Cosmological Perturbations

Metric Perturbations

Consider small perturbations $h_{\mu\nu}$ to the flat FRW metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

It is useful to decompose the perturbed metric in a scalar, vectorial and tensorial part. (Compare to vector fields in 3D-Euclidean space: $\vec{V} = \vec{V}_\perp + \vec{V}_\parallel$ with $\vec{\nabla}\cdot\vec{V}_\perp = 0$ and $\vec{\nabla}\times\vec{V}_\parallel = 0 \Rightarrow \vec{V}_\parallel = \vec{\nabla}\alpha$, $\vec{V}_\perp = \vec{\nabla}\times\vec{A}$)

$$\Rightarrow \quad h_{00} = -E$$
$$h_{i0} = a\left[\frac{\partial F}{\partial x^i} + G_i\right]$$
$$h_{ij} = a^2\left[A\delta_{ij} + \frac{\partial^2 B}{\partial x_i \partial x_j} + \frac{\partial C_i}{\partial x_j} + \frac{\partial C_j}{\partial x_i} + D_{ij}\right]$$

A, B, E, F scalars, C_i, G_i divergenceless vectors and D_{ij} traceless, symmetric, divergenceless tensor.
$h_{\mu\nu}$ has 10 degrees of freedom.

The decomposition theorem states that a scalar, vector and tensor perturbation modes do not couple to first order, i.e. they evolve independently.
$\Rightarrow$ Einstein's field equations can be solved for each mode separately.

Gauge Transformations

Consider a spacetime coordinate transformation $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$
A coordinate transformation affects both the coordinates of perturbed and unperturbed fields. For gauge transformations, we keep the coordinate system but change the metric tensor in a way that leaves the physics invariant.

Gauge transformations correspond to changes in the metric perturbations:

$$\Delta h_{\mu\nu} = h'_{\mu\nu} - h_{\mu\nu} = -\frac{\partial \varepsilon^\lambda}{\partial x^\mu}\bar{g}_{\lambda\nu} - \frac{\partial \varepsilon^\kappa}{\partial x^\nu}\bar{g}_{\kappa\mu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\lambda}\varepsilon^\lambda$$

$$\Rightarrow \left.\begin{aligned} \Delta h_{00} &= -2\frac{\partial \varepsilon_0}{\partial t} \\ \Delta h_{i0} &= -\frac{\partial \varepsilon_i}{\partial t} + \frac{2}{a}\frac{da}{dt}\varepsilon_i - \frac{\partial \varepsilon_0}{\partial x^i} \\ \Delta h_{ij} &= -\frac{\partial \varepsilon^i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} + 2a\frac{da}{dt}\delta_{ij}\varepsilon_0 \end{aligned}\right\} \text{transformation of metric perturbations under gauge transformations}$$

By decomposing $\varepsilon_i = \frac{\partial \varepsilon^{(s)}}{\partial x^i} + \varepsilon_i^{(v)}$ into a scalar and vectorial part, we can relate the decomposed parts of the perturbation to it:

$$\Delta A = \frac{2}{a}\frac{da}{dt}\varepsilon_0$$
$$\Delta B = -\frac{2}{a^2}\varepsilon^s$$
$$\Delta E = 2\frac{d\varepsilon_0}{dt}$$
$$\Delta F = \frac{1}{a}\left(-\varepsilon_0 - \frac{d\varepsilon^s}{dt} + \frac{2}{a}\frac{da}{dt}\varepsilon^s\right)$$

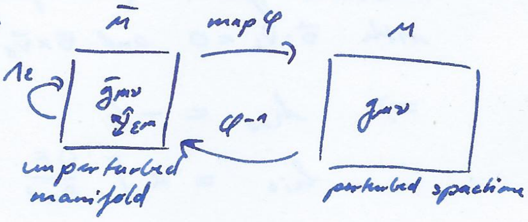
Choice of Gauge:

i) Newtonian gauge: $g_{00} = -1-2\Psi, \quad g_{0i} = 0, \quad g_{ij} = a^2\delta_{ij}\left[1+2\Phi\right]$

ii) Synchronous gauge: $g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = a^2\left[(1+A)\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j}\right]$

The gauges can be "chosen" by choosing ε^0, ε^s and ε^v appropriately.

Geometrical Interpretation of Gauge Transformation:



To define perturbation, we need to compare two manifolds to each other. Then we can define $h_{\mu\nu} \equiv (\varphi^{-1} g_{\mu\nu}) - \bar{g}_{\mu\nu}$

The choice of φ is not strictly constrained, which gives rise to the freedom of gauge choice. There are many permissible mappings φ so that the perturbation is small.

