gravitating sphere of perfect gas that is isothermal is impossible, the mass diverges. However, it can be shown that in the outer layers of a gravitational system is approximately isothermal.

The Pre-Stellar Phase

Overview and Signatures of Star Formation

The star formation process represents a jump by a factor of about 10^{23} for the density. The typical density of the interstellar medium goes from about $10^{-23}\,g/cm^3$ to an average solar density of about $1.4\,g/cm^3$.

Major changes in the matter properties occur during such a density change. It is meaningful to distinguish 3 phases of star formation:

- The pre-stellar phase

  contraction and fragmentation phases of an interstellar cloud under its gravitation. Essentially isothermal.

- Protostellar phase

  evolution of the fragment up to the stage where growth of internal pressure in the central core stops the fast contraction and fragmentation. The central core reaches hydrostatic equilibrium and evolves nearly adiabatically.

- Pre-Main sequence phase
  Phase of the evolution of the central object from the Hayashi line up to the zero-age main sequence.

The physical conditions, timescales, and observational properties are different in these three phases.

The range of sizes and masses of the star-forming clouds is quite large:

- small: $\sim 10\,pc$, $\sim 10^2\,M_\odot$
- large: $\sim 100\,pc$, $\sim 10^4\,M_\odot$ [Orion]
- giant: $\sim 10\,kpc$, $\sim 10^6\,M_\odot$ [100 × Orion]
- starbursts: $\sim 10^4 - 10^5 \times$ Orion; originate from interacting galaxies

How do we observe SF regions?

- radio emissions: large molecular clouds H_2 is traced by the 2.6 mm CO rotational line.

- IR: dust grains in contracting clouds, as well as grains heated by newly formed O-stars

- visible light: dark clouds

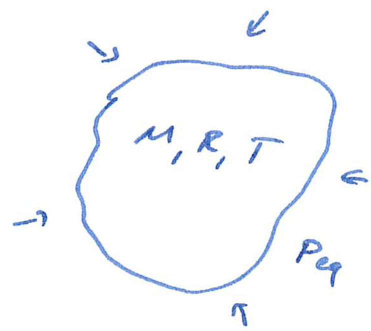
- emission lines as signatures of recent star formation in nebulas and Wolf-Rayet stars in other galaxies

- Broad Resonance UV lines: signature of massive star formation

The Jeans Criterion

An interstellar cloud starts contracting when gravity forces overcome the forces due to the gradient of internal pressure. The cloud becomes gravitationally unstable.

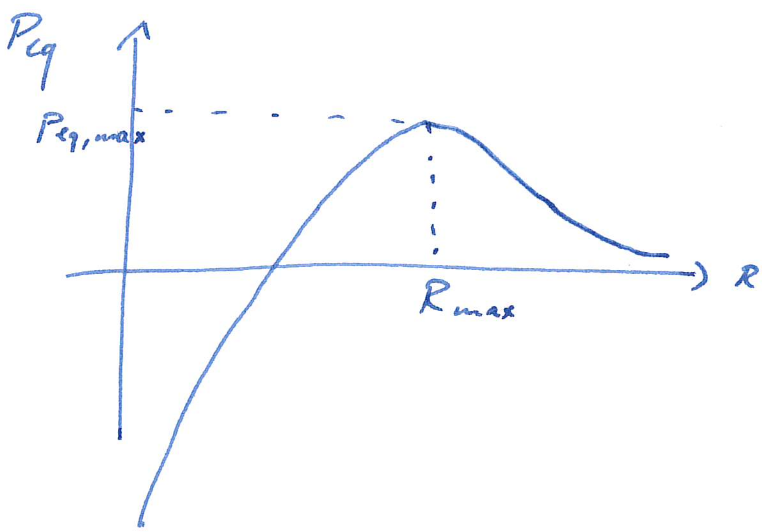
Let us consider an isothermal sphere of mass M, radius R and temperature T. Let P_{eq} be the ambient pressure with which the cloud is in equilibrium. The virial theorem states that



$$2E_{kin} + \Omega = 3PV = 4\pi R^3 P_{eq} = 2C_V M T - q\frac{GM^2}{R}$$

$$\Rightarrow P_{eq} = \frac{C_V M T}{2\pi R^3} - q\frac{GM^2}{4\pi R^4}$$

for a fixed M, T, q the P_{eq} behaves like this:



P_{eq} reaches a maximum at $R_{max} = R_J$ where

$$\frac{dP_{eq}}{dR} = 0 = -\frac{3 C_V M T}{2\pi R_{max}^4} + \frac{9 G M^2}{\pi R_{max}^5}$$

$$\Rightarrow R_J = \frac{9 G M}{3/2\, C_V T} \qquad \Big| \; C_V = \frac{3}{2}\frac{k}{\mu m_u}$$

$$\boxed{R_J = \frac{4}{9}\,\frac{\mu m_u}{k}\,\frac{GM}{T}}$$

R_J is the radius of the maximum pressure that the cloud in equilibrium can sustain.

The stability depends on the value of the actual radius compared to R_J:

If $R < R_J$:

a reduction of R produces a reduction of P_{eq}: Thus, if the cloud was initially in equilibrium with its surrounding medium, after a small decrease of R the sustainable pressure becomes smaller than the actual pressure and the cloud starts contracting. This leads to even smaller R, thus instability.

If $R > R_J$:

An increase of R makes a reduction of P_{eq}. If the cloud was initially in equilibrium with the ambient medium, the external pressure is now too strong and the cloud contracts, recovering its original size.

We can also write, for $\rho \approx$ const:

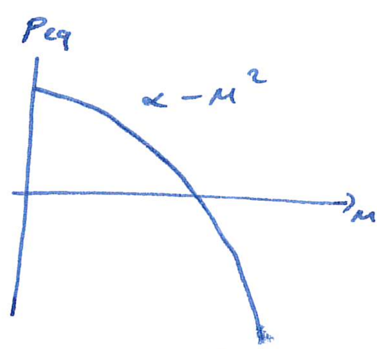
$$M_{jeans} = \frac{4}{3}\pi R_J^3 \rho$$

$$= \frac{4}{3}\pi \left(\frac{4}{9} G \frac{\mu m_u}{k} \frac{M_{jeans}}{T}\right)^3 \rho$$

$$\Rightarrow M_{jeans}^2 = \frac{3}{4\pi} \frac{9^3}{4^3} \left(\frac{k}{\mu m_u}\right)^3 G^{-3} \frac{T^3}{\rho}$$

$$\Rightarrow \boxed{M_{jeans} = \frac{27}{16} \left(\frac{3}{\pi G^3}\right)^{1/2} \left(\frac{k}{\mu m_u}\right)^{3/2} \frac{T^{3/2}}{\rho^{1/2}}}$$

- If the actual mass is higher than the Jeans mass for given T, ρ and μ, the cloud will collapse. You can do the analogue thought process to the Jeans radius:



If you perturb the mass a bit towards higher values, the maximal pressure that the cloud can withstand decreases. Assuming that the external pressure, the one that the cloud initially was in equilibrium with, remains the same, then the external pressure is too high for the cloud to handle and the cloud will collapse.

- $M_{jeans} \propto T^{3/2}$: Higher temperatures allow for higher masses to initiating the collapse, since gravity has to overcome a larger internal gas pressure.

- $M_{jeans} \propto \rho^{-1/2}$: Higher densities allow for smaller clouds to collapse.

