The Expanding Universe

The Cosmological Principle

Viewed on a sufficiently large scale (~ 100 Mpc), the properties of the Universe are the same for all observers $\Leftrightarrow$ the Universe is homogeneous and isotropic.

Friedmann - Robertson - Walker Metric

Describes a non-static Universe which is spatially homogeneous and isotropic, but evolves in time.

$\Rightarrow$ Consider spacetime to be $\mathbb{R} \times \Sigma$, with Σ = maximally symmetric three-manifold and $\mathbb{R}$ is the time direction.

$\Rightarrow$ Σ has maximal number of Killing vectors and is invariant under translations and rotations.

$\Rightarrow$ A comoving observer is defined to be at rest in the frame of the Universe $\Leftrightarrow$ has no peculiar velocity.

FRW metric:

$$\boxed{ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}$$

The curvature of space is defined by k:

$k=0$ $\quad d\sigma^2 = dr^2 + r^2 d\Omega^2$
$\quad\quad = dx^2 + dy^2 + dz^2$ $\quad$ flat Universe

$k=1$ $\quad d\sigma^2 = \dfrac{dr^2}{1-kr^2} + r^2 d\Omega^2$ $\quad$ closed Universe
$\quad\quad = d\chi^2 + \sin^2\chi \, d\Omega^2$

$k=-1$ $\quad d\sigma^2 = \dfrac{dr^2}{1+kr^2} + r^2 d\Omega^2$ $\quad$ open Universe
$\quad\quad = d\chi^2 + \sinh^2\chi \, d\Omega^2$

The curvature of the space is not the curvature of the spacetime!

Friedmann Equations

The Friedmann equations give the behaviour of $a(t)$ by introducing physics via the Einstein equations for the FRW metric.

Ansatz: homogeneous, isotropic Universe $\Rightarrow$ homogeneous, isotropic matter/energy distribution $\Rightarrow$ Ideal Fluid: $T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}$

Fluid is in rest frame $\Rightarrow$ $u^\mu = (1,0,0,0)$

1) Conservation of energy equation

$\nabla_\mu T^\mu{}_\nu = 0 \quad \Rightarrow$ For $\nu = 0$: $\boxed{\dfrac{\partial \rho}{\partial t} = -3\dfrac{\dot a}{a}(\rho + w)}$

2) Second Friedmann equation:

Using Einstein Field Equation $R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu})$

For $\mu = \nu = 0$:

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

3) Third Friedmann equation:

Einstein Field equs for $\mu = i;\ \nu = j$:

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}}$$

Definitions:

$H \equiv \dfrac{\dot a}{a}$ Hubble parameter

$H_0 \equiv 100h$ km/sec / Mpc Hubble constant; Hubble parameter today. $h \approx 0.7$

$\rho_{crit} \equiv \dfrac{3H^2}{8\pi G}$ Critical density; determines k in the absence of Λ

$\Omega_i \equiv \dfrac{\rho_i}{\rho_{crit}}$ Density parameter. $\Sigma\Omega_i \approx 1$ from observations

$\rho_c \equiv \dfrac{-3k}{8\pi G a^2}$ Curvature density term, fictional.

On Densities

For further progress, we need to assume an equation of state: $p = w\rho$ assuming w is a constant.

From the energy cons. eqn.: $\dot{\rho} = -3\frac{\dot{a}}{a}(1+w)\rho \Rightarrow \rho \propto a^{-3(1+w)}$

$\Rightarrow$ We can describe the evolution of the densities as power laws:

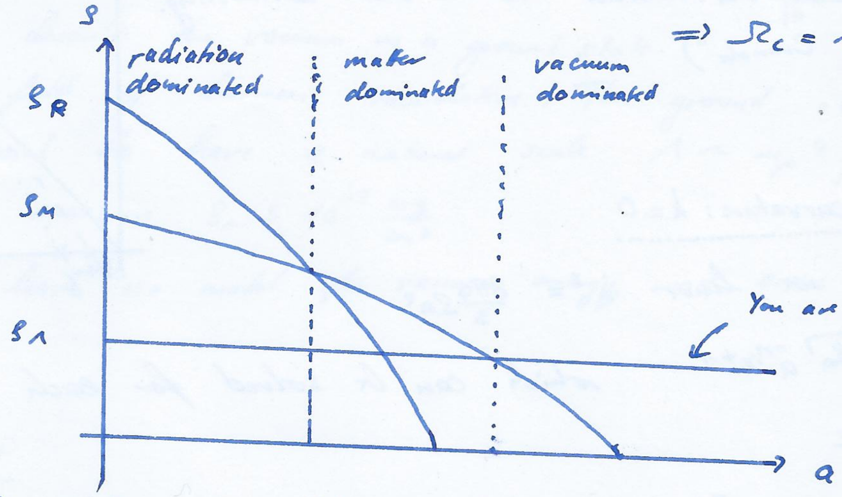
$$\rho_i = \rho_{i,0}\, a^{-n_i}, \quad \text{with } w_i = \frac{\rho_i}{p_i} = \text{const}$$

For the typical contents of the Universe:

matter	$p = 0$	$\rho = \rho_0 a^{-3}$
radiation	$p = \frac{1}{3}\rho$	$\rho = \rho_0 a^{-4}$
curvature		$\rho = \rho_0 a^{-2}$
vacuum	$p = -\rho$	$\rho = \text{const} = \rho_0$

The Friedmann eqns can be written as $H^2 = \frac{8\pi G}{3}\sum_i \rho_i \iff 1 = \sum_i \Omega_i$ (includes ρ_c)

$\Rightarrow \Omega_c = 1 - \sum_i \Omega_i \setminus \{\Omega_c\}$



Initially, all matter was relativistic $\hat{=}$ radiation. With cooling, gradually the particles became non-relativistic.

Solutions of the Friedmann Equation

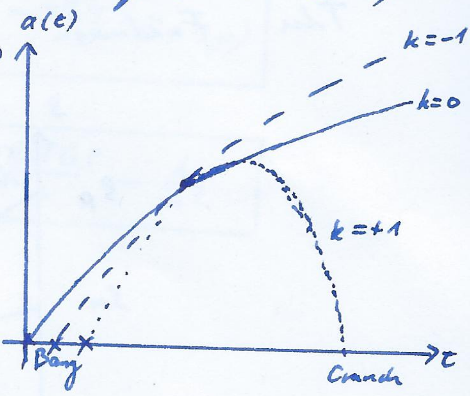
1) Positive Densities

Assuming all densities (including curvature) ≥ 0 $\quad (\hat{=}\ k \leq 0)$

Using the Friedmann equations, we get

$$\boxed{\dot{H} = -4\pi G \sum_i (1+w_i)\, \rho_i}$$

- With $|w_i| \leq 1$ and $\rho_i \geq 0$ $\Rightarrow$ $\dot{H} \leq 0$ always, meaning that the expansion rate always decreases.
- With $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k|$ it follows that $\dot{a}$ is exactly then zero for an empty, flat Universe $\Rightarrow$ The sign of $\dot{a}$ never changes.
- From observations today, we know that $\dot{a} > 0$.
- Asymptotic behaviour: Using the energy conservation equation, one can show that $\frac{d}{dt}(\rho a^3) \leq 0$ $\Rightarrow \rho a^2 \xrightarrow{a\to\infty} 0$ $\Rightarrow \boxed{\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \xrightarrow{a\to\infty} |k|}$
  - For $k > 0$ $(k=1)$ we get $\dot{a}^2 = -1$, which isn't possible. Instead, the expansion has to stop at some finite a_{max}, after which the Universe has to start contracting. (= "Big Crunch")