Consider map $\Lambda : \bar{M} \to \bar{M}$ induced by a vector field ε^μ on $\bar{M}$, i.e. $x^\mu \to x^\mu + \varepsilon^\mu$

Then $(\varphi \circ \Lambda^{-1})$ is a new mapping between M to $\bar{M}$ which induces new metric perturbations:

$$h^{(\varepsilon)}_{\mu\nu} = \left[(\varphi\circ\Lambda)^{-1} g\right]_{\mu\nu} - \bar{g}_{\mu\nu} = \left[\Lambda^{-1}\circ\varphi^{-1} g\right]_{\mu\nu} - \bar{g}_{\mu\nu} \neq h_{\mu\nu} = (\varphi^{-1} g_{\mu\nu}) - \bar{g}_{\mu\nu}$$

All possible infinitesimal Λ_ε correspond to possible gauge transformations.

A coordinate transformation corresponds to relabelling of the coordinates. A gauge transformation is a choice of mapping between M and $\bar{M}$: The problem is that some gauge transformations look like physical perturbations, but aren't. You can think of gauge transformations as redefining the reference potential.

Einstein Equations

Using the Newtonian gauge: $g_{00} = -1-2\Psi, \quad g_{i0} = 0, \quad g_{ij} = (1+2\Phi)a^2\delta_{ij}$

The Einstein equations can be decomposed in perturbed and unperturbed parts:

$$(\bar{G}^\mu{}_\nu + \Delta G^\mu{}_\nu) = 8\pi G(\bar{T}^\mu{}_\nu + \Delta T^\mu{}_\nu)$$

We only need 2 equations for Φ and Ψ, and take the easiest choices: $G^0{}_0$ and the traceless part of $G^i{}_j$ (by applying a projection operator)

Perturbed Boltzmann Equation

Non-Relativistic Boltzmann Equation

The distribution function $f(x^i, p^i, t)\, d^3x\, d^3p$ describes the number of particles in the phase space volume $d^3x\, d^3p$ at the time t centered on x^i, p^i.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt}$$

Moments:

$$n = \int d^3p\, f$$

$$\rho_m = m \int d^3p\, f$$

$$v = \frac{1}{n}\int d^3p\, \frac{\vec{p}}{E} f$$

$$\rho_E = \int d^3p\, E f$$

$$P = \int d^3p\, \frac{p^2}{3E} f$$

When collisions are included: $\frac{df}{dt} = C[f]$

$$C[f] = \int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \times$$ sum over all momentum space/ interactions

$$\times\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \times$$ conserves momentum

$$\times\, \delta^3(E_1 + E_2 - E_3 - E_4) \times$$ conserves energy

$$\times\, |M|^2 \times$$ Scattering matrix

$$\times\, (f_3 f_4 - f_1 f_2)$$ sources - sinks

In equilibrium, the distribution function for non-relativistic particles is Maxwell-Boltzmann: $f \propto e^{-E/T}$

Relativistic Boltzmann Equation

In GR: 7 dof with $p^\mu p_\mu = -m^2$

Boltzmann equation is the same as for non-relativistic.

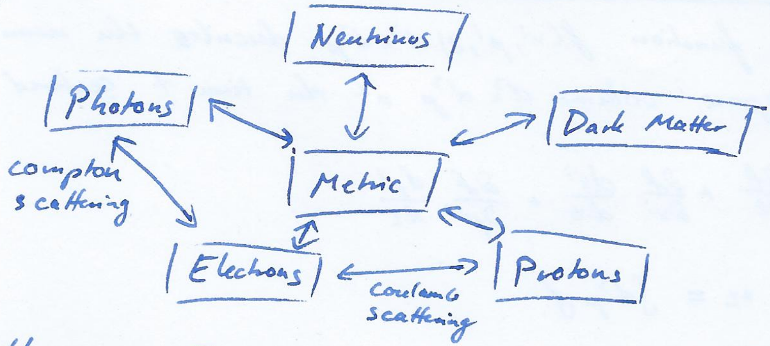
Moments are the same with factor $\frac{1}{(2\pi)^3}$

$$T^{\mu\nu} = \frac{1}{(2\pi)^3}\frac{1}{E}\int d^3p\, p^\mu p^\nu f$$

Collision term is also the same with factor $\frac{1}{E_i}\frac{1}{(2\pi)^3}$

Interactions between particles

The distributions of the particle species are coupled to each other through interactions:



Einstein - Boltzmann Equations

$$\frac{df_i}{dt} = C[f_i]$$

$$g_{00} = -(1+2\Psi), \quad g_{i0} = 0, \quad g_{ij} = \delta_{ij} a^2 (1+2\Phi)$$

$$G_{\mu\nu} = \overline{G_{\mu\nu}} + \Delta G_{\mu\nu} = 8\pi G (\overline{T_{\mu\nu}} + \Delta T_{\mu\nu})$$

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} p^\mu p^\nu f$$

Collisionless Boltzmann Equation for Photons

Keeping terms up to first order. For photons: $p^\mu p_\mu = 0$

$\frac{\partial f}{\partial x^i}$, $\frac{\partial f}{\partial p^i}$, $\frac{\partial \hat{p}^i}{\partial t}$ are already first order for an isotropic, homogeneous universe

Use the geodesic equation.