- $M_{jeans} \propto q^{-3}$: q increases for more peaky (centrally condensed) density profiles. For higher q, the less massive a cloud needs to be to initiate collapse.

For neutral monoatomic gas in a homogeneous density ($q = 3/5$) and solar composition ($\mu = 0.77$) we get

$$M_{Jeans} = 1.16 \times 10^5 M_\odot \left(\frac{T}{100K}\right)^{3/2} \left(\frac{\rho}{10^{-24} g/cm^3}\right)^{-1/2}$$

Thus the masses of collapsing clouds are rather large. For dense clouds with $T = 10K$ and $\rho = 10^{-22} g/cm^3$, $M_{Jeans} \approx 367 M_\odot$.

Since only clouds with masses much higher than the current stellar mass can start contraction, but typical stars masses are $\sim 0.8 M_\odot$, we clearly need another process than simple collapse to get stars of appropriate masses.

This process is fragmentation. It will enable the formation of many stars from a given cloud.

Over a large interval of densities from $\rho \approx 10^{-23}$ to $\approx 10^{-13}$ g/cm^3 the collapsing cloud remains isothermal. An isothermal cloud with an actual mass $M < M_{Jeans}$ will not start gravitational contraction, unless some external effect compresses the gas. If this happens, the increase of ρ produces a decrease of $M_{Jeans} \propto \rho^{-1/2}$, and thus the actual cloud mass M may happen to be larger than the corresponding M_{Jeans}. When this occurs, collapse is initiated.

Several mechanisms are able to produce the necessary density increase:

- contagious star formation
  shocks due to ionization fronts around newborn massive stars, or SNe
- density wave in spiral galaxies
- galaxy interactions
- cloud collisions

The dynamical timescale, or free-fall time, characterizes the changes of mechanical equilibrium in a gravitational configuration.

We have

$$\ddot{r} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM_r}{r^2}$$

In an isothermal situation, $\frac{dT}{dr} = 0$, and $P \propto T \Rightarrow \frac{dP}{dr} = 0$

so we have

$$\ddot{r} = -\frac{GM_r}{r^2}$$

$$\Rightarrow \ddot{r}\dot{r} = -\frac{GM_r}{r^2}\dot{r}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) = \frac{d}{dt}\left(\frac{GM_r}{r}\right)$$

$$\Rightarrow \frac{1}{2}\dot{r}^2(t) - \frac{1}{2}\dot{r}^2(t_{ini}) = GM_r\left(\frac{1}{r} - \frac{1}{r_{ini}}\right)$$

Assume $t_{ini} = 0$, $r_{ini} = \infty$, $\dot{r}(t_{ini}) = 0$,

and $M = \frac{4}{3}\pi r^3 \bar{\rho}$

$$\Rightarrow \frac{1}{2}\dot{r}^2(t) = \frac{G}{r}\frac{4}{3}\pi r^3 \bar{\rho}$$

$$\Rightarrow \dot{r}^2(t) = \frac{8\pi G\bar{\rho}}{3} r^2$$

$$\Rightarrow \quad \left(\frac{\dot r(t)}{r}\right)^2 = \frac{8\pi G \bar\rho}{3}$$

$$\Rightarrow \quad \frac{\dot r(t)}{r} = -\sqrt{\frac{8\pi G \bar\rho}{3}}$$

where we use the negative solution because our system is contracting.

Since the mass remains constant during the collapse, we can write

$$M_r = 4\pi \rho r^3 = \text{const}$$

$$\Rightarrow \quad \frac{dM_r}{M_r} = 0 = \frac{4\pi r^3 d\rho}{4\pi r^3 \rho} + \frac{4\pi\rho \cdot 3r^2 dr}{4\pi r^3 \rho}$$

$$\Rightarrow \quad \frac{d\rho}{\rho} = -3\frac{dr}{r}$$

We use this to express

$$\frac{\dot r}{r} = \frac{1}{r}\frac{dr}{dt} = -\frac{1}{3}\frac{1}{\rho}\frac{d\rho}{dt} = -\sqrt{\frac{8\pi G\rho}{3}}$$

$$\Rightarrow \quad \frac{1}{\rho}\frac{d\rho}{dt} = \sqrt{24\pi G\rho}$$

We define

$$\boxed{\tau_{ff} = \rho\frac{dt}{d\rho} = \frac{1}{\sqrt{24\pi G\rho}}}$$

Here we assumed that we have no counterforce for gravity, and that the contraction is isothermal, i.e. that the cooling is so efficient that the cloud doesn't heat up.

The Role of Magnetic Fields and Turbulence

Observations indicate that the interstellar clouds are often in a stage of equilibrium. If all clouds with masses higher than the Jeans mass would collapse, the resulting star formation rate would be much larger than the current estimate of $\sim 1\ \mathrm{kpc}^{-2}\ \mathrm{yr}^{-1}$ in the galactic plane. This suggests that the gravity is balanced by various effects, such as turbulence and magnetic fields.

The energy density of a magnetic field is

$$u_B = \frac{B^2}{8\pi}$$

and of gravity is (for homogeneous $q = 3/5$)

$$u_G = \frac{3}{5}\frac{GM^2}{R\left(\frac{4}{3}\pi R^3\right)} = \frac{9}{20\pi}\frac{GM^2}{R^4}$$

If the two energy densities are of the same order, the mass M_B above which gravitation dominates over the magnetic field is

$$u_B = u_G$$

$$\frac{B^2}{8\pi} = \frac{9}{20\pi} \frac{G M^2}{R^4}$$

$$\Rightarrow M^2 = \frac{B^2}{8\pi} \frac{20\pi}{9} G R^4 = \frac{B^2 5}{18} G R^4$$

$$\Rightarrow \boxed{M_B = \sqrt{\frac{5}{18} G}\, B R^2}$$

$$\approx 900 \cdot \left(\frac{B}{30\,\mu G}\right)\left(\frac{R}{2\,pc}\right)^2 M_\odot$$

If $M > M_B$, the magnetic field is insufficient to prevent cloud collapse. The timescale of the collapse is not much increased w.r.t. t_{ff}.

If $M < M_B$, the collapse is delayed. The magnetic field is attached to ions, and over time, they drift away from the neutral gas, i.e. they diffuse apart, and the neutral cloud can collapse. Delay $\sim 10^7$ yrs.

An additional pressure in a cloud results from internal turbulent motions. They contribute to the cloud support and may inhibit collapse.

However, there are issues with simple turbulence models:

- turbulence is something that needs to be fuelled
- turbulence is supersonic, introducing shocks, which should generate overpressure and initiate collapse

Cloud Fragmentation

In a collapsing cloud with average density ρ, regions with locally higher density ρ' collapse faster than the rest of the cloud. Remember that the collapse is characterized by the free-fall time $\tau_{ff} \sim \frac{1}{\sqrt{\rho}}$, so it follows that

$$\tau_{ff} \sim \frac{1}{\sqrt{\rho}} > \tau_{ff}' \sim \frac{1}{\sqrt{\rho'}} \quad \text{for } \rho' > \rho$$

Since $M_{Jeans} \propto T^{3/2}$, we need sufficient cooling present such that the collapse is isothermal. Otherwise, with the increasing temperature the Jeans mass increases, and the collapse can't continue.