2) Fixed spatial curvature: $k=0$

With $k=0$, we have $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2$

$\Rightarrow \quad \dot{a} = \sqrt{\frac{8\pi G \rho_0}{3}}\, a^{-n/2+1}$ $\quad$ which can be solved for each component:

$t \propto a^{n/2}$

Dust: $\quad t \propto a^{3/2}$ $\qquad$ Radiation: $\quad t \propto a^2$ $\qquad$ Vacuum: $\quad t \propto \ln(a)$

$a \propto \exp(t)$

3) Static Universe Solutions

For a static Universe, we demand $\ddot{a} = \dot{a} = 0$

For simplification, consider a matter dominated, static Universe with a cosmological const.:

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) + \Lambda \overset{p=0 \text{ for matter}}{=} -4\pi G\rho + \Lambda \overset{!}{=} 0 \quad (\text{static}) \Rightarrow \Lambda = 4\pi G\rho$$

Using the first Friedmann eqn: $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \overset{!}{=} 0 \quad \Rightarrow \frac{k}{a^2} = 4\pi G\rho$

$\Rightarrow k > 0$ with $\rho > 0$

But this solution is not stable.

If we don't assume a cosmological constant, we get $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = 0$

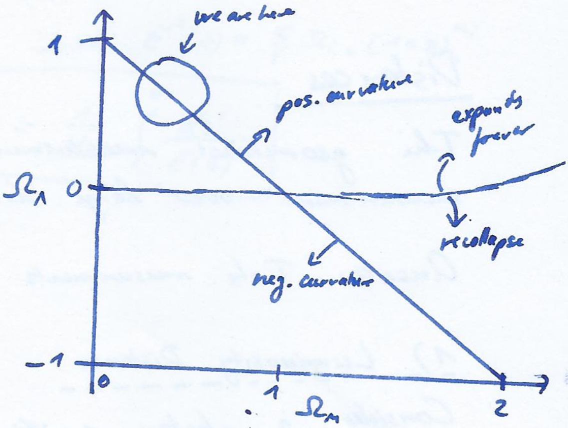
$\Rightarrow p = -\frac{1}{3}\rho$ $\quad$ which is a component other than matter or radiation... Some unknown component.

4) Matter and Vacuum Dominated Universe

In the present Universe, the radiation density is negligible. $\Rightarrow \Omega_M + \Omega_\Lambda + \Omega_c = 1$
As the Universe expands, the relative influences of matter, curvature and vacuum are altered as the corresponding densities evolve differently.

$a \to 0$: Ω_c, Ω_Λ negligible ; $a \to \infty$: $\Omega_c, \Omega_m, \Omega_r$ negligible

Λ will eventually dominate in every scenario because it remains constant.
$\rightarrow$ Recollapse will always occur if the Λ is negative.
Recollapse is also possible for positive Ω_Λ, provided that Ω_M is large enough to halt and reverse expansion before Ω_Λ can take over.
The condition for recollapse is given through the condition for a turnaround in a, therefore a has a local maximum $\Rightarrow \dot{a} = 0$



Toy Model for Vacuum Energy

Try to describe the vacuum as a ground state of a field of harmonic oscillators. The ground state oscillations would be expected to have a natural scale $\Lambda \sim m_p{}^4$, giving $\rho_{vac} \sim 10^{112} \frac{erg}{cm^3}$ but we measure $\rho_\Lambda \lesssim 10^{-3} \frac{erg}{cm^3}$

$\Rightarrow$ We have no model for vacuum energy, not even close.

Redshift

The FRW metric is a maximally symmetric space, not spacetime. There is no timelike Killing vector to give us a notion of energy conservation for the entire Universe (not for objects inside the Universe though). There is however a Killing tensor: $K_{\mu\nu} = a^2 (g_{\mu\nu} + U_\mu U_\nu)$ with $U_\mu = (1,0,0,0)$ is a Killing tensor $\Leftrightarrow \nabla_{(\sigma} K_{\mu\nu)} = 0$.
Then for a particle with four-velocity/momentum $V^\mu = \frac{dx^\mu}{d\lambda}$ the quantity $K^2 \equiv K_{\mu\nu} V^\mu V^\nu$ will be a constant along a geodesic.
$= a^2 [V_\mu V^\mu + (U_\mu V^\mu)^2]$

1) Massive Particles

For massive particles: $V_\mu V^\mu = -1 = (V^0)^2 + |\vec{V}|^2 \Rightarrow (V^0)^2 = 1 + |\vec{V}|$, $U_\mu V^\mu = -V^0$
$\Rightarrow |\vec{V}| = \frac{K}{a}$ $\Rightarrow$ The particle slows down with respect to the comoving coordinates; A gas will cool down with time.

2) Massless Particles

For photons: $V_\mu V^\mu = 0$ and $U_\mu V^\mu = U_0 V^0 = -E = -\hbar\omega = -\omega = \frac{K}{a}$

The frequency ω measured by a comoving observer will be observed with a lower frequency $\omega(t_{obs})$ as the Universe expands:

$$\frac{\omega(t_{obs})}{\omega(t_{em})} = \frac{a(t_{em})}{a(t_{obs})} \iff \omega(t) \propto \frac{1}{a}$$

Define redshift $z_{em} = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \overset{a_{obs} \overset{!}{=} 1}{=} \frac{1}{a_{em}} - 1 \Rightarrow a_{em} = \frac{1}{1+z_{em}}$

The redshift of an object tells us the scale factor when the photon was emitted.

Distances

The geometrical measurement of distances is impossible: It requires simultaneous measurement over large distances.

Ansatz: Take measurements as in Euklidean space and apply corrections for cosmology.

1) Luminosity Distance

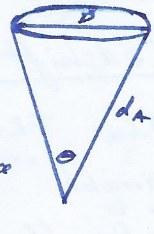
Consider a photon emitting source at a distance d_L, when L is the absolute luminosity of the source and F is the flux measured by the observer:

$$L = FA \Rightarrow d_L^2 = \frac{L}{4\pi F} \qquad \text{with } L = \frac{N\hbar\omega}{\Delta t}$$

Corrections: i) Photons redshift: $\omega \to \omega/(1+z)$

ii) Photons hit sphere $4\pi d_L^2$ less frequently: $\Delta t \to \Delta t(1+z)$

$$\Rightarrow \boxed{d_L^2 = \frac{L}{4\pi F(1+z)^2}}$$



2) Angular Diameter Distance

= distance we infer from the intrinsic and observed size of the source

$\tan\theta \approx \theta = \frac{D}{d_A}$

Assume the entire observed galaxy is described by the same comoving coordinate.