Assume Bose - Einstein distribution for first order: $f = \frac{1}{\exp\left[\frac{p}{T(1+\Theta)}\right] - 1} = f^{(0)}(p, T(1+$

with $\Theta = \frac{\delta T}{T} = \Theta(\vec{x}, \hat{p}, t)$.

Insert into Boltzmann equ, then separate terms by order, get 2 equations.

$\left.\frac{df}{dt}\right|_{zero} = 0$: There is no zeroth order collision term, because the zero-order distribution function is set precisely by the requirement that the collisions vanish, as equilibrium is expected.

For photons, the dominant scattering process is Compton scattering.
For non-relativistic electrons, we can approximate the scattering matrix to be Thomson scattering: $|M|^2 = 8\pi \sigma_T m_e^2 \doteq const$, neglecting polarisation and angle dependance of $|M|^2$.

Boltzmann Equation for CDM

Here $p_\mu p^\mu = -m^2$, always non-relativistic. By solving, we see that there is a factor $\frac{p}{E}$ in the Boltzmann function which suppresses free streaming.
Integrating the first moment and separating the equations by orders gives two equations.

The Boltzmann Equation for Baryons

= for electrons and baryons. Electrons and protons are coupled by Coulomb scattering. Their Coulomb scattering rate is much larger than the expansion rate at all times of interest and forces the electrons and protons overdensities to a common value:

$$\delta_b = \frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \,, \qquad \vec{v}_b \equiv \vec{v}_e \simeq \vec{v}_p$$

Starting point: Unintegrated equations for electrons and protons:

$$\frac{df_e}{dt} = \langle C_{ep} \rangle_{QQ'q'} + \langle C_{e\gamma} \rangle_{pp'q'}$$

$$\frac{df_p}{dt} = \langle C_{ep} \rangle_{qq'Q'}$$

We get the equations by taking the zeroth and first moment of the Boltzmann equ up to first order.

Einstein Equations

First we compute the stress-energy tensor from the distribution function:

$$T^{\mu\nu}_{\text{universe}} = \sum_{s \in \text{species}} T^{\mu\nu}_{(s)} \qquad T^{\mu\nu}_{(s)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_s} p^\mu p^\nu f_s$$

Again we only look at the $T^0{}_0$ and the $T^i{}_j$ components to obtain 2 equs. We only considered the longitudinal, traceless part of $G^i{}_j$ which can be extracted by contracting $G^i{}_j$ with the projection operator $\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^j_i$, which kills all terms $\propto \delta^i_j$.

The gravitational potentials Φ, Ψ are equal and opposite unless the photons have appreciable quadrupole moments.

Inhomogeneities

Simplified Einstein-Boltzmann Equations

Simplifying assumptions: Focus on dark matter ($\Omega_b \ll \Omega_m$), thus neglect baryons and neutrinos. Furthermore neglect all $l > 2$ moments for photons and polarisation. This is an ok approximation for early times $a \ll a_{eq}$, while in late = matter dominated times, radiation has little impact on the metric, its only connection to dark matter.

$\rightarrow \dot{\tau} \propto n_e \propto x_e \approx 0 \ , \ \phi \simeq \psi$

Simplified Equations:

$$\left.\begin{aligned} \dot{\Theta}_0 - k\Theta_1 &= -\dot{\phi} \\ \dot{\Theta}_1 - \frac{k}{3}\Theta_0 &= \frac{k}{3}\phi \end{aligned}\right\} \text{by taking moments of Boltzmann eq}$$

$$\dot{\delta} + ikv = -3\dot{\phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\phi$$

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} + \frac{\dot{a}}{a}\phi\right) = 4\pi G a^2 (\rho_{dm}\delta + 4\rho_r\Theta_0)$$

(Second Einstein eqn only contains radiation and neutrinos)

with $v \sim \frac{p}{m}$, $\rho_{dm} = \rho_{dm}^{(0)}(1+\delta)$, ϕ from metric, $\Theta = \frac{\delta T}{T}$, $\Theta_l = (-1)^l \int_{-1}^{1} d\mu\, P_l(\mu)\Theta(\mu)$

P_l Legendre polynomes, $\mu = \frac{\vec{p}\cdot\vec{k}}{p}$

Initial Conditions

Inflation provides a mechanism to produce primordial inhomogeneities, where microscopic quantum fluctuations are promoted to macroscopic fluctuations.

The perturbations are best described in terms of the Fourier modes. The mean of a Fourier mode $\langle\phi(k)\rangle = 0$. The perturbations to one Fourier mode are uncorrelated with those to another. A given mode however has nonzero variance.

$$\langle\phi(\vec{k})\,\phi^*(\vec{k}')\rangle = (2\pi)^3 P_\phi(k)\,\delta^3(\vec{k}-\vec{k}')$$

For scalar perturbations: $P_\phi \propto k^{n-4}$, $n \simeq 1$

Primordial fluctuations are quantified by a power spectrum which gives the power of the variation as a spatial scale.