At some stage, the medium becomes opaque and contraction ceases to be isothermal. At $\rho \sim 10^{-14} - 10^{-13}$ g/cm³, the contraction starts to become adiabatic.

In the adiabatic phase, the Jeans mass no longer decreases for higher densities, no smaller cloud fragment collapses and fragmentation comes to an end. The fact that the opacity sets a limit on the smallest fragment is known as the <u>opacity-limited fragmentation.</u>

We can estimate the lower mass limit given by the opacity-limited fragmentation:

The produced gravitational power is

$$\dot{E}_{grav} \simeq \frac{E_{grav}}{\tau_{ff}} = \frac{GM^2}{R}(G\rho)^{1/2}$$

and the maximal radiated away energy is that of a black body:

$$\dot{E}_{rad} = f\, 4\pi f\, \sigma T^4 R^2$$

where f is some "efficiency" factor ≤ 1.

$\dot{E}_{rad}$ grows $\propto R^2$, so large objects can always radiate away sufficiently much energy.

The transition towards adiabatic collapse occurs when $\dot{E}_{rad} \approx \dot{E}_{grav}$.

We get

$$M_{lim} \approx 0.018 \frac{T^{1/4}}{f^{1/2}} M_\odot$$

or for $T = 10K$, $f = 1$:

$$M_{lim} = 0.03 M_\odot \approx 30 M_{Jupiter}$$

$\Rightarrow$ planets can't form through gravitational collapse of gas clouds in the ISM.
$\rightarrow$ Accumulation instead of contraction.

$\Rightarrow$ the minimal mass for Hydrogen burning is $\approx 0.08 M_\odot$. These type of stars are called Brown Dwarfs.

The Initial Stellar Mass Spectrum

Also called the initial mass function.
It gives the (average) number of stars in a given mass interval. It can be approximated as

$$\frac{dN}{dM} = A M^{-(1+x)}$$

or $$\phi(M) \equiv \frac{dN}{d\ln M} = \frac{dN}{dM/M} = M\frac{dN}{dM} = AM^{-x}$$

$\Rightarrow$ there will be more low-mass stars than high mass stars.

Evolution in the H-burning phase

The hydrogen burning phase or main sequence (MS) phase is the longest phase of stellar evolution, in which stars spend 85-90% of their nuclear life. There are several reasons why this phase is the longest one:

1) H → He fusion produces more nuclear energy per nucleon

2) The stars are less luminous than in later phases

3) Convective cores, when present, are larger than in later phases, thus the nuclear fuel is available for longer periods.

Hydrogen Burning

H-burning converts four protons into an α-particle, with liberation of energy due to a relative mass defect, i.e. a minor fraction of mass is transformed into energy during the burning process.

There are two major chains of reactions: proton-proton (pp) chains dominating stars with $M \leq 1.2 M_\odot$, and the CNO cycle for the masses above.

In a nuclear reaction of the form

$$a + X \rightarrow b + Y$$

the energy conservation is given by

$$E_{aX} + (M_a + M_X)c^2 = E_{bY} + (M_b + M_Y)c^2$$

where E_{aX}, E_{bY} are the kinetic energies of the (aX), (b, Y) systems.

The binding energy of an atom with atomic number A and Z protons is

$$\Delta M_{X(A,Z)} = M_{X(A,Z)} - Z m_p - (A-Z) m_n \approx 930 \, \text{MeV} \, (M_{AZ} - A)$$

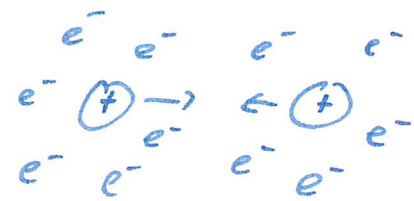
Then the energy released by a nuclear reaction is

$$Q = \underbrace{(\Delta M_a + \Delta M_X)}_{\text{first need to break up}} - \underbrace{(\Delta M_b + \Delta M_Y)}_{\text{released by new elements}}$$

You can derive an expression for the energy released per unit time per unit mass by a reaction $a + X \rightarrow b + Y$ via collisional cross-sections between the particles a and X.

You may assume that the particles follow a Maxwellian velocity distribution. However, you also need to include a screening factor for charged particles: The cross-section will depend on the charges of the particles in the reaction, which we assume are going to be fully ionized in the core of a star.

In the star, the electrons will gather around the charged ions, reducing the effective charge of the ions. This is called a "screening effect".



The effective electrostatic potential between two protons is $\sim 700\ keV$, however the estimated average thermal energy is $kT \sim k \cdot 10^7 K \sim 1 keV$.

So how do we get high enough energies for two protons to undergo nuclear fusion? They need to be able to collide in order to fuse.

There are two possibilities:

1) The protons follow a Maxwellian velocity distribution. Thus there are particles with much higher velocities present than the average velocity. The energy distribution is given by

$$\phi(E) \propto e^{-E/kT}$$

2) Gamov / Tunneling effect

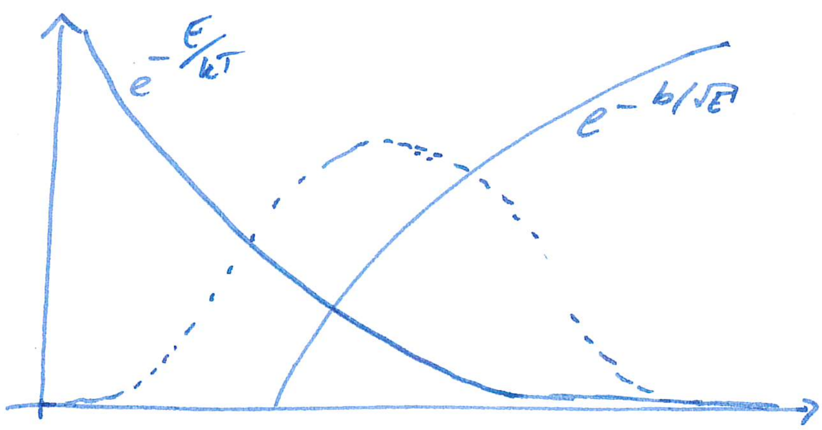
The particles may come into contact via the tunneling effect, where energy conservation can be violated over a short time.

The collisional cross-section is given by

$$\sigma(E) = S(E) \frac{1}{E} e^{-b/\sqrt{E}}$$

The slower the particles ($\propto 1/E$), the greater the chance of interaction. But the particles also need enough energy to be able to come close.

The average energy production rate will depend on $\langle \sigma v \rangle = \int \sigma v f(E)\, dE \propto \int_0^\infty S(E)\, e^{-E/kT - \frac{b}{\sqrt{E}}}\, dE$



this puts a lower and upper limit on energies which allow for tunneling successfully into a nuclear reaction. This range of energies is called the "Gamow Window".

With higher densities, the electron screening is more effective and lower energies are required for the reactions to take place.

The energy generation of H-burning can be schematically estimated by

$$\varepsilon = \varepsilon_0 \, \rho \, X^2 T^\nu$$

with ε = energy production rate, i.e. the power produced by unit mass:

$$\varepsilon = \frac{[\text{Energy released by reaction}] \cdot [\text{rate of reactions}]}{[\text{unit mass}]}$$

The reaction rate depends on the reactands density, so for a pp-chain, we get $\varepsilon \propto X^2$. That's where the X^2 comes in.