Looking at an object in the sky: $dt = dr = d\varphi = 0$

$$\Rightarrow ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right] = a^2 r^2 d\theta^2$$

$$\Rightarrow D = \int_0^b ds = a r \Delta\theta$$

$$\Rightarrow d_A = \frac{D}{\Delta\theta} = ar = (1+z)^{-1} r$$

Relations

Expressed in coordinates of the metric $ds^2 = -dt^2 + a^2[d\chi^2 + S_k^2(\chi)\, d\Omega^2]$ with $S_k(\chi) = \sin\chi$ for $k=1$, χ for $k=0$, $\sinh(\chi)$ for $k=-1$, the area for luminosity distance is $\frac{L}{F} = \frac{1}{A(1+z)^2}$ with $A = 4\pi S_k^2(\chi)$. But χ (comoving distance) can't be measured directly so we need an other way. So we try to relate $S_k(\chi)$ with the densities Ω:

Consider a radially infalling photon: $ds^2 = 0 = a^2 d\chi^2 - dt^2 \Rightarrow dt = a\, d\chi$

$$\Rightarrow d\chi = \frac{dt}{a} = \frac{1}{a}\frac{dt}{da}\, da = \frac{1}{a}\frac{1}{aH}\, da = \frac{da}{a^2 H} \quad \Rightarrow \chi = \int_{a_{em}}^{a_{obs}} \frac{da}{a^2 H}$$

Using the Friedmann equation:

$$H^2 = \frac{8\pi G}{3} \sum \rho_{i,0} (1+z)^{n_i} \equiv H_0^2 E^2(z) \quad \text{with } E^2(z) = \sum_i \Omega_{i,0}(1+z)^{n_i}$$

with $\chi(a = \frac{1}{1+z} \rightarrow dz = -\frac{1}{a^2} da \Rightarrow \boxed{\chi = \frac{1}{H_0} \int_0^z \frac{dz'}{E(z')}}$

Also: $d_L = (1+z)^2 d_A$

Measuring Distances

Different techniques are necessary for different distances:

- Solar system: radar ranging
- Nearby stars: parallax
- Milky Way: Main sequence fitting gives standard candles
- Nearby galaxies: Cephid variable stars give standard candles
- Galaxy Clusters: White dwarf supernovae and other standard candles

Hubble's law:

The comoving distance between two object is given by $d_{com} = \frac{r}{a} = \frac{r'}{a'} = \frac{r_{today}}{a_{today}}$

Assuming the galaxy is not moving with respect to the background Universe, we get

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(a \frac{r_{today}}{a_{today}}\right) \overset{a_0=1}{=} \frac{d}{dt}(ar) = \dot{a} r_{tod} + a \underbrace{\dot{r}_{tod}}_{=0} = \dot{a}\frac{a_{tod}}{a} r = Hr$$

Almost all galaxies appear to move away from us with velocity proportional to their distance from us, which is very strong evidence for an expanding Universe.

Thermal History of the Universe

Temperature Relations

- Dust: $k_B T \propto \langle v \rangle^2 \propto \frac{1}{a^2} \quad \Rightarrow \quad \boxed{T \propto 1/a^2}$
- Radiation: $\rho_r = \frac{k_B \sigma T^4}{c} \propto \frac{1}{a^4} \quad \Rightarrow \quad \boxed{T \propto 1/a}$

Hot Big Bang Model

Standard cosmology: The temperature of the Universe was arbitrarily high at the beginning of the Big Bang and has decreased continuously as the expansion progresses.

Key concepts:

i) At any given time, particles can be created if their rest mass is such that $k_B T_r \gg mc^2$, where T_r is the temperature of radiation.

ii) The early Universe was filled with radiation and hot plasma. The initial soup of elementary particles and radiation is in thermal equilibrium. The equilibrium is maintained by electroweak interactions between particles and radiation.

iii) As the Universe expands, the interaction rate $\Gamma_i = n_i \langle \sigma_i v_i \rangle$ decreases, because n and v decrease. When $\Gamma_i < H$, the particle species decouples from the photon fluid and the particle density and distribution "freezes out" to the value at first decoupling.

The Chronology of the Hot Big Bang

1) Planck Era $\quad t \sim 10^{-44}\,s$

Smallest possible physical timescale we can describe without a quantum theory of gravity. The unified force of the standard model is assumed to be unified with gravitation.

2) Grand Unification Time to Electroweak Epoch: $t \sim 10^{-35}\,s$ to $t \sim 10^{-24}\,s$

Gravity has separated from electronuclear force at the end of the Planck era. Then after GUT, the strong force separated from the electroweak force.

2 important processes:

i) Inflation

ii) Baryogenesis, which contained some spontaneous symmetry breaking that lead to a much higher abundance of particles than antiparticles

3) Particle Era

After electroweak era, all known elementary particles of the standard model have been produced. The Universe is made up from quarks, leptons and photons in equilibrium.

i) Quark-Hadron transition $t \sim 10^{-5}s$, $T \sim 3\times10^{12}K$

Quarks get confined into hadrons once the temperature is low enough.

ii) Nucleons era $t \sim 10^{-4}s$, $T \sim 10^{12}K$

Pions annihilate and decay. Nucleons remain as only hadrons left.

iii) Neutron - Proton assymmetry $T < 10^{11}K$

The production of slightly heavier neutrons is suppressed exponentionally by the factor $\exp(-\Delta m/T)$. The assymmetry grows until the reaction rates involving protons and neutrons becomes negligible.

iv) e^+e^- pair annihilation, neutrinos decoupling $t \sim 4s$, $T \sim 5\times10^9K$

The annihilations heat the photons, but not the neutrinos.

v) Nucleosynthesis $t \sim$ few minutes, $T \sim 10^9K$

Nucleosynthesis starts, producing D, He and others. Due to high temperature all the elements are highly ionized.

vi) Recombination $t \sim 2\times10^5$ yrs, $T \sim 4000K$

Electrons combine with ions to produce neutral atoms. With the number of free electrons decreasing, the Universe becomes transparent to photons. The photons that decouple from matter become the CMB. Most particles are non-relativistic and the Universe enters the matter-dominated era.

Matter-Radiation Thermal Equilibrium

Evidence of equilibrium: Blackbody spectrum of CMB (Blackbody needs matter-radiation equil.)