Solving Simplified Einstein-Boltzmann equations

There are no known general analytic solutions. $\rightarrow$ Do it numerically, or consider limits.

First identify scale of the problem:

i) particle horizon $\eta = \int \frac{da}{a}\frac{1}{aH}$, $\frac{1}{aH}$ = comoving Hubble radius

ii) matter-radiation equality $a_{eq} \approx 1.5 \cdot 10^{-4}$

$\Rightarrow$ Superhorizon modes: $k \ll \eta^{-1}$ not subject to causal physics

Subhorizon modes: $k \gg \eta^{-1}$

Small scale modes: $k \gg k_{eq}$ cross horizon while $a < a_{eq}$ $\qquad k_{eq} \simeq \frac{1}{a_{eq}H(a_{eq})} \approx 0.073\ \mathrm{Mpc}^{-1}\Omega_m h^2$

Large scale modes: $k \ll k_{eq}$ cross horizon while $a > a_{eq}$

Comoving wavelength $\lambda \sim \frac{2\pi}{k}$ is constant over time, but the horizon grows. Once it enters the horizon, causal physics begin to [illegible]

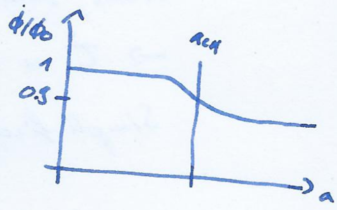
Large Scales

Large scales are superhorizon scales: $k\eta \ll 1$. In this limit, drop all terms that contain k. We get $\delta - 3\Theta_0 = \text{const}$, which is given by the initial conditions:

- const $= 0$ adiabatic perturbations
- const $\neq 0$ isocurvature perturbations (not successful models)

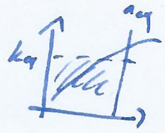
Solution: Early times $\Phi \to \Phi_0$

Late times $\Phi \to \frac{9}{10}\Phi_0$



Φ is constant in radiation and matter dominated epochs and drops 10% slowly in the transition.

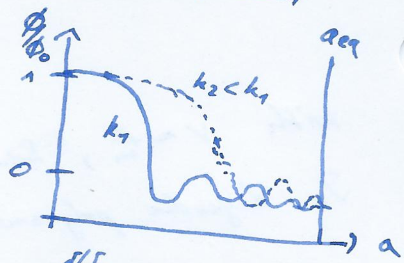
Small Scales and Early Times

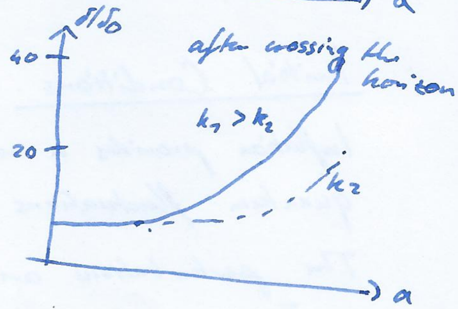


Early times: $a \ll a_{eq}$

In radiation dominated Universe, the potential is determined by the perturbation to the radiation. The dark matter perturbations are influenced by the potential, but do not themselves influence it significantly. $\Rightarrow$ DM terms drop out

$\rightarrow$ First solve eqns for Θ_0, Θ_1 and Φ, then eqn for matter evolution using the potential as an external driving force.





Get $$\boxed{\Phi(\eta) = 3\Phi_0 \frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}}$$

$\Rightarrow$ As soon as a mode enters the horizon ($k\eta = 1$) its potential starts to decay $(k\eta/\sqrt{3})^{-3}$. After decaying, it oscillates.

Smaller k enter the horizon later.

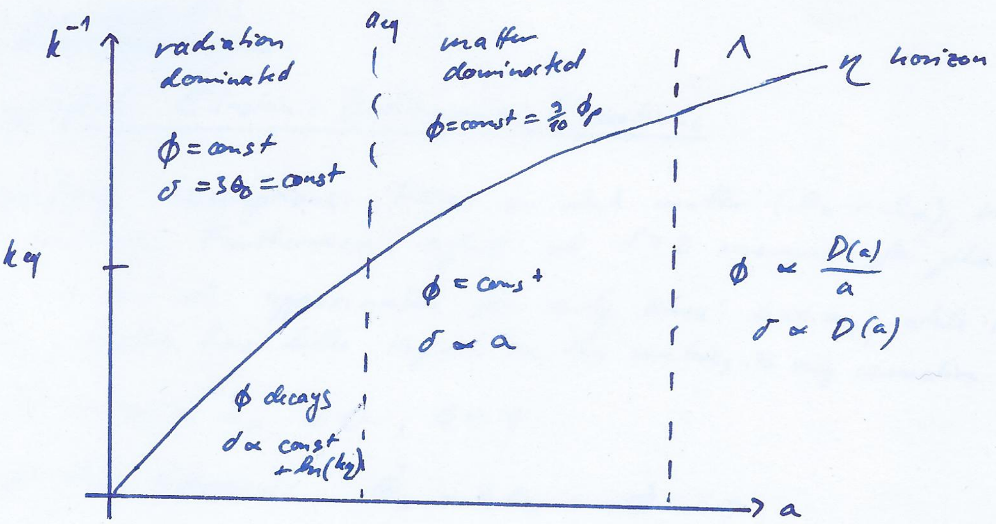
The evolution of the matter perturbation can now be computed, giving $\delta \propto \text{const} + \ln(k\eta)$