We have

$\varepsilon \propto T^{4-5}$ for pp-chains

$\varepsilon \propto T^{17}$ for CNO-cycles

There are 3 pp-chain reactions possible.

The pp I chain is

$$p + p \rightarrow D + e^+ + \nu_e$$

$$D + p \rightarrow {}^3He + \gamma$$

$${}^3He + {}^3He \rightarrow {}^4He + 2p$$

Since we need $2 \times {}^3He$ to get a ${}^4He$, we also need two pp-chain reactions to obtain one ${}^4He$ atom. The total released energy is $\approx 26\,MeV$, thus the energy released per nucleon is $\frac{24\,MeV}{4\,proton} \approx 6\,MeV$

In case there is an isotope present in the reaction that is destroyed rapidly, then very soon an equilibrium value for the abundance of that element will be reached. This is the case for Deuterium.

When $T > 17 \cdot 10^6\,K$ for a standard composition, the H burning occurs mainly through the CNO cycles. C, N, and O act as catalysts for the He synthesis and must be initially present. The individual abundances X_C, X_N, X_O will change over time, but their sum won't.

There is a basic cycle, the CN cycle, to which two loops of ON reactions are added. The ratios reach equilibrium values at some point, which are different from the cosmic values that are initially present. This is interesting because it has observational counter-parts. In convective stars, e.g. red giants, absorption lines on the surface can be observed

The zero-order approximation of the energy generation rate is of the form

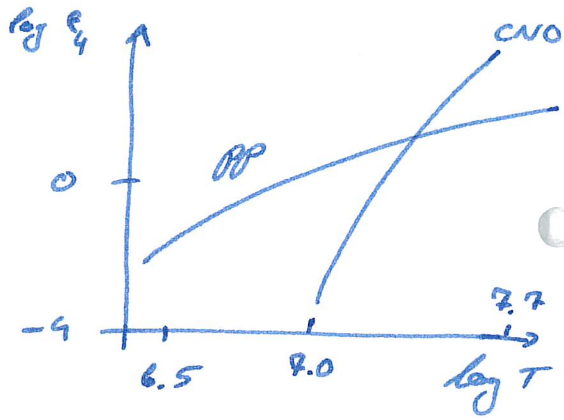
$$\varepsilon = \varepsilon_0 \, \rho \, T^{\nu}$$

with

$\nu \approx 17$ at $T \sim 25 \cdot 10^6$ K

$\nu \approx 13$ at $T \sim 50 \cdot 10^6$ K

The CNO cycle has a much higher temperature dependence with $\nu \sim 17$ than the pp-chains with $\nu \sim 4-5$.



Basic Properties of MS stars

High- and low-mass stars have very different internal structures. The nuclear energy generation rate ε strongly depends on T, and about above $1.2\,M_\odot$, the CNO cycle dominates the energy production. This means that the luminosity L is rapidly built near the center; $\nabla_{rad} \propto \frac{L_r}{M_r}$ is large, larger than ∇_{ad}, which means that the criterion for convection is satisfied. The size of the convective core determines the mass fraction which participates in the nuclear burning, covering larger mass fractions in more massive stars. Stars where the pp-chains are the dominant process have a milder dependance on T and thus show no convective cores, their deep interior being fully radiative.

Below $\sim 0.4\,M_\odot$, the opacities are large enough everywhere to make the stars fully convective.

From models with standard composition, we can find approximate relations for L, M, R and T_c over a wide range of masses:

$$L \sim M^3$$

$$R \sim M^{0.6}$$

$$\bar{\rho} \propto M/R^3 \quad , \quad T_c \propto \frac{M}{R}$$

The lifetime of a star is mainly determined by the amount of fuel available divided by the power emitted:

$$t_H \simeq \frac{\text{fuel available}}{\text{power emitted}} \simeq \frac{\Delta M c^2}{L}$$

$$\simeq \frac{0.007\, q_c M c^2}{L}$$

q_c: mass fraction of convective core

0.007: relative mass defect $\frac{\Delta m}{m}$ for $4H \rightarrow {}^4He$

Accounting for the mass-luminosity relation

$$L \sim M^{\alpha} \sim M^{3},$$

we have

$$t_H \sim \frac{M}{L} \sim M^{1-\alpha} \approx M^{-2}$$

so more massive stars have shorter lifetimes. The reason is that more massive stars produce more energy, but radiate away much much more, as $L \sim M^{\alpha}$.

For the sun, we have $t_H \approx 10^{10}$ yrs.

Solar Neutrinos

Due to their very small cross-sections, neutrinos do not deposit their energy in the sun, but escape in about 2s.

You can estimate the number of neutrinos produced. The solar luminosity is $L_\odot = 3.8 \, 10^{33}$ erg/s, the energy produced by the pp chain is 26.23 MeV, giving $\sim 9 \times 10^{37}$ reactions/s. Each reaction (pp chain) produces two neutrinos, hence

the neutrino production rate is $\sim 1.8 \cdot 10^{38}/s$,
so the neutrino flux on the earth is

$$F_\nu \sim 6.5 \times 10^{10} \frac{\text{neutrinos}}{\text{cm}^2\, s}$$

The "Solar Neutrino Problem" was the fact that the various experiments since 1967 observed only about 1/3 – 1/2 of the neutrinos predicted by solar models.

The problem is solved by neutrino oscillations,

i.e. $\nu_e \to \nu_\mu \to \nu_\tau$

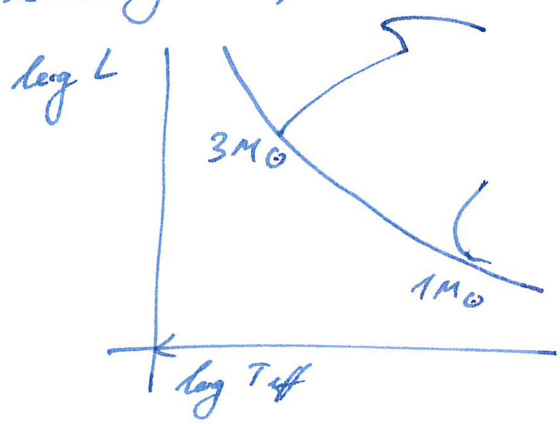
which implies that neutrinos have nonzero mass $10^{-4} - 10^{-2}$ eV.

The End of the Main Sequence

During the MS phase, the fusion of hydrogen into helium progressively modifies the H and He profiles in the deep interior and thus the mean molecular weight μ. The average μ is increased, resulting in a higher luminosity. Remember from homology relations:

$$L \propto \frac{M^3 \mu^4}{\kappa_0}$$

The growth of the central L favours a global expansion of the outer layers, leading to a cooling by this expansion, which increases the opacity and decreasing T_{ef}.



As the star evolves, it is displaced upwards to the right on the HR diagram.

The main sequence phase ends when H is exhausted in the center. After that, the H burning migrates into a shell around the new He core, which then grows in mass due to nuclear burning in the shell, which produces new He. With T not high enough for He ignition, the He is deprived of nuclear energy source and becomes isothermal. There is a maximum mass fraction q_{sc} for an isothermal core of perfect gas called the <u>Chandrasekhar–Schönberg limit</u>. Above this limit, the core can't sustain the upper layers and contracts. The star then evolves fast toward the RGB.

As P increases toward the center and T is constant in an isothermal core, the density must provide the whole pressure and above the SC limit, this becomes no longer possible. The isothermal core collapses.

