The Cosmological Principle

Viewed on a sufficiently large scale, the properties of the Universe are the same for all observers.
The physical laws valid on earth are also valid everywhere.

The Universe is homogeneous and isotropic.

"Large scales" means ~ 100 Mpc.

Robertson Walker Metric

The goal is to describe a non-static Universe which is spatially homogenic and isotropic, but evolving in time.

→ Consider spacetime to be $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ is the time direction and Σ a maximally symmetric three-manifold. (Subspace; Hypersurface)

⇒ Σ has maximal number of Killing vectors and is thus invariant under translations and rotations.

The spacetime takes the form $ds^2 = -dt^2 + a^2(t)\, d\sigma^2$
with $d\sigma^2 = \gamma_{ij}(u)\, du^i du^j$

The choice of coordinates needs to be consistent with the cosmological principle. The Universe needs to be homogeneous and isotropic. If an observer has non-negligible velocity with respect to the Universe (= non-negligible peculiar velocity), then it changes the local density, making the Universe locally non-isotropic. Therefore we need comoving observers, which are defined by being at rest in the frame of the Universe. The distance between two comoving observers only changes because of the change in $a(t)$:

$$d_{AB}(t_1) = d_{AB}(t_0)\, \frac{a(t_1)}{a(t_0)}$$

Ansatz: Start from general solution for Schwarzschild:

$$ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2$$

and demand $\alpha = 0$, $\partial_t \beta = 0$ giving:

$$ds^2 = -dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

Next use the properties of the Ricci tensor for a maximally symmetric space: $R_{ijkl} = \kappa (\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk})$; $R_{jl} = 2\kappa \gamma_{jl}$

The non-vanishing compounds of the Ricci tensor are:

$R_{11} = \frac{2}{r} \partial_1 \beta$, $R_{22} = e^{-2\beta} (r \partial_1 \beta - 1) + 1$

$R_{33} = [e^{-2\beta} (r \partial_1 \beta - 1) + 1] \sin^2\theta$

Together with $R_{jl} = 2\kappa \gamma_{jl}$, we get:

$$d\sigma^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2$$

Yielding the FLRW - metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]$$

κ can be renormalised such that there are only three options.
$\kappa = \{0, \pm 1\}$ through clever coordinate transformations

The curvature of the space is determined by K:

K = 0:
$$ds^2 = dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2$$

flat Universe, Euclidean metric on Σ

K = 1:
$$ds^2 = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2 \overset{r \to \sin\chi}{=} d\chi^2 + \sin^2\chi \, d\Omega^2$$

3-sphere; bounded space

K = -1:
$$ds^2 = \frac{dr^2}{1+r^2} + r^2 d\Omega^2 \overset{r \to \sinh\chi}{=} d\chi^2 + \sinh^2\chi \, d\Omega^2$$

3d-hyperbolic surface; unbound space

Note that the curvature of the space is not the curvature of spacetime!

$$\mathcal{R} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)$$

Friedmann Equations

Goal: Determine (the behaviour) of $a(t)$ using physics/ Einstein equations:

A homogeneous, isotropic Universe must have a homogeneous, isotropic matter/ energy distribution. Such a distribution is described by an ideal fluid in its rest frame. So we will describe the Universe as an ideal fluid in comoving coordinates:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \quad \text{with} \quad u^\mu = (1,0,0,0)$$

1) Conservation of energy equation

Conservation law: $\nabla_\mu T^\mu{}_\nu = 0$

Take 0-component:
$$\begin{aligned} 0 &= \nabla_\mu T^\mu{}_0 \\ &= \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \\ &= -\partial_0 \rho - 3\frac{\dot a}{a}(\rho + p) \end{aligned}$$

For further progress, you need to assume an equation of state: $p = w\rho$. Assuming w is a constant:

$$\frac{\partial \rho}{\partial t} = -3\frac{\dot a}{a}(\rho + p) = -3\frac{\dot a}{a}(1+w)\rho$$