For homogeneous, isotropic Universe: The evolution of a particle species can be written in terms of a distribution function $f(\vec{x},\vec{p},t) = f(p,t)$

Then
$$n = 4\pi \int f\, p^2 dp$$
$$S = 4\pi \int E(p)\, f\, p^2 dp$$
$$P = 4\pi \int \frac{p^2}{3E(p)} f\, p^2 dp$$

For a particle species in thermal equilibrium:

$$f(\vec{p},t)\,d^3\vec{p} = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{E-\mu}{T}\right) \pm 1\right]^{-1} d^3\vec{p}$$, + for Fermi-Dirac

− for Bose-Einstein

Solutions for special cases:

i) Non-relativistic particles $m \gg T$

$$n_{eq} = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(\frac{\mu - m}{T}\right)$$

ii) Relativistic particles $m \ll T$

$$n_{eq} = \begin{cases} \zeta(3)/\pi^2 \, g T^3 & \text{for bosons} \\ \frac{3}{4}\zeta(3)/\pi^2 \, g T^3 & \text{for fermions} \end{cases} \qquad \rho_{eq} = \begin{cases} \frac{\pi^2}{30} g T^4 \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 \end{cases}$$

More massive particles transition earlier to non-relativistic statistics.
The mass also determines the abundance: With $T \gg m$ the photons have sufficient energy to create a thermal background number density of particle-antiparticle pairs.
Particles in thermal equilibrium with the photon gas can only contribute significantly to the energy density and pressure when they are relativistic:

$$\rho_{Universe} \approx \sum_i \rho_{i,\text{relativistic}}$$

Chemical potential:

= additive quantity, conserved during a chemical reaction: $\mu_1 + \mu_2 = \mu_3 + \mu_4$; $\mu_i = \frac{\partial U_i}{\partial N_i}$
Radiation in thermodynamical equilibrium is described by black body radiation. The equilibrium is achieved through interactions between matter and radiation, which includes spontaneous absorption and emission. $\Rightarrow$ Number of photons can't be a conserved quantity $\Rightarrow \mu_\gamma = 0 = \frac{dU_\gamma}{dN_\gamma}$
With number densities of baryons and leptons are found to be much smaller than the number density of photons, the $\mu_b \approx 0$ to good approximation.

Entropy of the Universe

$$TdS = dU + PdV \qquad \Rightarrow \quad S = \frac{V}{T}(\rho + P) \quad + \text{const}$$

Define entropy density $s \equiv \frac{S}{V} = \frac{\rho + P}{T}$

The cosmological principle allows us to apply the entropy on the entire Universe by considering it to have a representative Volume element dV.
Non-relativistic particles contribute negligibly to the energy density because $\propto e^{-m/T}$.
$\Rightarrow$ Sum only relativistic particles:

$$\boxed{s_{eq,i} = \frac{2\pi^2}{45} g_i T^3 \quad \text{and} \quad s(T) = \frac{2\pi^2}{45} g_{*s} T^3}$$

with $\boxed{g_{*s} = \sum_{i \in \text{Bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i \in \text{Fermions}} \left(\frac{T_i}{T}\right)^3}$ ← allows for already decoupled, but relativistic particle species

Entropy conservation: $\boxed{sa^3 = \text{const} = g_{*s} T^3 a^3; \quad \frac{dS}{dt} = 0}$

Distribution Function for Decoupled Particles

After a particle species decouples, the mean interaction rate $\Gamma < H$ and the particles basically move freely, i.e. on geodesics. Its distribution function remains fixed for the rest of all times, because there are no more collisions by constructions. So it keeps the form by scaling the time dependent momentum $p \propto \frac{1}{a}$ back to freeze-out:

$$f(\vec{p},t) = f\left(\vec{p}\,\frac{a(t)}{a(t_f)}, t_f\right)$$

Depending on the state of the particle at decoupling, we can differentiate:

i) Hot relics: Relativistic decoupling
$E \approx p, \ \mu \approx 0$
Distribution function is self-similar to that of a relativistic species in thermal equilibrium, but with a temperature $T = T_f \frac{a(t_f)}{a(t)}$

ii) Cold relics: Non-relativistic decoupling
$E = m + p^2/2m$, we can ignore ± 1 term
Has same form as Maxwell-Boltzmann distribution for $T = T_f \frac{a^2(t_f)}{a^2(t)}$

Neutrino Decoupling

As shown: $s = g_{*s} T^3 \propto a^{-3} \Rightarrow T \propto \frac{1}{a}$ as long as g_{*s} remains constant.
Neutrinos don't couple directly to the photon field, but are kept in equilibrium through neutral current weak interactions $e^+ + e^- \rightleftarrows \nu_e + \bar{\nu}_e$.
The neutrinos decouple at $T \sim 1\,MeV$ when the reaction rate becomes inefficient. Their temperature from this point on decreases as $T_\nu \propto 1/a$, but g_{*s} remains constant as the neutrinos still remain relativistic. The neutrino temperature remains the same like the photons', despite being decoupled, until $T_\gamma \sim 0.51\,MeV$: The reaction $e^+ + e^- \rightleftarrows \gamma + \gamma$ gets suppressed, which heats the photon field. Neutrinos conserve energy separately, so $T_\gamma > T_\nu$.
Their ratio follows from the entropy conservation law: [after = after heating]

$$T_{\nu,after} = T_{\nu,before} = T_{\gamma,before} \Rightarrow \frac{T_{\gamma,after}}{T_{\nu,after}} = \frac{T_{\gamma,after}}{T_{\gamma,before}} = \left[\frac{g_{*s}(T_{before})}{g_{*s}(T_{after})}\right]^{1/3}$$

$$\Rightarrow T_{\nu,after} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,after} \approx 1.95\,K$$

Freeze-Out of Stable Particles

Stable particles = particles with lifetime much larger than the age of the Universe.
The evolution of the particle number density is governed by the Boltzmann equ.:

$$\frac{df_i}{dt} = C_i[f] = \frac{\partial f_i}{\partial t} + \dot{p}\frac{\partial f}{\partial p} = \frac{\partial f_i}{\partial t} \underbrace{- Hp\frac{\partial f_i}{\partial p}}_{\text{Hubble drag term}} \Rightarrow \frac{dn_i}{dt} + 3Hn_i = \int C_i[f]\,d^3p$$

The Hubble drag term describes the dilution of the number density due to the expansion of the Universe.

Single Interaction Case

Consider a case where species i only takes part in the following two-body interaction:

$i+j \rightleftarrows a+b$ (e.g. $e^+ + e^- \rightleftarrows \gamma + \gamma$)

The evolution of the matter content of the Universe is described by a coupled set of Boltzmann equations: $C[f_i]$ depends on f_i and on the distribution function of all other species that interact with species i; Other species can still affect the distribution via their contributions to the general expansion of the Universe.