This can be shown from the virial theorem applied to an isothermal core. Let the index 1 denote the quantities in/of the core.

The virial theorem writes

$$2E_{kin,1} + \Omega_1 - 3P_1V_1 = 0$$

$$= 3(\Gamma_1 - 1)U_1 + \Omega_1 - 3P_1V_1 = 0$$

for a fully ionized perfect gas, $\Gamma_1 = 5/3$

and $U_1 = E_{kin,1} = \frac{3}{2}\frac{M_1}{\mu_1 m_u} kT_1$

we take

$$V_1 = \frac{4}{3}\pi R_1^3$$

$$\Omega_1 = -q\frac{GM_1^2}{R_1}$$

This gives us:

$$3(\Gamma_1 - 1)U_1 + \Omega_1 - 3P_1V_1 = 0$$

$$\Rightarrow P_1 = \frac{1}{3V_1}\left(3(\Gamma_1 - 1)U_1 + \Omega_1\right)$$

$$= \frac{1}{4\pi R_1^3}\left(3\left(\frac{5}{3} - 1\right)\frac{3}{2}\frac{kT_1}{\mu_1 m_u}M_1 - q\frac{GM_1^2}{R_1}\right)$$

$$= \frac{3}{4\pi}\frac{kT_1}{\mu_1 m_u}\frac{M_1}{R_1^3} - q\frac{GM_1^2}{4\pi R_1^4}$$

Analoguely to the derivation of the Jeans mass, we have a local maximum of $P_1(R_1)$ at

$$\frac{\partial P_1}{\partial R_1} = 0$$

$$\Rightarrow R_{1,max} = \frac{4}{9} \frac{\mu_1 m_u q G M_1}{k T_1}$$

$$\Rightarrow P_{1,max} \approx \frac{3^4}{2^7 \pi} \left(\frac{k T_1}{\mu_1 m_u} \right)^4 \frac{1}{G^3 M_1^2}$$

for $q = 3/2$

So for a given T_1, P_{max} is smaller for higher masses, as $P_{1,max} \propto M_1^{-2}$. P_{max} is also the maximal pressure that the core can handle.

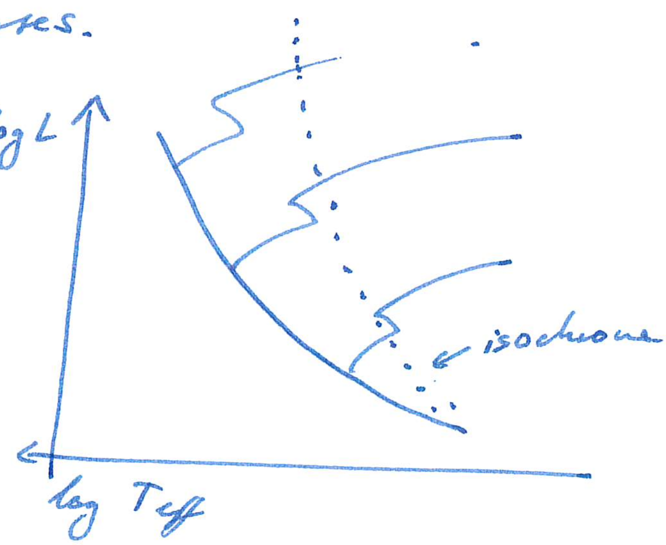
If we take $P_{env} = a P_c$, it can be shown that

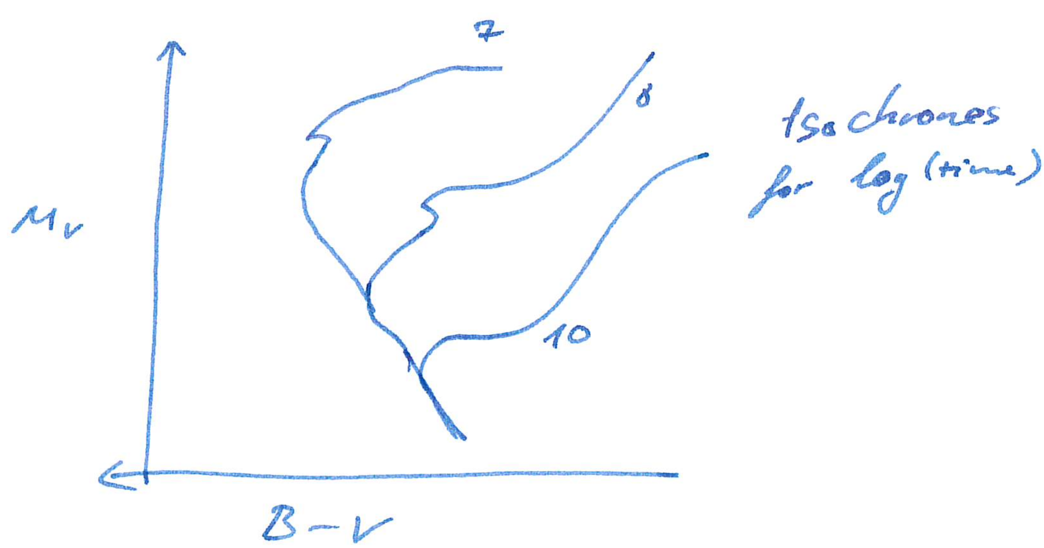
$$q_{sc} = \frac{M_1}{M} \approx \frac{0.162}{\sqrt{a}} \left(\frac{\bar{\mu}}{\mu_1} \right)^2$$

$$\approx 0.10 \quad \text{for} \quad a = 0.5, \quad \mu_1 = 4/3, \quad \bar{\mu} = 0.9$$

If at the end of the MS the He core is smaller than q_{sc}, H-shell burning proceeds increasing the core mass fraction until it reaches q_{sc}, then central contraction occurs.

Isochrones in the log L vs. log Teff diagram are obtained by connecting the points of the same ages on the tracks of various masses.





Nature produces isochrones in clusters, where all stars are approximately born at the same time. You can estimate the age of clusters by estimating / fitting isochrones.

Scatters in isochrone diagrams are probably produced by unresolved binaries. (This is for observed isochrones of clusters.)

Helium Burning

A temperature of 10^8 K is necessary. The central stellar regions contract and heat until this temperature is reached. The fusion of three nuclei of helium into carbon constitutes the entrance door to the further nucleosynthesis.

A number of remarkable types of objects are in the He-burning phase: red giants, red supergiants with radii up to $200 R_\odot$, variable Cepheids, AGB stars, most Wolf-Rayet stars, etc.

The He burning phase is the last stage of evolution observed before the stars become white dwarfs or explode as supernovae.

There are no stable nuclei of atomic mass $A = 5$ or 8, so to build elements heavier than ^4He, other processes than proton or α-capture are needed.

A tiny concentration of ^8Be is present, but the nuclei are short-lived. ^8Be captures another α-particle, thus leading to ^{12}C formation.

So the reaction proceeds in three steps:

$$^4He + {}^4He \rightleftarrows {}^8Be + \gamma \qquad (\text{lifetime} \sim 2 \cdot 10^{-16}\,s)$$

$$^8Be + {}^4He \rightleftarrows {}^{12}C^*$$

$$^{12}C^* \to {}^{12}C + \gamma$$

The total energy of the reaction, called the triple-α-reaction, is ~ 7.3 MeV for 12 nucleons, giving ~ 0.606 MeV per nucleon, about 10 times less than the H-burning.