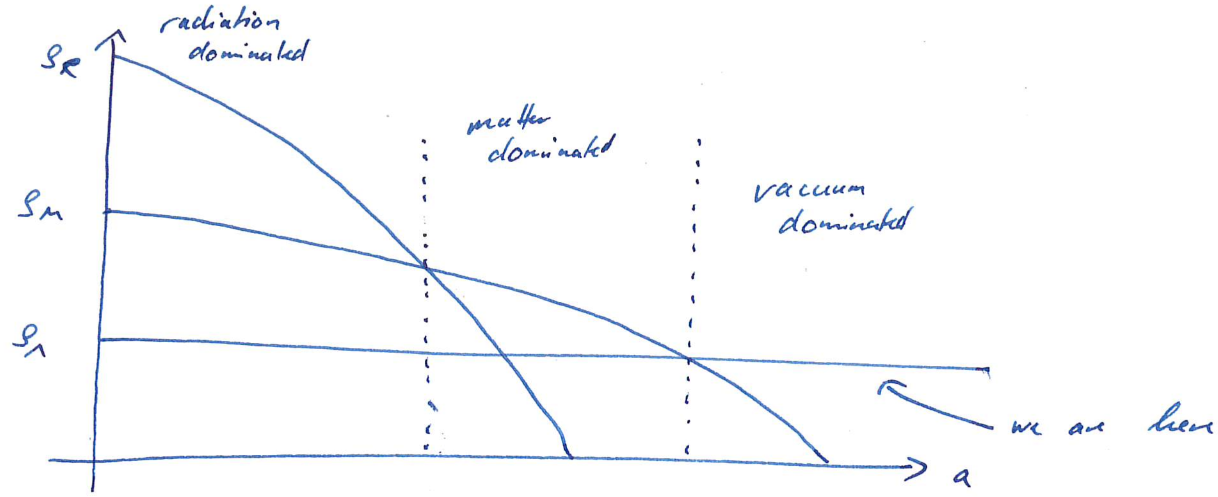
$$\Rightarrow \quad \rho \propto a^{-3(1+w)}$$

For non-interacting, collisionless matter (dust): $w = 0 \Rightarrow \rho_M \propto a^{-3}$

For radiation: $w = \frac{1}{3} \Rightarrow \rho_R \propto a^{-4}$

($\rightarrow$ photons also lose energy with a^{-1} as they redshift)

For cosmological constant: $w = -1 \Rightarrow \rho_\Lambda \propto a^0$



(Initially, all matter was relativistic → radiation density dominates. With cooling, particles gradually became non-relativistic)

2) Second Friedmann Equation

Using the Einstein Field equations $R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$ for the 00 component:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

3) Third Friedmann equation

Use ij-components; eliminate $\ddot{a}$ term with second Friedmann equ

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

To summarise:

$$\frac{\partial \rho}{\partial t} = -3(p+\rho)\frac{\dot{a}}{a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Definitions:

$H \equiv \frac{\dot a}{a}$ Hubble parameter

$H_0 \equiv 100h$ km/sec/Mpc Hubble constant; The value of the Hubble parameter at the present epoch $h \approx 0.7$

$q \equiv \frac{-a\ddot a}{\dot a^2}$ deceleration parameter; measures the rate of change of the rate of expansion

$\rho_{crit} \equiv \frac{3H^2}{8\pi G}$ Critical density; Determines the value of k in the absence of Λ in the Friedmann equation:

$$H^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2}$$

$$\Rightarrow \frac{k}{H^2 a^2} = \frac{8\pi G}{3H^2}\rho - 1 \Rightarrow \begin{cases} \rho < \rho_{crit} \Rightarrow k = -1 \\ \rho = \rho_{crit} \Rightarrow k = 0 \\ \rho > \rho_{crit} \Rightarrow k = 1 \end{cases}$$

$\Omega_i \equiv \frac{\rho_i}{\rho_{crit}}$ density parameter. Observations tell us $\Omega \simeq 1$

$\rho_c \equiv \frac{-3k}{8\pi G a^2}$ "curvature density" term; Fictional density so that