Define α: Production rate of i,j per unit volume $a+b \to i+j$

β: Annihilation rate of i,j per unit volume $i+j \to a+b$

$$\Rightarrow \quad \frac{dn_i}{dt} + 3Hn_i = \alpha(T)\, n_a n_b - \beta(T)\, n_i n_j = \int C_i[f]$$

Writing the same eqn for species j gives $(n_i - n_j)a^3 = \text{const.}$

In equilibrium: $C_i = 0 \iff \beta\, n_{i,eq}\, n_{j,eq} = \alpha\, n_a n_b$

$$\Rightarrow \quad \frac{dn_i}{dt} + 3Hn_i = \beta\left(n_{i,eq}^2 - n_i^2\right)$$

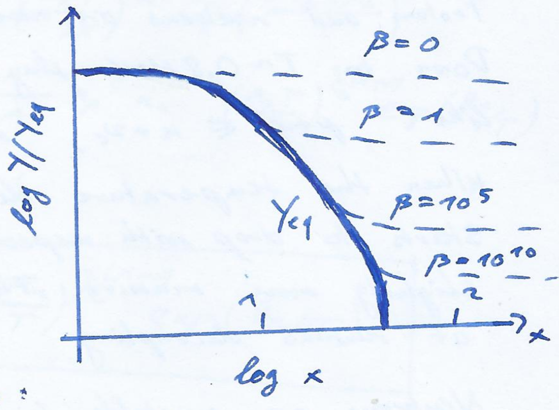
With $n_i \propto a^{-3}$ and $s \propto a^{-3}$, it is convenient to define

$$Y_i \equiv \frac{n_i}{s}, \quad x_i \equiv \frac{m_i}{T}$$

$$\Rightarrow \quad \boxed{\frac{x}{Y_{i,eq}}\frac{dY_i}{dx} = -\frac{\Gamma}{H}\, Y_{i,eq}\left(\frac{Y_i^2}{Y_{i,eq}^2} - 1\right)} \quad \text{The rate equation}$$

The rate eqn can be solved numerically for given m_i and β (through standard model of particle physics prediction)

In the relativistic limit $m_i \ll T \Rightarrow x \ll 1 \Rightarrow Y_i \to Y_{i,eq}$, which gives us initial conditions.



- A larger β means a higher reaction rate Γ (larger cross section), which implies that the species can maintain thermal equilibrium for a longer time and decouple later.
- For a β such that decoupling occurs in the relativistic regime, ($x \ll 1$), the final freeze-out abundance will be comparable to the number density of photons.
- For a sufficiently large β the particles remain in thermal equilibrium well into the non-relativistic regime ($x \gg 1$), causing an exponential suppression in their final freeze-out abundance.

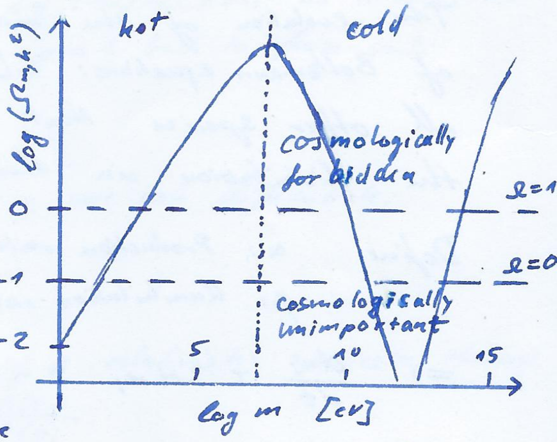
Estimate of freeze-out abundance: Assume freeze-out at $\Gamma = H$, which gives x_f, T_f.

$$\Rightarrow \quad Y_i = \begin{cases} \dfrac{45\,\zeta(3)}{2\pi^4}\dfrac{g_{i,eff}}{g_{*s}} & x \ll 1 \quad \text{hot relics} \\ \dfrac{90}{(2\pi)^{7/2}}\dfrac{g_i}{g_{*s}}\, x^{3/2} e^{-x} & x \gg 1 \quad \text{cold relics} \end{cases}$$

Where it was assumed that $Y_i(x \to \infty) = Y_{i,eq}(x_f)$ and $g_{i,eff} = g_i$ for bosons and $\frac{3}{4} g_i$ for

$\Omega_\nu \sim 10^{-5} h^2$ as measured from the CMB. Relativistic dark matter candidates should have same order of magnitude relic abundance, which $\ll \Omega_{DM} \approx 0.3$, and are therefore not interesting.

Mass Boundaries for Dark Matter



We can estimate boundaries for the mass of dark matter candidates using the relic abundance equation:

$$\rho_{i,0} = m_i n_{i,eq} = m_i \left(Y_{i,eq} s_0 \right)$$

using the present-day value of the entropy s_0.

The candidates are WIMPs: weakly interacting massive particles, like massive neutrinos and light supersymmetric particles. The limits are found by demanding $\Omega_{i,0} h^2 \lesssim 1$

i) Hot relics: $m_i \lesssim 94\,eV$

ii) Cold relics: $m_i \gtrsim 1.4\,GeV$; $\Omega_{i,0}$ decreases with increasing particle mass ($\propto m^{-2}$)
For particles with $m_i \gg m_Z \approx 100\,GeV$ (Z-boson) the cross section actually decreases with particle mass as m^{-2}, giving the upper limit $m_i \lesssim 3\,TeV$

Nucleosynthesis

The observed mass fraction of Helium is roughly a constant everywhere in the Universe, suggesting that most of the Helium is in fact primordial and that there is an universal process responsible for it.
It is possible that all heavier elements are synthesized in stars.

Protons and neutrons are non-relativistic at early times ($t \approx 10^{-6}s$, $T \approx 10^{13}K$)
Down to $T \sim 0.8\,MeV$ they maintain thermal equilibrium through weak interactions like $p + e^- \rightleftarrows n + \nu_e$, $n + e^+ \rightleftarrows p + \bar{\nu}_e$, which is up until the neutrinos decouple.
When the temperature decreases towards $\sim 1\,MeV$, the number density of neutrons starts to drop with respect to the number density of protons because it is slightly more massive. The ratio will eventually freeze out at $\exp\left(\frac{-\Delta m}{0.8\,MeV}\right) \approx 0.2$ at neutrino decoupling.

Neutrons are unstable to beta decay: $n \to p + e^+ + \bar{\nu}_e$ so that even after freeze-out the ratio $\frac{n_n}{n_p}$ continues to decrease: $X_n \equiv \frac{n_n}{n_n + n_p} \propto \exp\left[\frac{-t}{\tau_n}\right]$ where $\tau_n \approx 900s$ is the mean lifetime of neutrons.
The mean lifetime of nucleosynthesis is $\tau \approx 200-300s$, so the neutron decay is negligible during nucleosynthesis. Locking neutrons into nuclei forbids further β decay due to Pauli's exclusion principle.

Why does nucleosynthesis is only start at ~MeV?

→ First step of nucleosynthesis: $p+n \to D+\gamma$. D however has a binding energy $B_D \sim 1\,\text{MeV}$, so if the Universe is hotter than that, photons will destroy D. Nucleosynthesis needs to wait until the Universe cools down enough, until the decay $D+\gamma \to p+n$ is inefficient, which is ~1 MeV.

This analogy is valid for nuclei with $A<7$ which also have a binding energy ~ MeV.