The nuclear rate ε has a strong T dependence:

$$\varepsilon = \varepsilon_0 \, \rho^2 Y^3 T^{\nu}$$

$$T_8 = 1 \to \nu \sim 40$$

$$T_8 = 2 \to \nu \sim 19$$

the triple collision creates a dependence in ρ^2.
With strong temperature sensitivity, convection is promoted.

He-burning occurs in a convective core.

The triple-α-reaction is accompanied by

$$^{12}C(\alpha,\gamma)^{16}O$$

$$(^{12}C + \alpha \rightleftarrows {}^{16}O + \gamma)$$

which destroys a lot of ^{12}C.

and

$$^{16}O(\alpha,\gamma)^{20}Ne$$

the $^{12}C(\alpha,\gamma)^{16}O$ reaction is more frequent for a given temperature than the $^{16}O(\alpha,\gamma)^{20}Ne$ reaction, because the latter has a higher Coulomb barrier to overcome.

The abundance of ^{12}C first increases during the He-burning phase, then has a peak and decreases because it gets destroyed to produce ^{16}O. The amount of ^{16}O formed increases with T and thus with stellar mass, since higher T are required when the electric charges of the particles are larger. Analogously, the fraction of ^{20}Ne formed also increases with stellar mass.

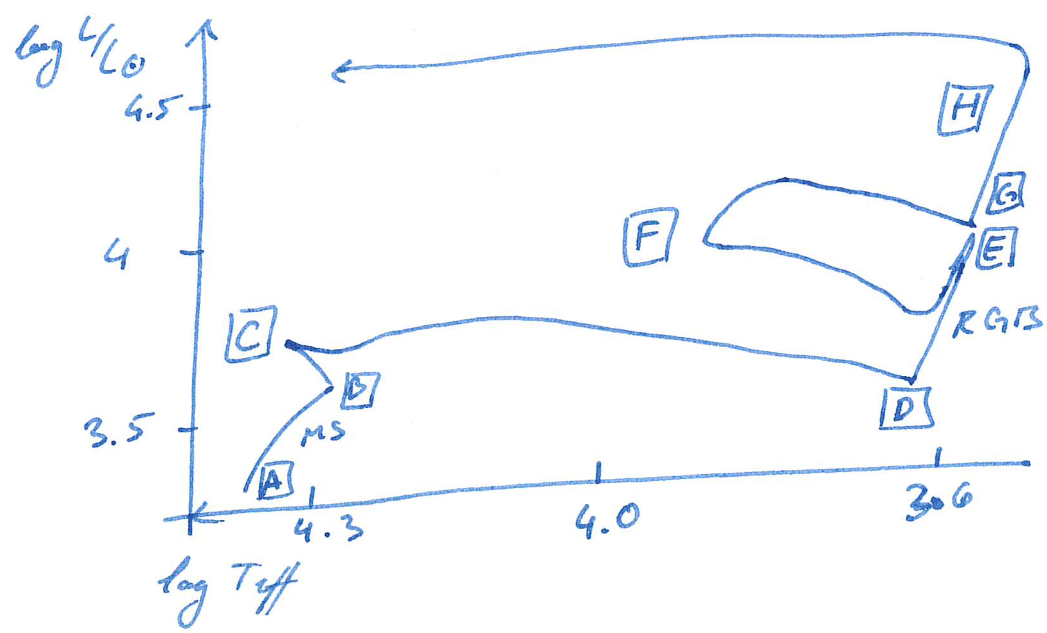
The above reactions are accompanied by some other ones, which are not important energetically, but have a nucleosynthetic importance, as they destroy ^{14}N (which is the second most important product of CNO reactions)

$$^{14}N(\alpha,\gamma)^{18}F(,e^+\nu_e)^{18}O(\alpha,\gamma)^{22}Ne \rightarrow (\alpha,n)^{25}Mg$$
$$\searrow (\alpha,\gamma)^{26}Mg$$

and $^{13}C(\alpha,n)^{16}O$

They also create new elements with many consequences. Especially the $^{22}Ne(\alpha,n)^{25}Mg$, which creates neutrons that can produce heavier elements (s-process).

From Main Sequence to Red Giants



We study the evolution of intermediate mass stars in the range of 2-9 $M_\odot$.

$\boxed{A}-\boxed{B}$: Main sequence, H-burning phase. Longest phase.

$\boxed{B}-\boxed{C}$: Contraction phase. The central H content becomes very low, the star contracts. In C, central hydrogen is exhausted.

$\boxed{C}-\boxed{D}$: Hertzsprung gap. Very short timescale $t_{KH} \sim 1\%$ MS; very few stars are found in this region on the HR diagram. The core contracts until central He ignition. The contraction leads to envelope expansion ("mirror effect"). Hydrogen is burning in shells. The initially thick H-shells become narrower towards D. Highly convective layers. A high density contrast develops between the core and the envelope.

$\boxed{D}-\boxed{E}$: Red giant branch. There is a deep convective envelope (up to 90% of stellar radius). Convection brings excesses of ^{14}N, ^{13}C and ^{4}He to the surface, together with a ^{12}C depletion.

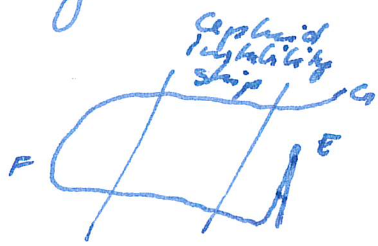
This carrying of atoms/heavier elements to the
surface is called a "dredge up". This is
the first dredge-up. The core is still
contracting until $\boxed{F}$, when He-burning ignites.

$\boxed{E}$ - $\boxed{F}$: He-burning, blue loops. The highly sensitive
ε produces a steep T gradient. A convective
core appears within the central regions and
the core expands while the envelope shrinks,
again due to the mirror effect. The H
shell progressively looses its energetic importance
and its outward migration slows down.

For stars with $\lesssim 2.3 M_\odot$ mass, He ignition
occurs in degenerate conditions. This produces
what is called the "helium flash". Helium
fusion increases the temperature, which increases
the reaction rate, which increases the temperature
again. This produces a flash of very intense
helium fusion that lasts only a few minutes.
The excess heat expands the star/core, so the
matter won't be degenerate anymore. The He
flash has no dramatic consequences, it
essentially takes the core out of its degenerate
conditions and allows the core He-burning
phase to occur in non-degenerate conditions

Typically one defines "low mass stars" as those who don't undergo a helium flash.

During the motion from $\boxed{E}$ to $\boxed{G}$, the star describes the "blue loops". This blue loop crosses the instability strip, where they pulsate radially and are observed as Cepheids. Cepheids have relatively short periods of ~ days.



$\boxed{F}$ - $\boxed{G}$: End of central He burning. Again the fuel of the core runs out, the core contracts, the envelope expands.
At $\boxed{G}$, the He core is exhausted.

$\boxed{G}$-$\boxed{H}$: We enter the phase of the asymptotic giant branch. The star evolves along the Hayashi line. In contrast to the red giant phase, there is no He core available. We have a partly degenerate CO core and a He-burning shell, which expands the outer layers. There is also a deep external convective envelope, producing the second dredge-up.

For stars below ≈ 6-$8 M_\odot$, the CO core is not massive enough to enter in a new nuclear phase (C burning). However, C burning rapidly dies out and the star nevertheless evolves like an AGB star.