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i$$

$$= \frac{8\pi G}{3}(\rho_M + \rho_R + \rho_{vac} + \rho_c)$$

with $\rho_{vac} = \frac{\Lambda}{8\pi G} = \text{const}$

Notes on Densities

We've defined: $\rho_{crit} = \frac{3H^2}{8\pi G}$, $\rho_{vac} = \frac{\Lambda}{8\pi G}$, $\rho_c = \frac{-k}{8\pi G a^2}$

We can describe the evolution of the densities as individual power laws:

$$\rho_i = \rho_{i,0}\, a^{-n_i}$$

with $w_i = \rho_i / p_i = \text{const}$ (EoS)

Using $\dot{\rho}_i = -3(1+w_i)\frac{\dot{a}}{a}\rho_i \Rightarrow \rho_i = K\, a^{-3(1+w_i)} = \rho_{i,0}\, a^{-n_i}$

$\Rightarrow \frac{1}{K} = \frac{a^{-3(1+w_i)+n_i}}{\rho_{i,0}} = \text{const} \Rightarrow n_i = 3(1+w_i) \Rightarrow w_i = \frac{1}{3}n_i -$

For the typical contents of the Universe, we then have:

matter		radiation		curvature		vacuum
$p = 0$	$w = 0$	$p = \frac{1}{3}\rho$	$w = \frac{1}{3}$		$w = -\frac{1}{3}$	$w = -1$
$\rho = \rho_0 a^{-3}$	$n = 3$	$\rho = \rho_0 a^{-4}$	$n = 4$	$\rho_c = \rho_{c,0} a^{-2}$	$n = 2$	$n = 0$, $\rho_{vac} = \text{const}$

The Friedmann equation can be written as:

$$H^2 = \frac{8\pi G}{3}\sum \rho_i$$

with $1 = \sum_i \Omega_i$, which includes the fictitious Ω_c curvature density. Therefore: $\Omega_c = 1 - \Omega$

Solutions of the Friedmann equations

1) Positive Densities

Assume all partial densities ≥ 0, including the curvature density. With $\rho_c = \frac{-3k}{8\pi G a^2} \Rightarrow k \leq 0$

Then from the Friedmann equation:

$$H^2 = \frac{8\pi G}{3} \sum \rho_i$$

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a}{a^2} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i)$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i) = \dot{H} + H^2$$

$$\Rightarrow \dot{H} = -\frac{4\pi G}{3} \sum (\rho_i + 3p_i) + H^2 = -\frac{4\pi G}{3} \sum (\rho_i + 3p_i) + \frac{8\pi G}{3} \sum \rho_i$$

$$= -\frac{4\pi G}{3} \sum (\rho_i + 3p_i + 2\rho_i)$$

$$= -4\pi G \sum_i (\rho_i + p_i)$$

$$= -4\pi G \sum_i (1 + w_i)\rho_i$$

$$\Rightarrow \boxed{\dot{H} = -4\pi G \sum_i (1 + w_i)\rho_i}$$

Observations:

i) With $|w_i| \leq 1$ and $\rho_i \geq 0$, $\dot{H} \leq 0$ always.
$\Rightarrow$ The expansion rate always decreases

ii) $\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 + |k| \Rightarrow \dot{a}$ is exactly then 0 for an empty, flat Universe
$\Rightarrow$ The sign of $\dot{a}$ never changes

Today we know from observations that $\dot{a} > 0$

iii) $\dot{H} = \frac{\ddot{a}}{a} - H^2 \Rightarrow \dot{H}$ may be negative while $\ddot{a}$ is positive

H and $\ddot{a}$ are the answers to two different questions.

- Set two particles at a fixed initial distance. How much have they separated after some time? $\rightarrow H$
- How does a fixed source appear to be moving away from us with time? $\rightarrow \ddot{a}$

iv) Asymptotic behaviour : Consider $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k|$

$\dot{\rho} a = -3(1+w)\rho \dot{a}$ (Energy conservation equation)

Then : $\frac{d}{dt}(\rho a^3) = \dot{\rho} a^3 + 3\rho a^2 \dot{a} = 3\rho a^2 \dot{a} - 3(\rho + p)\dot{a} a^2$

$$= \underbrace{-3 p a^2 \dot{a}}_{\geq 0}$$

$\Rightarrow \frac{d}{dt}(\rho a^3) \leq 0$ (assuming $\dot{a} > 0$ for our Universe)