At such low temperatures the number densities of protons and neutrons are already much too low to form heavy elements by direct multibody reactions such as $2n+2p \to {}^4He$. Nucleosynthesis must proceed through a chain of two-body reactions, which are hindered by the D-bottleneck. Even with $T<B_D$, the high energy tail of the Planckian can destroy D efficiently enough.

Nucleosynthesis can't produce any elements heavier than Lithium ($A=7$) due to the fact that there aren't any stable nuclei with $A=5$ or $A=8$. We can assume nuclei with $A>7$ are produced in stars, where the short life of ^8Be can capture another ^4He fast enough due to the high densities allowing for further nuclear reactions.

D, ^3H, ^3He, ^4He, ^9Be, ^7Li are nuclei that can be used for cosmological probes to measure $\Omega_{b,0}$ from predicted abundancies. All but ^4He are very rare, but D is a particularly good probe because it is easily destroyed and therefore its production in stars suppressed.

Calculation of Abundancies of Nuclei

For a non-relativistic particle species: $n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp\left(\frac{-m_A+\mu_A}{T}\right)$

with $\mu_A = Z\mu_p + (A-Z)\mu_n$ $\qquad = \frac{g_A A^{3/2}}{2^A} n_p^Z n_n^{A-Z} \left(\frac{m_n T}{2\pi}\right)^{\frac{3}{2}(1-A)} \cdot \exp(B_A/T)$

Define abundance: $X_A \equiv \frac{A n_A}{n_b}$ with $\sum_i X_{A_i} = 1$

$$\Rightarrow \boxed{X_A = \frac{g_A}{2} A^{5/2} \left[\frac{4\zeta(3)}{\sqrt{2\pi}}\right]^{A-1} X_p^Z X_n^{A-Z} \eta^{A-1} \left(\frac{m_n}{T}\right)^{\frac{3}{2}(1-A)} \exp(B_A/T)}$$

with $\eta = \frac{n_b}{n_\gamma} \approx 2.7 \times 10^{-8}\, \Omega_{b,0} h^2$

Short version: $X_A \propto \left(\frac{m_n}{T}\right)^{3/2(1-A)} \exp(B_A/T)$

Species with $A>1$ can only be produced in appreciable amounts ($\hat{=} X_A \lesssim 1$) once the temperature drops to

$$T_A \sim \frac{|B_A|}{(A-1)\left[|\ln\eta| + \frac{3}{2}\ln\left(\frac{m_n}{T}\right)\right]} \sim 0.1\,|B_A|$$

Hydrogen Recombination

Immediately after primordial nucleosynthesis ($T \sim 0.1\,\text{MeV}$), the Universe consists mainly of protons, ^4He, electrons, photons and decoupled neutrinos. Baryons and electrons are non-relativistic.

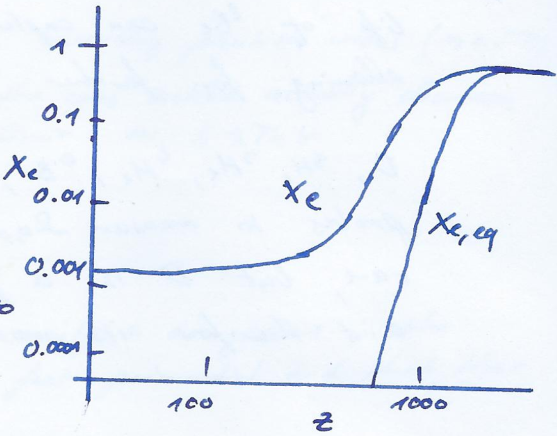
As soon as the temperature of the Universe drops below $B_H = 13.6\,\text{eV}$, electrons and protons start to combine to form hydrogen atoms. At $T \sim 10\,\text{eV}$, $\Gamma_{rec} > H$ and the particles are still in thermal equilibrium.

The ionisation fraction X_e in thermal equilibrium is given by the Saha equation

$$\boxed{\frac{1 - X_{e,eq}}{X_{e,eq}^2} = \sqrt{\frac{32}{\pi}}\, \zeta(3)\, \eta \left(\frac{m_e}{T}\right)^{-3/2} \exp\left(\frac{B_H}{T}\right)} \quad \text{with } B_H \approx -13.6\,\text{eV}$$

Assuming thermal equilibrium holds, we can compute T_{rec} and z_{rec}:
$z_{rec} \approx 1300$, $T_{rec} \approx 0.3\,\text{eV} \ll B_H$ because of Planckian tail.

Main reactions of recombination:



i) direct recombination
but: A Lyman continuum photon is produced with $E_\gamma > 13.6\,\text{eV}$ which can ionize other hydrogen, so no net recombination

ii) Capture to excited state
e^- cascades down to ground state and emit photons with $E_\gamma < 13.6\,\text{eV}$, but multiple absorptions still lead to re-ionisation.

iii) Two-photon decay
From meta-stable $H^{*\,2s} \to H^{1s} + \gamma + \gamma$ two photons must be emitted to conserve angular momentum and energy. The energies of the emitted photons may fall below the ionisation threshold, but this process has a slow reaction rate.

iv) Redshift
Once redshifted to a lower energy, the Lyα photons produced in the cascade will no longer be able to excite hydrogen atoms from their ground state.

Plot: $X_{e,eq}$ is what we computed, X_e is carefully computed numerically. It shows that still free electrons remain and freeze-out.

Decoupling and Origin of the CMB

Charged particles and photons interact through Thomson scattering. Scattering with ions can be neglected since they have a much smaller interaction rate $\Gamma_{Th} = n_e \sigma_T c$ with $\sigma_T = \frac{8\pi}{3}\left(\frac{q e^2}{m_e c^2}\right)^2 \approx 6 \cdot 10^{-25}\,\mathrm{cm}^2$

We can estimate z_{dec} and T_{dec} using the Saha equation to compute n_e for $X_e \ll 1$, which gives us $\Gamma_{dec} = H$ to obtain z_{dec} and H_{dec}.

$\Rightarrow$ $T_{dec} \approx 0.26\,\mathrm{eV}$, $z_{dec} \approx 1100$

A more accurate derivation comes from defining optical depth τ of Thomson scattering from an observer at $z=0$ to a surface at redshift z_{dec}:

mean free path $= \frac{1}{\text{absorption coefficient}}$ $\Rightarrow$ $l = \frac{1}{\alpha_S} = \frac{1}{n_e \sigma_{Th}}$

Remember $\frac{dI}{ds} = -\alpha_S I \longleftrightarrow dI = -d\tau I \longleftrightarrow d\tau = \alpha_S ds$

$$\Rightarrow \tau = \int_0^s \alpha_S\, ds' \approx \alpha_S \int_0^s ds' = c\int n_e \sigma_{Th}\, dt = c\int X_e n_\gamma \eta\, \sigma_T \frac{dt}{dz} dz \stackrel{!}{=} 1$$

$\Rightarrow$ $z_{LS} = 1067$ [Last surface]

No information carried by photons originating at $z \gtrsim 1100$ can reach the earth, as the photons will be scattered many times. The photons are freestreaming, but can still keep the free electrons to same temperature. To ~~find~~ when the temperatures decouple

Now look at Compton scattering. Comparing mean free paths:

$$l_{e^-} = \frac{1}{n_\gamma \sigma_T}, \quad l_\gamma = \frac{1}{n_e \sigma_T} \Rightarrow \frac{l_e}{l_\gamma} = \frac{n_e}{n_\gamma} = X_e \eta \ll 1 \Rightarrow l_\gamma \gg l_e$$

$\Rightarrow$ It is much easier for photons to change the energy distributions of electrons than the other way around.