For a while, only the He-shell will be burning. The core will increase its mass because of this shell burning. The extinct H-shell re-ignites when it is almost reached by the He-burning shell. The H- and He-burning shells are towards the end very thin and very close to each other. Nuclear burning dominates in the H-shell, however the thin He-shell experiences instabilities because of the degenerate core which is now isothermal and because it is thin. This leads to violent amplifications of the He-burning and to thermal pulses. The star gets rid of the envelope revealing the central core.

The last shell ejection makes a planetary nebula, which finally reveals a hot white dwarf.

Some Mass Limits

Minimal mass:

$\sim 0.007 M_\odot$ from opacity-limited fragmentation

Minimal mass for H burning:

$\approx 0.08 M_\odot$

Maximal mass for helium flash:

$\sim 2.3 M_\odot$

below: No He ignition

Maximal initial mass for White Dwarfs:

$\sim 8 M_\odot$

Intermediate mass stars

Lower limit for C ignition

$\sim 6-9 M_\odot$

Maximum initial mass for electron capture:

$\sim 8-10 M_\odot$

above, stars go through all phases of stable nuclear burning (H, He, C, Ne, O, Si) and form an onion skin model with an iron core.

Minimal initial mass to form Wolf-Rayet stars:

$\sim 25 M_\odot$

Massive Stars

Massive stars play a major role in the spectral and chemical evolution of galaxies.

- Main source of UV radiation, power IR radiation by heating dust
- Progenitors of interesting objects, like luminous blue variables, Wolf-Rayet stars, supernovae, black holes.
- Main nuclear reactors forming the heavy elements

The lower mass limit for a 'massive star' is $\sim 6.6 - 8 M_\odot$, the condition to ignite C-burning.

C ignites on the outside part of the core. The core at that point is isothermal, but neutrino escape cools the inner regions.

After the core C-burning phase, the core is degenerate. Its mass is regulated by three competing mechanisms:

- M_{core} increase through deepening of the outer convective zone
- Mass loss of the star decreases M_{core}
- Activity of shell-burnings increases M_{core}

If the degenerate core never exceeds the Chandrasekhar mass of $\sim 1.2\, M_\odot$, the core evolves into an ONe white dwarf. Otherwise, it explodes in an electron capture (type II) supernova.

This gives a lower limit on the mass of a star to be able to produce a type II supernova, which is $\sim 8 \pm 1\, M_\odot$.

A theoretical upper bound on the stellar mass is given by the Eddington luminosity

$$\frac{L}{L_\odot} \sim 4 \times 10^4 \frac{M}{M_\odot}$$

giving an upper limit of $\sim 135\, M_\odot$.

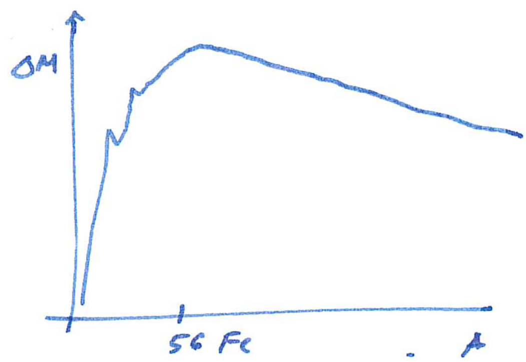
Nuclear Phases After the core He-burning phase

In massive enough stars, the core can go through a whole sequence of nuclear burning stages, ending up in layers of different burnings and an iron core. The sequence is

$H \to He \to C \to Ne \to O \to Si$

As evolution proceeds from H to Si burning, there is less and less nuclear energy available.

Heavier nuclei give less energy per nucleon in their reaction, thus assuming the star's luminosity is ≈ constant, the reactions need to progress faster. The core contracts also to compensate for the energy loss. Above iron, there is no fusion that releases energy; Instead, energy is gained through nuclear fission. So no elements above Fe can be used as fuel for the core.



Stellar Winds

"The radiation observed to be emitted must work it's way through the star, and if there were too much obstruction, it would blow up the star."

Let us calculate the mass loss rate of a star due to stellar winds.

The continuum and line absorptions in stellar atmospheres transfer the radiation momentum to the stellar atmosphere, accelerating it outward. The momentum is mainly transferred by UV photons in resonance lines of Fe and CNO elements.

The Doppler effect is saving the stellar winds. Without the Doppler shift, the already slightly accelerated ions in the wind could not be further accelerated because all the flux at the efficient absorption wavelengths would have already been absorbed in lower layers.

The equation of motion of the stellar wind in the stationary regime is

$$\underbrace{\frac{\partial u}{\partial t}}_{\approx 0\,:\ \text{stationary}} + u\frac{du}{dr} = g_{rad} - \frac{GM(1-\Gamma)}{r^2}$$

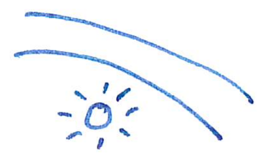
g_{rad} is the acceleration due to gravity

$-\frac{GM}{r^2}$ is the gravitational acceleration

$\frac{GM\Gamma}{r^2}$ is the Eddington acceleration

we can express g_{rad} as

$$g_{rad} = \frac{\text{absorbed momentum in a shell}}{\Delta t\,\Delta m}$$



where we consider a shell right outside / on the edge of a star.

and express the absorbed momentum as

$$\frac{\text{abs. mom.}}{\Delta t} = \frac{\text{Energy}}{\Delta t}\cdot\frac{1}{c} = \underbrace{\frac{L}{c}}_{\substack{\text{total momentum}\\ \text{provided by}\\ \text{the star}}}\cdot\underbrace{\sum_i \frac{L_{\nu_i}\Delta\nu_i}{L}}_{\substack{\text{fraction of the luminosity}\\ \text{absorbed by the}\\ \text{line } i \text{ of frequency}\\ \nu_i \text{ and interval } \Delta\nu_i}}$$

the line bandwidth is determined by the Doppler effect: $\Delta\nu_i = \frac{\Delta u}{c}\nu_i$

with $\Delta m = 4\pi r^2 \varrho \Delta r$, this gives us

$$g_{rad} = \frac{\frac{L}{c} \sum_i \frac{L_{\nu_i} \Delta\nu_i}{L}}{4\pi r^2 \varrho \Delta r} = \frac{\frac{L}{c} \sum_i \frac{L_{\nu_i}}{L} \frac{\Delta u}{c} \nu_i}{4\pi r^2 \varrho \Delta r}$$

$$= \frac{L}{c^2} \underbrace{\sum_i \frac{L_{\nu_i}}{L} \nu_i}_{\equiv N_{eff}} \frac{\Delta u}{\Delta r} \cdot \frac{1}{4\pi r^2 \varrho}$$

$$= \frac{L}{c^2} N_{eff} \frac{1}{4\pi r^2 \varrho} \frac{du}{dr}$$

$$\Rightarrow g_{rad} \propto L, \quad g_{rad} \propto \frac{du}{dr}$$

Now using the equation of motion

$$u \frac{du}{dr} = g_{rad} - \frac{GM(1-\Gamma)}{r^2}$$

and the continuity equation

$$dM = 4\pi \varrho r^2 dr$$

we can write

$$\dot{M} = \frac{dM}{dt} = 4\pi \rho r^2 \frac{dr}{dt} = 4\pi \rho r^2 u$$

$$= \frac{1}{du/dr} \left(g_{rad} - \frac{GM(1-\Gamma)}{r^2} \right) 4\pi \rho r^2$$

$$= \frac{L}{c^2} N_{eff} \frac{4\pi \rho r^2}{4\pi \rho r^2} - \frac{GM(1-\Gamma) 4\pi \rho r^2}{r^2 \, du/dr}$$

$$= \boxed{\frac{L}{c^2} N_{eff} (1 - \varepsilon)}$$

with $\varepsilon = \frac{GM(1-\Gamma) 4\pi \rho}{du/dr} \frac{c^2}{N_{eff} L}$

$\Rightarrow$ the mass loss is proportional to the number of strong lines, which are also proportional to proton number Z

$\Rightarrow$ higher Z leads to higher mass loss

We assumed that the wind is homogeneously distributed around the star. If it instead is clumpy, the mass loss is overestimated.