$\Rightarrow \rho a^3 \leq \text{const} \Rightarrow \underbrace{\rho}_{\geq 0} \underbrace{a^2}_{\geq 0,\ \text{square}} \leq \frac{\text{Const}}{a}$

$\Rightarrow \rho a^2 \xrightarrow{a \to \infty} 0$

This gives us for $a \to \infty$:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \xrightarrow{a\to\infty} |k|$$

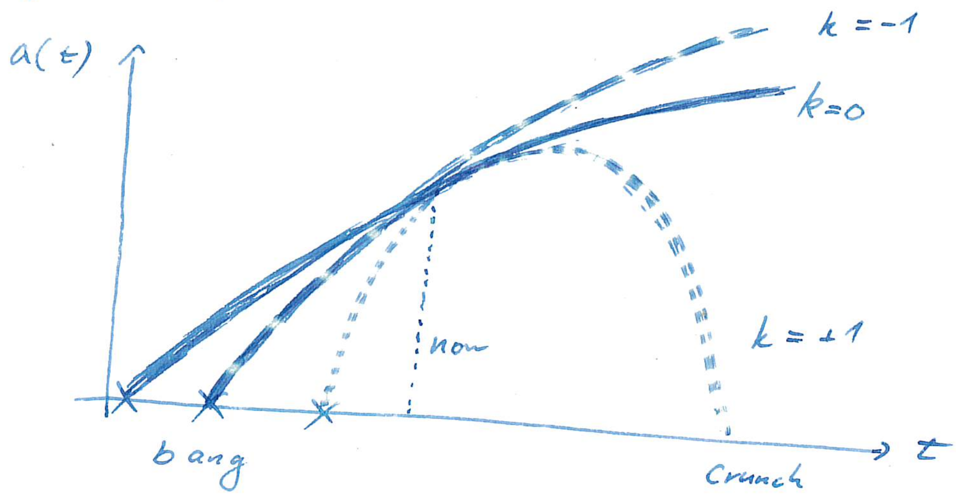
v) $k = +1$: $(k > 0)$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1$$

Using the same argument $\rho a^2 \xrightarrow[a\to\infty]{} 0$, we get $\dot{a}^2 = -1$, which isn't possible. So there is no asymptotic behaviour: Instead, the expansion has to stop at some finite $a = a_{max}$ when $\ddot{a}|_{a=a_{max}} = -\frac{4\pi G}{3}(\rho + 3p)\, a_{max} < 0$

The Universe has to decelerate.

With $\ddot{a} = -\underbrace{\frac{4\pi G}{3}(\rho + 3p)}_{\geq 0} \underbrace{a}_{\geq 0} < 0$, the deceleration remains negative, so the Universe inevitably continues to contract to zero. (= "Big Crunch")



2) Solutions for fixed spatial curvature

We now consider the case of a flat Universe ($k=0$) for different density dominations:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad \Rightarrow \quad \dot{a} \overset{k=0}{=} \sqrt{\frac{8\pi G}{3}\rho}\, a = \sqrt{\frac{8\pi G}{3}}\, a^{-\frac{n}{2}+1}$$

with $\rho = C a^{-n}$ where $n = 3$ for dust, $n = 4$ for radiation and $n = 0$ for vacuum energy.

Neglecting all other densities, we can solve this differential equation for each component:

$$\frac{da}{dt} = \sqrt{\frac{8\pi G}{3}C}\, a^{-\frac{n}{2}+1} \quad \Rightarrow \int a^{+\frac{n}{2}-1}\, da \propto \int dt$$

$$\Rightarrow t \propto a^{\frac{n}{2}} \qquad \left[t = \sqrt{\frac{3}{2\pi G C\, n^2}}\, a^{n/2} \right]$$

Dust: $t \propto a^{3/2}$ Einstein – de Sitter Universe

Radiation: $t \propto a^2$

Vacuum: $\rho_{vac} = \text{const} \Rightarrow \dot{a} = \sqrt{\frac{8\pi G \rho_{vac}}{3}}\, a \Rightarrow \frac{1}{\sqrt{\;}}\frac{da}{a} = dt$