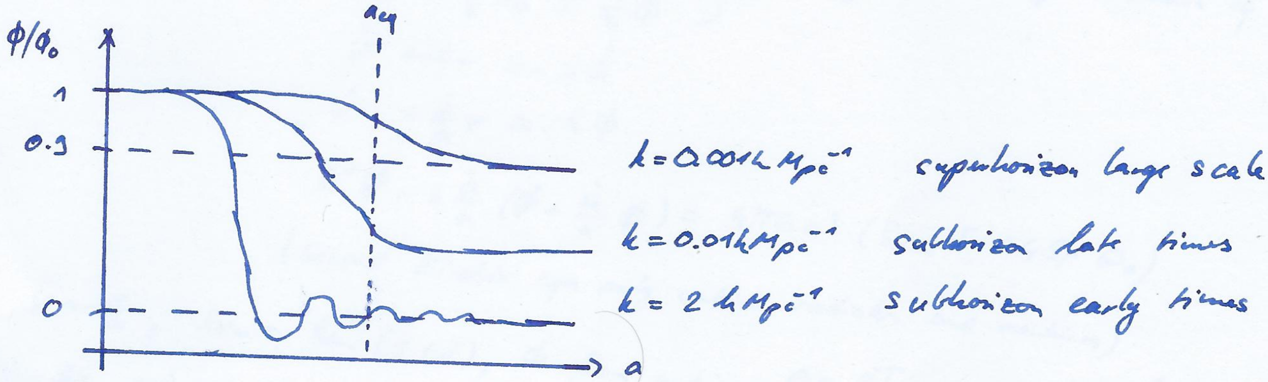
Late Times and Subhorizon Case

$k\eta \gg 1$, $a \gg a_{eq}$. Neglect Θ (radiation negligible), neglect $\frac{1}{k}$ terms in Einstein equation.

Solution: $\delta \propto H \propto a$ in matter dominated era $\Rightarrow \delta \propto t^{-3/2}$

$\Phi \propto \frac{\delta}{a} = \text{const}$ (from Einstein eqn)





- The earlier the mode enters the horizon $\Leftrightarrow$ The smaller the k, the more it decays
- ϕ only evolves during transitions and in Λ-dominated era

Overview

Gravitational instability is most likely responsible for structure formation in our Universe.
Schematically: $\ddot{\delta} + [\text{Pressure} - \text{Gravity}]\delta = 0$

Gravity increases overdensities, but random thermal motion in overdense regions causes net particle loss. $\Rightarrow$ For strong pressure, inhomogeneities don't grow, but oscillate.

At late times, all modes evolve identically. To relate the potential during these times to the primordial ϕ_p set up during inflation:

$$\phi(\vec{k},a) = \Phi_p(\vec{k}) \times \underbrace{\{\text{Transfer function }(k)\}}_{\text{horizon crossing, radiation/matter transition}} \times \underbrace{\{\text{Growth function }(a)\}}_{\text{wavelength-independent growth at late times}}$$

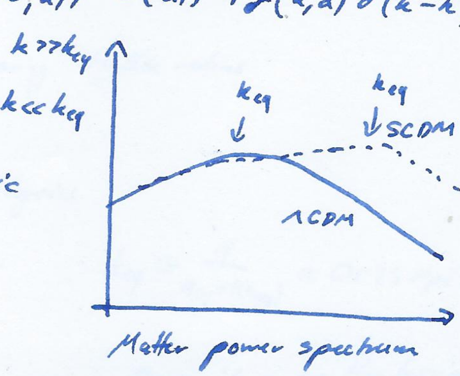
ϕ and δ are related through the Einstein equations.

We can define a matter power spectrum $\langle \delta(\vec{k},a)\,\delta^*(\vec{k}',a)\rangle = (2\pi)^3 P_\delta(k,a)\,\delta^3(\vec{k}-\vec{k}')$

We have $P_\delta(k) \propto k^n T(k)^2 \propto \begin{cases} \propto k^n & k \gg k_{eq} \\ \propto k^{n-4} \ln^2\left(\frac{k}{8k_{eq}}\right) & k \ll k_{eq} \end{cases}$ peaks at $k = k_{eq}$

The matter power spectrum can be probed using spectroscopic galaxy surveys.