To compute redshift of matter decoupling via Compton effect:

Consider rate of change of matter energy density ε_m due to compton scattering:

$$\frac{d\varepsilon_m}{dt} = n_e n_\gamma \sigma_T c \Delta E$$

With $\Gamma_{\gamma \to e} = \frac{1}{\varepsilon_m}\frac{d\varepsilon_m}{dt}$ $\Rightarrow$ $\Gamma = H$ $\Rightarrow$ $z_{dec,\gamma} \approx 150$

Much lower than decoupling defined by optical depth! The electron temperature can remain coupled to that of photons even if only a tiny fraction are scattered by the electrons. This is thermal decoupling.

Compton scattering can significantly modify the energy distribution of photons (photon fluid) at $z \gtrsim 5 \times 10^4$ ($\Gamma_{e \to \gamma}$) if the $T_e > T_\gamma$ for some reason. Since Compton scattering does not change the number of photons, this process alone moves the photon fluid from a Planckian to a Bose-Einstein distribution with $\mu \neq 0$ ($\equiv \mu$-distortion). $\Rightarrow$ Increase of electron temperature leads to observable μ-distortion in the CMB.

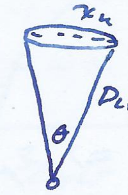
Processes that produce photons are Bremsstrahlung, double Compton emission (is dominant)

Inflation

Problems of Standard Cosmology

i) Horizon Problem

Rough estimate of the particle horizon at the time of decoupling (at time of CMB origin): $z \sim 1100$



Assume for simplicity $\Omega_{m,0} = 1$

$$\chi_{h,dec} = \int_0^{a_{dec}} \frac{c\,dt}{a} = \frac{c}{H_0} \int_{\infty}^{z_{dec}} \frac{dz}{(1+z)^{3/2}} \simeq 180\ h^{-1}\ \mathrm{Mpc}$$

The (comoving) distance to the surface of last scattering is

$$D_{LS} = \int_{z_{dec}}^{0} \frac{dz}{(1+z)^{3/2}} = 5820\ h^{-1}\ \mathrm{Mpc}$$

So that the particle horizon at decoupling subtended an angle

$$\theta \sim \frac{180}{5820} \approx 0.03 \sim 1.8^\circ \ll 360^\circ$$

That implies that many regions that we observe of the CMB couldn't have been in thermal equilibrium / causal contact, yet still have extremely similar temperatures. The CMB shows nearly uniform temperature, implying that all regions were causally connected.

Note: For a radiation or matter dominated universe, the particle horizon χ_h converges $\Leftrightarrow$ encloses some finite region. $\Leftrightarrow$ not all observers are causally connected. χ_h diverges for Λ-dominated Universe.

ii) Monopole Problem

Grand Unified Theories predict magnetic monopoles = massive particles with net magnetic charge as a result of spontaneous symmetry breaking during the phase transition at the threshold at which the GUT is expected. The expected density at formation is about one per horizon volume at that time with $m \sim T_{GUT}$.

This predicts an cosmological parameter $\Omega_{mono,0} \sim 10^{16}\,\Omega_{r,0} \sim 2.5 \times 10^{11}\,h^{-2}$ which would absolutely dominate the present-day matter density and is in fatal conflict with observations of the flat Universe.

iii) Flatness problem

Age, density and size of the Universe are assumed to arise as initial conditions at the Planck time, when the Universe emerged from the quantum gravity epoch.
The problem arises if $\Omega = \Omega_m + \Omega_\Lambda + \Omega_r$ differs mildly from unity at the present time, because such a Universe requires extreme fine-tuning of Ω in the Planck time:

$$\frac{\Omega_{Pl}^{-1} - 1}{\Omega_0^{-1} - 1} \sim \frac{T_0}{T_{eq}} \left(\frac{T_{eq}}{T_{Pl}}\right)^2 \sim 10^{-60}$$

$\Rightarrow$ Ω_{pl} is 60 orders of magnitude closer to Unity than the present-day value Ω_0. For example: If $\Omega_0 = 0.1$, then $\Omega_{pl} = 1 - 10^{59}$, which is way to exact to be possible.
Other solution: postulate $\Omega = 1$, but this has no plausible physical explaination.

iv) Structure formation problem

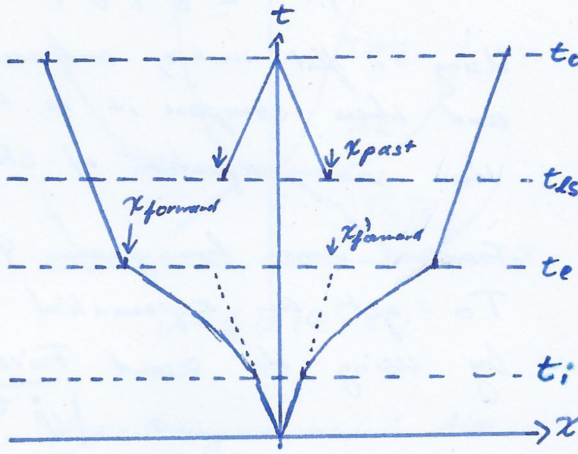
The largest gravitationally bound structures, galaxy clusters, have size $\sim 10\,Mpc$ and binding energy $E/c^2 \sim 10^{-5}$. They are presumed to have grown via gravitational instabilities from small initial perturbations, where the gravity amplified small perturbations or inhomogeneities in the initial conditions.
Parts of a cluster needed to be in causal contact, which can happen only at $z \leq 10^6$, but at this high redshift there is no way to achieve these high binding energies.
$\rightarrow$ We need a way to start perturbations much earlier but inflate them fast so that they acquire a much larger scale very early and still remain causally connected.

The Concept of Inflation

Inflation describes an early phase of exponential growth of the scale factor after $T \sim T_{GUT}$. It is phenomenologically similar to an Λ-dominated Universe, but inflation needs to end.
After inflation, the Universe evolves as in the standard Big Bang cosmology. Also the energy density of matter and radiation is nearly zero ($\sim 1/a^3$ with a grown exponentially) but the temperature can be high because of energy released by the inflation (reheating). Reheating is necessary because otherwise the temperature is too low for the creation of standard model particles.

How Inflation solves Problems

i) Solution of the particle horizon problem



Suppose inflation starts at some $t=t_i$ and ends with some $t=t_e$ with $\Delta t = t_e - t_i$, during which the forward light cone χ_f expands exponentially.