$\Rightarrow t \propto \ln(a) \iff a \propto \exp(t)$

3) Static Universe Solutions

For a static Universe, we demand: $\ddot{a} = \dot{a} = 0$

Now consider a radiation-free Universe with the cosmological constant:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

Simplifications: For matter: $p \sim 0$; Staticity: $\ddot{a} = 0$

$$\Rightarrow \Lambda = 4\pi G \rho$$

Furthermore:

$$0 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$= \frac{8\pi G}{3}\rho + \frac{4\pi G}{3}\rho - \frac{k}{a^2} = \frac{12\pi G}{3}\rho - \frac{k}{a^2}$$

$$\Rightarrow \frac{k}{a^2} = \frac{12\pi}{3} G \rho = 4\pi G \rho$$

$$\Rightarrow k > 0 \quad \text{with} \quad \rho > 0$$

But: This solution is not stable. Small perturbations in $\frac{\dot{a}}{a}$ "explode". For a small perturbation > 0, the density will go down, which will make $\ddot{a} < 0$ etc ...

Without cosmological constant:

Static Universe $\Leftrightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = 0$

$$\Rightarrow p = -\frac{1}{3}\rho$$

$\Rightarrow$ The Universe is not dominated by dust/matter or radiation ...

4) Matter and Vacuum dominated Universe

A realistic cosmology will feature several forms of energy-momentum. In the current Universe, radiation density is significantly lower than matter density, but both matter and vacuum density are both dynamically important.

$$\rightarrow \quad \Omega_M + \Omega_\Lambda + \Omega_c = 1$$

As these Universes expand, the relative influences of matter, curvature, and vacuum are altered, since the corresponding densities evolve at different rates: $\Omega_\Lambda \propto \Omega_c a^2 \propto \Omega_M a^3$

For $a \rightarrow 0$: $\quad \Omega_c, \Omega_\Lambda$ negligible

$a \rightarrow \infty$: $\quad \Omega_c, \Omega_M$ negligible

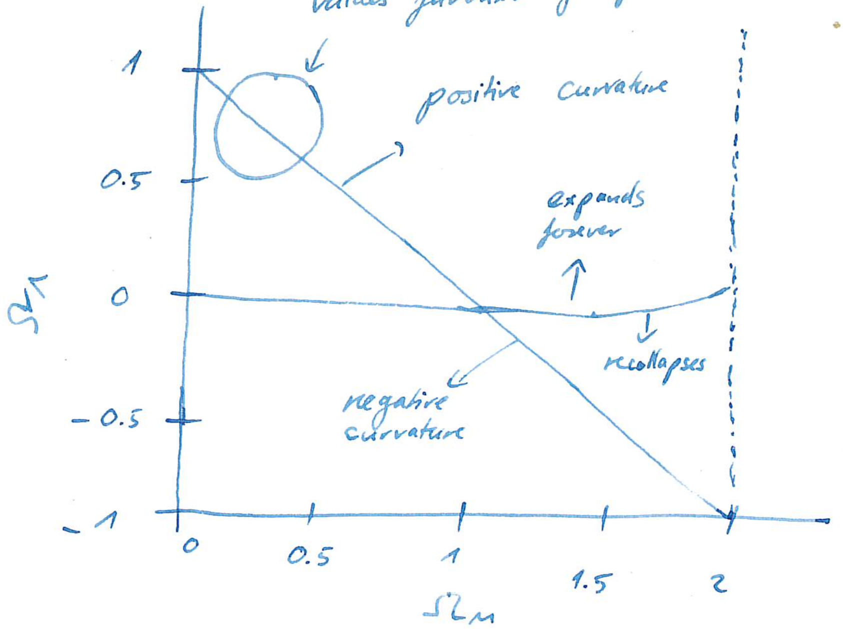
Recollapse will always occur if the vacuum energy is negative. As the Universe expands, the vacuum energy will eventually dominate. It will cause deceleration and recollapse.

Recollapse is also possible for $\Omega_\Lambda \geq 0$, provided that Ω_M is sufficiently large that it halts the expansion before Ω_Λ has a chance to take over.