Different models of dark energy predict different growth factors, giving different k_{eq}, where the peak is. This was a strong piece of evidence for dark energy.



Matter power spectrum

Anisotropies

Large Scales

From Einstein-Boltzmann eqns for superhorizon modes and adiabatic initial conditions:

$$\Theta_0 + \Psi = \text{const}$$

At time of recombination (origin of CMB) η_*:

$$\boxed{(\Theta_0 + \Psi)(k, \eta_*) = \frac{1}{3}\Psi(k, \eta_*) = -\frac{1}{6}\delta(\eta_*)}$$

for the observed anisotropy at recombination

The observed anisotropy of an overdense region will be negative.
For large scale perturbations, overdense regions contain hotter photons. However, to get to us today they first must travel out of their potential wells. In doing so, they lose energy, which more than compensates for the initial "overhotness".
$\Rightarrow$ When we observe hot spots on the sky today, we are actually observing underdense regions at the time of recombination.

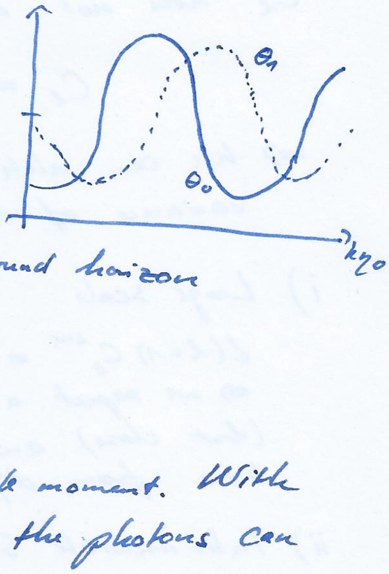
It also allows us to relate: $\frac{\delta T}{T} \propto -\frac{1}{6}\frac{\delta \rho}{\rho}$

Tightly Coupled Limit

Tightly coupled limit: $\tau \gg 1$, before recombination at η_*, Compton scattering caused the electron-proton fluid to be tightly coupled. $\tau \equiv \int_\eta^{\eta_0} n_e \sigma_T \, a(\eta') d\eta'$

In this limit, we can neglect high multiples $l \geq 2$.

We get a damped harmonic oscillator equation for Θ_0.



Neglecting the damping, we obtain oscillating solutions for Θ_0 and Θ_1: $S_1 = \sin(k r_s)$, $S_2 = \cos(k r_s)$

$$r_s = r_s(\eta) = \int_0^\eta d\eta \, c_s(\eta) = \int d\eta \sqrt{\frac{1}{3\left(1 + \frac{\rho_b^{(0)}}{\rho_\gamma^{(0)}}\right)}} \equiv \text{sound horizon}$$

Θ_0 and Θ_1 are completely out of phase.

Diffusion Damping

Diffusion is characterized by a small but nonnegligible quadrupole moment. With diffusion considered, modes with small k are damped because the photons can diffuse out of the potential wells and dilute the latter.
We get exponential dampening.

Inhomogeneities to Anisotropies

Relating inhomogeneities to anisotropies.
Define visibility function $g(\eta) \equiv -\dot{\tau} e^{-\tau} \approx \delta(\eta_*)$: τ is large at early times (→ exponential suppression), and $\dot{\tau}$ is small at late times (= scattering rate)

We get
$$\Theta_\ell(k,\mu,\eta) = \int_0^{\eta_0} d\eta\, g(\eta)\left[\Theta_0(k,\eta) + \Psi(k,\eta)\right] j_\ell\left[k(\eta_0-\eta)\right] \quad \text{monopole}$$
$$- \int_0^{\eta_0} d\eta\, g(\eta) \frac{i v_b(k,\eta)}{k} \frac{d}{d\eta} j_\ell\left[k(\eta_0-\eta)\right] \quad \text{dipole}$$
$$+ \int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] j_\ell\left[k(\eta-\eta_0)\right] \quad \text{ISW}$$

ISW: Integrated Sachs-Wolfe term; arises as a small correction if the potentials are not time independent.
We got this equation from full EB-equation.