The past light cone with $t > t_e$ is not affected by this expansion.

If Δt is sufficiently large, the size of the forward light cone can be much larger than the size of the past light cone.

For a model with $a \propto e^{Ht}$ and comparing the future and past light cone $\chi_f > \chi_p$, we get that $\Delta t \gtrsim \frac{60}{H} \Leftrightarrow \frac{a_e}{a_i} = e^{60}$ in order to solve the horizon problem.

ii) Solution of the Flatness Problem

Using the same concept as for the horizon problem, we get:

$$\frac{\Omega^{-1}(t_e)-1}{\Omega^{-1}(t_i)-1} = \left(\frac{a(t_i)}{a(t_e)}\right)^2 \lesssim 10^{-52}$$

Even if $\Omega(t_i)$ deviates substantially from unity, at the end of inflation $\Omega(t_e) \simeq 1$ to very high accuracy. The same number of e-foldings also solves the flatness problem. Inflation "flattens the Universe out".

iii) Solution of the monopole problem

Assuming monopoles are produced before inflation, their number density will be decreased by $\propto a^3 \sim \exp(60)^3 \simeq 10^{78}$ making the contribution of monopoles to the cosmic density completely negligible.

Realisation of Inflation

Requirements:
- Exponential growth of scale factor
- Needs to stop after some time
- Need reheating that creates matter and radiation at the end of inflation

This can be realised in a natural way with a scalar field.

Ansatz: At $t \simeq t_{GUT}$, the scalar field $\varphi(x)$ is everywhere in the Universe with $\langle\varphi\rangle = 0$, and the potential $V(\varphi)$ goes from a finite value to zero. Inflation occurs as $V(\varphi)$ decays from initial $V(\varphi)$ to $V(\varphi)=0$. If there is a coupling between the inflaton field and the thermal bath, the required reheating is possible.

$\langle\varphi\rangle = 0$: Assuming Gaussian

For a scalar field, the stress-energy tensor is given by

$$T^{\mu\nu} = \partial^\mu \varphi \, \partial^\nu \varphi - g^{\mu\nu} \mathcal{L} \qquad \text{with } \mathcal{L} = \tfrac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \quad \text{Lagrange density}$$

Using a flat metric, and assuming φ is homogeneous (small inhomogeneities) $\partial_i \varphi = 0$ and then compare it to the stress-energy tensor of a perfect fluid, we get the same equation of state for the cosmological constant: $P = -\varrho$

For a non-homogeneous φ, this equation of state is not valid anymore. To get an exponential expansion, we demand the "slow roll approximation" by using the second Friedmann equation:

$$\boxed{\dot{\varphi}^2 \ll V(\varphi)} \qquad \text{slow roll approximation}$$

The field

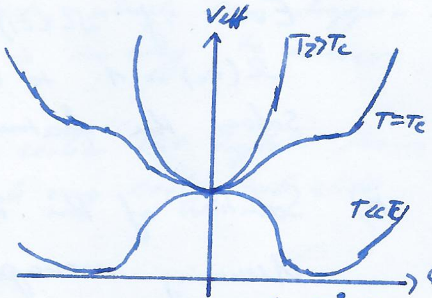
$$\boxed{2[\dot{\varphi}^2 - V(\varphi)] < 0} \qquad \text{condition for accelerated expansion}$$

The field is required to move slowly over the potential well ($\equiv$ small slope). Inflation has to stop at the zero-point energy minimum well because then the slow-roll condition isn't satisfied anymore as $V \to 0$. The lost energy is transferred to the matter-energy content (reheating).

Constraints on the potential V can be obtained by Euler-Lagrange equations on $S = \int \mathcal{L} \sqrt{-g} \, d^4x$, giving $\ddot{\varphi} \ll \frac{dV}{d\varphi}$; $\left|\frac{dV}{d\varphi}\right| \ll 1$ and $\frac{1}{V} \frac{d^2V}{d\varphi^2} \ll 1$

Models of Inflation

i) Old Inflation



$V_{eff} = V(\varphi) + \frac{1}{2} \lambda T^2 \varphi^2$

A scalar field is initially trapped in a false vacuum (local minimum) at $\varphi = 0$ which at some point undergoes spontaneous symmetry breaking to its true vacuum state via a first order phase transition.

Starting with only one minimum at $\varphi = 0$ and develops two other minima at $T = T_c$ as the temperature decreases. When the temperature drops below T_c, the field gets trapped in the false vacuum at $\varphi = 0$, when the slow-roll condition is also satisfied and inflation can progress.

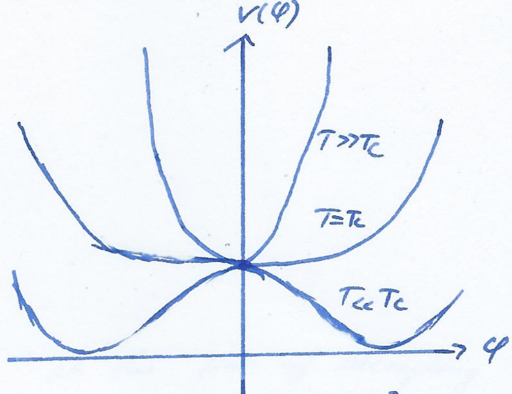
Inflation ends when when thermal fluctuations or quantum tunneling moves φ across the barrier so that it can proceed towards the true vacuum.

During this transition $V(\varphi = 0)$ is released and can be used for reheating.

If the system stays in the false vacuum state sufficiently long, the Universe can be inflated by a sufficiently large number of e-foldings.

But the old inflation has a 'graceful exit' problem: Because the transition is of first order, it proceeds through the nucleation of bubbles of true vacuum, which must grow in a causal way. The model will predict too large inhomogeneities to match the CMB and too little reheating. Inflation either goes on indefinitely or is too short.

ii) New Inflation



$V = \frac{1}{2}\lambda(\varphi^2 - \sigma^2)^2$

At high temperatures, V_{eff} has a single minimum at $\varphi = 0$ which disappears and becomes a local maximum below $T = T_c$, while two new minima develop. This time, the field configuration evolves smoothly, making the spontaneous symmetry breaking occur via a second-order phase transition.

New inflation doesn't have a graceful exit problem, but in order to achieve a large enough e-folding, fine-tuning is needed. Also it requires unphysical conditions with $\sigma \gg m_{pl}$ so that we obtain inflation.

iii) Chaotic Inflation

No phase transition is involved here, so that no initial thermal bath is required. At any given point, the initial field configuration is assumed to be set by some chaotic processes. Inflation only occurs in those regions where the conditions for inflation are met, other regions never inflate. In a region where inflation persists for a sufficiently long period, the boundary of this region can be blown out of the current particle horizon, leaving a Universe in which the initial inhomogeneities generated by the chaotic processes have no observable consequences.

Problems:
- not natural conditions: $m \ll m_{pl}$ at $t \approx t_{pl}$ needed
- monopole problem: If inflation starts too early, it might also finish too early, which might also be an inconsistency.