The separating curve between expanding and collapsing Universe is given through the condition that a turns around, therefore has a local maximum: $\dot{a} = 0$

$$\Rightarrow \quad H^2 = 0 = \frac{8\pi G}{3}\left(\rho_M + \rho_\Lambda + \rho_c\right)$$

$$= \frac{8\pi G}{3}\left(\rho_{M,0}\, a^{-3} + \rho_{\Lambda,0} + \rho_{c,0}\, a^{-2}\right)$$



Open and flat Universes: $\Omega_{M0} = \leq 1$ expand forever for $\Omega_\Lambda = 0$. Closed Universes ($\Omega_{M0} > 1$) will recollapse for $\Omega_\Lambda = 0$

Toy model for vacuum energy

Consider a free quantum field that doesn't couple with any other field and doesn't interact.

Try to describe vacuum as zero-point energy of a field of harmonic oscillator ($\hat{=}$ not excited, simplest model). The ground state oscillations would be expected to have a natural scale.

$$\Lambda \sim m_p^4$$

with $m_p \sim 10^{19}$ GeV

But we measure $\rho_\Lambda \leq 10^{-8} \frac{erg}{cm^3}$

$$\rho_{vac} \sim 10^{112} \frac{erg}{cm^3}$$

Redshift and Distances

Geodesics in FRW Universe

The FRW metric is a maximally symmetric space, not spacetime. There are a number of spacelike Killing vectors, but no timelike Killing vector to give us a notion of energy conservation. This is valid for the metric and the Universe, not for objects inside the Universe though.

There is however a Killing tensor: let $U^\mu = (1,0,0,0)$ be the comoving velocity. Then

$$K_{\mu\nu} = a^2 (g_{\mu\nu} + U_\mu U_\nu) \quad \text{is a Killing tensor} \Leftrightarrow \nabla_{(\sigma} K_{\mu\nu)} = 0$$

Then for a particle with four-velocity $V^\mu = \frac{dx^\mu}{d\lambda}$ the quantity $K^2 \equiv K_{\mu\nu} V^\mu V^\nu = a^2 \left[V_\mu V^\mu + (U_\mu V^\mu)^2 \right]$ will be a constant along geodesics.

Massive Particles in FRW Universe

Consider massive particles: $V_\mu V^\mu = -1 = -(V^0)^2 + |\vec{V}|^2$

$\Rightarrow (V^0)^2 = 1 + |\vec{V}|^2$

also: $U_\mu V^\mu = -V^0$

$$\Rightarrow \quad K^2 = a^2 \left[-1 + (V^0)^2 \right] = a^2 \left[-1 + (1 + |\vec{V}|^2) \right] = a^2 |\vec{V}|^2$$

$$\Rightarrow \quad \boxed{|\vec{V}| = \frac{K}{a}} \quad \text{with } K = \text{const. (along geodesics)}$$

$\Rightarrow$ The particle slows down with respect to the comoving coordinates; A gas will cool down as the Universe expands. This will depend on the choice of FRW solution, as they determine the behaviour of a.

Massless particles in the FRW Universe

For photons: $V_\mu V^\mu = 0$

$\Rightarrow \quad K^2 = a^2 \left[V_\mu V^\mu + (U_\mu V^\mu)^2 \right] = a^2 (U_\mu V^\mu)^2$

$\Rightarrow \quad U_\mu V^\mu = \frac{K}{a}$

For a photon:

$U_\mu V^\mu = U_0 V^0 = -E = -\hbar\omega$

So the frequency measured by a comoving observer is $\omega = -U_\mu V^\mu$, so the frequency $\omega(t_{em})$ at emission will be observed with a lower frequency $\omega(t_{obs})$ as the Universe expands:

$$\frac{\omega(t_{obs})}{\omega(t_{em})} = \frac{a(t_{em})}{a(t_{obs})} \quad \Leftrightarrow \quad \omega(t) \propto \frac{1}{a}$$

We can define the redshift z: (between two events)

$$z_{em} = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

Assuming the observation takes place today ($a = a_0 \equiv 1$):

$$z_{em} = \frac{\lambda_{obs}}{\lambda_{em}} - 1 = \frac{1/\omega_{obs}}{1/\omega_{em}} - 1 = \frac{\omega_{em}}{\omega_{obs}} - 1 = \frac{a_{obs}}{a_{em}} - 1 \overset{a_{obs}=1}{=} \frac{1}{a_{em}} - 1$$

$$\Rightarrow \boxed{a_{em} = \frac{1}{1 + z_{em}}}$$

The redshift of an object tells us the scale factor when the photon was emitted.