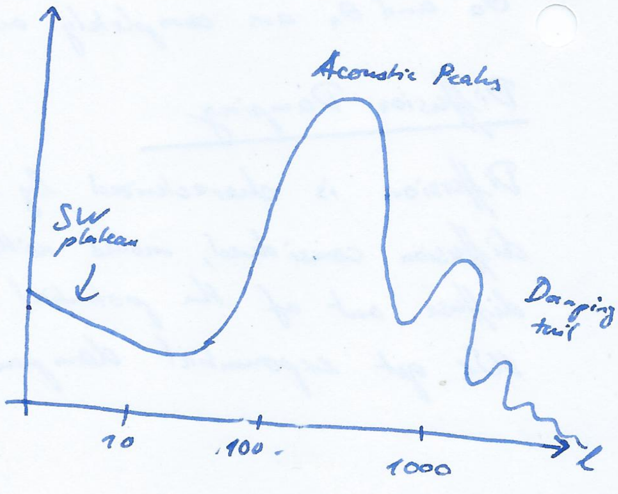
Power Spectrum of the CMB

Ansatz: Decompose Θ [from $T = \bar{T}(\eta)(1+\Theta)$] in terms of spherical harmonics:
$$\Theta(\vec{x},\hat{p},\eta) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\vec{x},\eta)\, Y_{\ell m}(\hat{p})$$
All the information in the temperature field T is also contained in the space-time dependent amplitude $a_{\ell m}$.
$\langle a_{\ell m} \rangle = 0$; $\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell$ = variance.
C_ℓ does not depend on m and can be related to the matter power spectrum:
$$C_\ell = \frac{2}{\pi}\int_0^\infty dk\, k^2 P(k) \left|\frac{\Theta_\ell(k)}{\delta(k)}\right|$$
⇒ We can relate the power spectrum and the multipole moments with the variance of the amplitudes of Θ.

i) Large scale
$\ell(\ell+1) C_\ell^{SW} = \text{const}$ for tilt $n=1$
⇒ we expect a plateau, but $n \neq 1$ in general (but close) and there are also small contributions from dipoles and SW-effects for ΛCDM.

ii) Intermediate Scales
Acoustic oscillations starting at $\ell \approx 200$

iii) Small scales
Diffusion damping starting at $\ell \approx 1000$



Cosmological Parameters

- Curvature causes a shift in the location of the peaks. 1st peak is at higher ℓ for an open Universe.
- δ_H, τ, r, n move spectrum up and down, hardly change its shape
- $\Omega_m, \Omega_\Lambda, \Omega_b$ induce small shifts in the location of peaks/throughs in the spectrum.

Initial Perturbations from Inflation

Principle of inflation: in an early Universe $t_i < t < t_f$ there is a period during which the Hubble constant was actually approximately a constant.

$$H = \frac{1}{a}\frac{da}{dt} \approx \text{const} \Rightarrow a(t) = a_0 e^{H(t-t_0)}$$

$\Rightarrow$ Hubble radius $\frac{1}{aH} \propto a^{-1}$ decreased. Modes of interest were initially causally connected, then left the hubble radius because it decreased, and re-entered it after inflation again.

Scalar Field Perturbations

Decompose scalar field: $\phi(\vec{x},t) = \phi^{(0)}(t) + \delta\phi(\vec{x},t)$

To zeroth order we have
$$\rho = -T^{(0)0}{}_0 = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi^{(0)})$$
$$P = T^{(0)i}{}_i = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi^{(0)})$$

With the slow roll condition $\left(\frac{d\phi^{(0)}}{dt}\right)^2 \ll V(\phi^{(0)})$: $H = \sqrt{\frac{8\pi G\rho}{3}} \approx \sqrt{\frac{8\pi G V}{3}} \approx \text{const}$

From the Einstein equations we get
$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + \frac{dV(\phi)}{d\phi^{(0)}} = 0$$
$$\Rightarrow \boxed{\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2 V' = 0}$$

Giving us a second order differential equation for the inflaton.

To quantify the slow roll, introduce two parameters:
$$\varepsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) \qquad \delta \equiv \frac{1}{H}\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} \qquad \varepsilon, \delta \xrightarrow{\phi = \text{const}} 0$$

The slow roll condition is given by $\varepsilon, \delta \ll 1$.

Tensor Modes

Vector, Scalar and Tensor modes evolve independently. Look at $h_{00} = h_{i0} = 0$, $h_{ij} = D_{ij}a^2$

D_{ij} has $9 - \underbrace{3}_{\text{symmetric}} - \underbrace{1}_{\text{traceless}} - \underbrace{3}_{\text{div. less}} = 2$ degrees of freedom.

Ansatz: Choose coordinate system so that $\vec{k} = k\vec{e}_z$. $\hat{h}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Computing the Einstein equations give $\boxed{\ddot{\tilde{h}}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0}$ $\alpha = +, \times$
(small radiation contribution neglected)

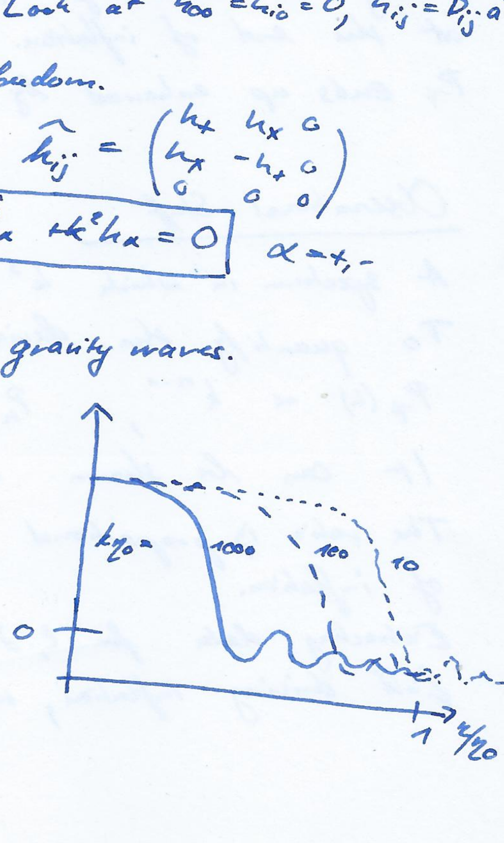
This is a wave equation, corresponding solutions are called gravity waves.