Wolf-Rayet Stars

WR stars are identified with 'bare cores' left over from the peeling of massive stars by stellar winds. Their strong emission lines show highly non-solar chemical abundances.

These are stars with emission lines!

WR stars have high mass loss rates and occur in young associations. They are also rare, about one in 10^8 stars.

The observed emissions and their relative luminosities provide a check for nucleosynthesis models.

The ejected masses provide mechanical energy to the ISM comparable to SNe.

Mass loss, by removing matter at an early stage of the evolution of the star, may save some elements from further destruction. This increases the abundance of these elements.

The Core Collapse

At $T \sim 10^{10}$ K, $\rho \approx 10^{10}$ g/cm^3, the energetic photons are able to photodisintegrate the nuclei of the Fe peak:

$$^{56}Fe + \gamma \rightarrow 13\alpha + 4n - 2.2 \text{ MeV/nucl}$$

This reaction is endothermic and removes a lot of thermal energy, dramatically accelerating the collapse. Equilibrium is broken, the adiabatic exponent $\Gamma < 4/3$, and the central regions collapse at the dynamical timescale.

When $\rho > 10^{11}$ g/cm^3, the Fermi energy of electrons is higher than the energy difference of 1.29 MeV between neutrons and protons, and the inverse of the β decay occurs:

$$p + e^- \rightarrow n + \nu_e$$

This reaction initiates the collapse for stars by electron captures. The removal of the degenerate electrons, which were the source of the pressure, also helps the collapse.

All these reactions lead to the catastrophic implosion of the core. It collapses as in free-fall, for $\rho \sim 10^{10}$ g/cm $\rightarrow \tau_{ff} \sim 10^{-?}$ s.

As the massive star nears its end, it takes an onion-like layer structure of chemical elements. Iron in the core does not undergo nuclear fusion, so the core becomes unable to generate heat. This leads to core contraction, increase of densities, and the processes mentioned above.

The core collapses to form a neutron star. Material rebounds off the neutron star, setting up a shock wave. The shock sweeps through the entire star, blowing it apart.

As the shock wave passes through the star, matter is briefly heated to temperatures far above what it would have experienced in hydrostatic equilibrium.

Through this heat, elements can be burned again during the explosion. Iron and most of the iron group elements can be produced by this explosion.

Neutron Capture

The nuclear burning can create enough neutrons that nuclei can grow through neutron capture.

Unlike protons, neutrons do not face Coulomb repulsion.

In the capture process, two types of reactions and two types of nuclei are involved: Neutron captures and β decays; Stable and unstable nuclei.

Stable nuclei may undergo only neutron captures, unstable ones both. The outcome depends on the timescales of the two processes.

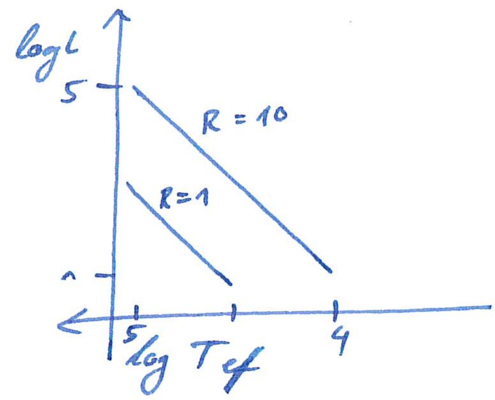
Neutron capture reactions may proceed more slowly or more rapidly than the competing β- decays. The different chains of reactions and products are called the s-process and the r-process.

R-processes can occur in

- neutrino powered winds from neutron stars
- merging neutron stars
- jets from stellar iron core collapse

Miscellaneous Stuff

Radii on HR-Diagrams



Why are the loci of constant radii straight lines?

Consider the Steffan-Boltzmann law:

$$F_{rad} = \sigma T_{eff}^4 = \frac{L}{4\pi R^2}$$

$$\Rightarrow L \propto R^2 T_{eff}^4$$

Now suppose we have a constant radius. Increasing $\log T_{eff}$ by 1 increases $\log L$ by 4. So the slope of the constant radii lines will be +4. Note that $\log T_{eff}$ increases to the left in HR diagrams.

It also means that for constant T_{eff}, increasing $\log L$ by 1 increases the radius by two orders of magnitude. As one would expect: A star with a bigger surface $\propto R^2$ radiates more.

White Dwarf Lifetime

Why can a White Dwarf persist for long times without much change?

White dwarfs are fully degenerate systems, held together by electron degenerate pressure. The degenerate pressure follows a different EoS:

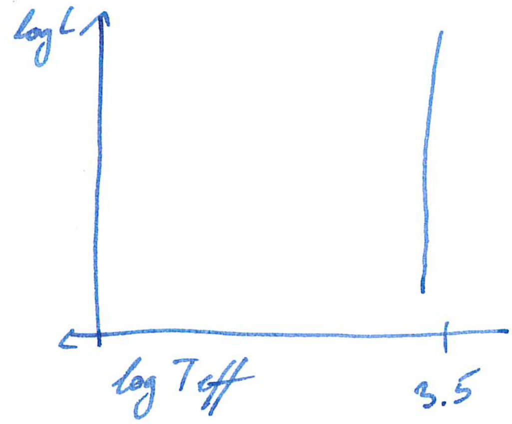
$$p = p(\rho), \text{ not } p = p(\rho, T) \text{ any more.}$$

$\Rightarrow$ there is no temperature gradient any more, and barely any energy is radiated away.
"They do not have enough energy to cool".

So they remain in this state over very long periods of time.

Hayashi line

The Hayashi line is an important concept in stellar evolution. It is the location of fully convective stars in the HR diagram. This locus is nearly vertical and depends on the mass of the star.



The Hayashi line is at about constant $T_{eff} \approx 3500$ K. This results from the high opacity for $T \lesssim 7000$ K. At low T, $\kappa \propto T^9$, thus if T increases, κ increases even faster, which in return reduces T. This produces a feedback maintaining $T \approx$ const.

The Hayashi line is the lower T_{eff} limit of convective stars in the HR diagram. To the right, there are no stable stars.

Mean Molecular Weight

Let us consider a medium with various elements j of atomic mass A_j. Let us call X_j the mass fraction of element j.

The mean molecular weight is

$$\frac{1}{\mu} = \sum_j \frac{X_j}{A_j} (1 + E_j)$$

where E_j is the number of free electrons. So for a fully ionized medium, we have

$$\frac{1}{\mu} = \sum_j \frac{X_j}{A_j} (1 + Z_j)$$