This redshift is not the same as the Doppler effect: It is the expansion of space, not the relative velocities of the observer and emitter.

Show $K_{\mu\nu}$ is Killing Tensor

Let $K_{\mu\nu} = a^2 (g_{\mu\nu} + U_\mu U_\nu)$

with $U^\mu = (1,0,0,0)$

Show $K_{\mu\nu}$ is Killing Tensor $\Leftrightarrow$ $D_{(\sigma} K_{\mu\nu)} = 0$

$$D_\sigma K_{\mu\nu} = (D_\sigma a^2)(g_{\mu\nu} + U_\mu U_\nu) + a^2 \Big(\underbrace{D_\sigma g_{\mu\nu}}_{=0} + (D_\sigma U_\mu) U_\nu + U_\mu (D_\sigma U_\nu)\Big)$$

$$= 2a\, \partial_\sigma a\, (g_{\mu\nu} + U_\mu U_\nu) +$$

$$+ a^2 \Big[(\underbrace{\partial_\sigma U_\mu}_{=0} - \Gamma^\lambda_{\sigma\mu} U_\lambda) U_\nu + U_\mu (\underbrace{\partial_\sigma U_\nu}_{=0} - \Gamma^\lambda_{\sigma\nu} U_\lambda) \Big]$$

$$= 2a\, \partial_\sigma a\, (g_{\mu\nu} + U_\mu U_\nu) - a^2 \big[\Gamma^\lambda_{\sigma\mu} U_\lambda U_\nu + \Gamma^\lambda_{\sigma\nu} U_\lambda U_\mu \big]$$

$$= 2a\, \partial_\sigma a\, (g_{\mu\nu} + U_\mu U_\nu) - a^2 \big[\Gamma^0_{\sigma\mu} U_0 U_\nu + \Gamma^0_{\sigma\nu} U_0 U_\mu \big]$$

Use $\Gamma^0_{ij} = \frac{\dot a}{a} g_{ij}$ and $\Gamma^0_{00} = 0$

Define $\hat g_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu = \begin{cases} 0 & \mu = \nu = 0 \\ 0 & \mu \neq \nu \\ g_{ij} & \mu = i,\ \nu = j \end{cases}$

Such that $\Gamma^0_{\mu\nu} = \frac{\dot a}{a} \hat g_{\mu\nu} = \Gamma^0_{ij}$

$$\rightarrow D_\sigma K_{\mu\nu} = 2a\, \partial_\sigma a\, (g_{\mu\nu} + U_\mu U_\nu) - a^2 \big[\Gamma^0_{\sigma\mu} U_0 U_\nu + \Gamma^0_{\sigma\nu} U_0 U_\mu \big]$$

$$= 2a\, \partial_\sigma a\, \hat g_{\mu\nu} - a^2 \Big[\frac{\dot a}{a} \hat g_{\sigma\mu} U_\nu + \frac{\dot a}{a} \hat g_{\sigma\nu} U_\mu \Big]$$

$$= 2a \dot a\, \delta^0_\sigma\, \hat g_{\mu\nu} - a \dot a\, \delta^0_\nu\, \hat g_{\sigma\mu} - a \dot a\, \delta^0_\mu\, \hat g_{\sigma\nu}$$