The expansion produces a friction term $\propto \dot{h}_\alpha$.

Superhorizon modes: konstant

Subhorizon modes: ostillating, but decaying.

Only large-scale anisotropies are impacted by gravity waves. At decoupling, $k\eta_0 \lesssim 100$ persist.



Generation of Tensor Modes during Inflation

Define $\tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}$ $\Rightarrow$ $\ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h} = 0$ harmonic oscillator eqn

It has no damping term. We quantize in in the Heisenberg picture $(\hat{x}(t), \Psi)$ and relate it to the quantum harmonic oscillator:

$$\hat{h}(\vec{k}, \eta) = v(k,\eta)\,\hat{a}_{\vec{k}} + v^*(\vec{k},\eta)\,\hat{a}^+_{\vec{k}}$$

Then $\langle \hat{h}^+(k,\eta)\,\hat{h}(k',\eta)\rangle = \frac{16\pi G}{a^2}|v(k,\eta)|^2 (2\pi)^3 \delta^3(\vec{k}-\vec{k}') = (2\pi)^3\delta(\vec{k}-\vec{k}')\,P_h$

$$\Rightarrow \boxed{P_h = 16\pi G\,\frac{|v(\vec{k},\eta)|^2}{a^2} = \frac{8\pi G H^2}{k^3}}$$ Primordial power spectrum of gravitational waves.

The primordial power spectrum is constant after inflation until the mode reaches the horizon.
With detection of gravitational waves amounts to the measurement of V of the inflation with $H \propto \sqrt{\rho} \approx \sqrt{V}$.

Generation of Scalar Perturbations.

Decomposing the inflation scalar field: $\phi = \phi^{(0)} + \delta\phi$

We get $\boxed{\ddot{\delta\phi} + 2aH\dot{\delta\phi} + k^2\,\delta\phi = 0}$ where we neglected a small term in V.

This is identical to the tensor perturbation equation to the metric.

The power spectrum is given by $\boxed{P_{\delta\phi} = \frac{H^2}{2k^3}}$

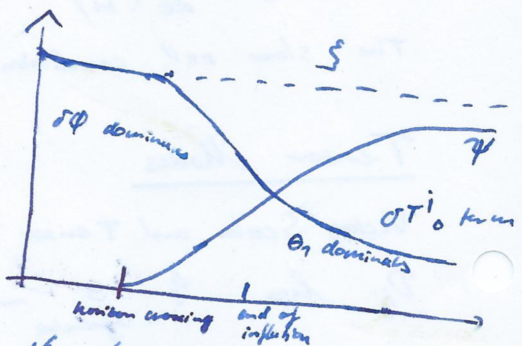
This are the perturbations to the inflation field. But how does it couple to the metric?

$\rightarrow$ Define $\xi = \frac{ik_i\,\delta T^0{}_i\,H}{k^2(\rho+P)} - \Psi = \text{const}$ which is conserved at large scales across horizon crossing.

(contains $\delta\phi$, Θ_1)

ε, H evaluated at horizon crossing

We get $\boxed{P_\Psi\big|_{after} = P_\xi\big|_{after} = \frac{8\pi G}{9k^3}\,\frac{H^2}{\varepsilon}}$



Perturbations start off as $\delta\phi$ dominated, up to horizon crossing and ends up Ψ and Θ_1 dominated at the end of inflation.

P_Ψ ends up enhanced by $\frac{1}{\varepsilon} = \left(\frac{d}{dt}\left(\frac{1}{H}\right)\right)^{-1}$ compared to the tensor power spectrum.

Observational Stuff

A spectrum in which $k^3 P$ is constant (ε does not depend on k) is called scale free.

To quantify the deviations from scale invariance, it is conventional to write

$P_\Psi(k) \propto k^{n-1}$, $P_h \propto k^{n_T - 3}$

It can be shown that $n = 1 - 4\varepsilon - \delta$, $n_T = -2\varepsilon$, $\varepsilon \propto \frac{P_h}{P_\phi}$

The ratio is proportional to the spectral indices n, n_T, which is a generic prediction of inflation.

Extracting data for ε, δ and $\frac{P_h}{P_\phi}$ is tantamount to probing the potential of the field driving inflation, which is thought to be at 10^{15} GeV scales.