Then $D_{(\sigma}K_{\mu\nu)} = \frac{1}{3!}\left(D_\sigma K_{\mu\nu} + D_\sigma K_{\nu\mu} + D_\mu K_{\sigma\nu} + D_\mu K_{\nu\sigma} + D_\nu K_{\sigma\mu} + D_\nu K_{\mu\sigma}\right)$

$$= \frac{1}{3}\left(D_\sigma K_{\mu\nu} + D_\mu K_{\sigma\nu} + D_\nu K_{\mu\sigma}\right)$$

$$= \frac{1}{3}\Big[\, 2a\dot{a}\,\delta^0_\sigma\,\hat{g}_{\mu\nu} - a\dot{a}\,\delta^0_\nu\,\hat{g}_{\sigma\mu} - a\dot{a}\,\delta^0_\mu\,\hat{g}_{\sigma\nu}$$
$$+ 2a\dot{a}\,\delta^0_\mu\,\hat{g}_{\sigma\nu} - a\dot{a}\,\delta^0_\nu\,\hat{g}_{\mu\sigma} - a\dot{a}\,\delta^0_\sigma\,\hat{g}_{\mu\nu}$$
$$+ 2a\dot{a}\,\delta^0_\nu\,\hat{g}_{\mu\sigma} - a\dot{a}\,\delta^0_\sigma\,\hat{g}_{\nu\mu} - a\dot{a}\,\delta^0_\mu\,\hat{g}_{\nu\sigma}\,\Big]$$

$$= \frac{1}{3}\Big[\delta^0_\sigma\left(2a\dot{a}\,\hat{g}_{\mu\nu} - a\dot{a}\,\hat{g}_{\mu\nu} - a\dot{a}\,\hat{g}_{\mu\nu}\right) + \delta^0_\mu(\ldots) + \delta^0_\nu(\ldots)\Big]$$
$$= 0$$

Show $K_{\mu\nu}V^\mu V^\nu$ is constant along geodesic

$V^\mu = \dfrac{dx^\mu}{d\lambda}$ where x^μ are integral curves

Then $D_{\dot{x}}\left(K_{\mu\nu}V^\mu V^\nu\right) = \dot{x}^\alpha D_\alpha\left(K_{\mu\nu}V^\mu V^\nu\right)$

$$= \dot{x}^\alpha D_\alpha\left(K_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu\right)$$

$$= \dot{x}^\alpha (D_\alpha K_{\mu\nu})\,\dot{x}^\mu\dot{x}^\nu + K_{\mu\nu}\underbrace{(D_\alpha \dot{x}^\mu)\dot{x}^\nu\dot{x}^\alpha}_{=0} + K_{\mu\nu}\underbrace{(D_\alpha \dot{x}^\nu)\dot{x}^\mu\dot{x}^\alpha}_{=0}$$

$\leftarrow$ geodesics $\rightarrow$

$$= \dot{x}^\alpha (D_\alpha K_{\mu\nu})\,\dot{x}^\mu\dot{x}^\nu$$

$$= \frac{1}{3!}\underbrace{D_{(\alpha}K_{\mu\nu)}}_{=0}\,\dot{x}^\alpha\dot{x}^\mu\dot{x}^\nu = 0$$

Distances

The geometrical measurement of distances is impossible: It requires simultaneous measurement over large distances. "Instantaneous" distance is not measurable, geometrical definitions of distance known in Euclidean space is not usable.

Ansatz: Take measurements as in Euclidean space, apply corrections to find alternatives like luminosity distance and angular diameter distance

1) Luminosity distance

Consider a photon emitting source at a distance d_L, where L is the absolute luminosity of the source and F is the flux measured by the observer.

$$L = FA \quad \Rightarrow \quad d_L^2 = \frac{L}{4\pi F} \qquad \text{with} \quad L = \frac{N\hbar\omega}{\delta t}$$

This is valid for Euclidean space. In cosmology, we have to apply two corrections:

i) The photons redshift by a factor $(1+z)$: $\omega \to \omega/(1+z)$

ii) The photons hit the sphere $(4\pi d_L^2)$ less frequently: Photons emitted at time δt apart will be measured at a time $(1+z)\delta t$ apart. $\Rightarrow \Delta t \to \Delta t(1+z)$

$$\Rightarrow \quad d_L^2 = \frac{L}{4\pi F(1+z)^2}$$

Expressed in coordinates of the metric $ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k(\chi)^2 d\Omega^2\right]$ with $S_k(\chi) = \sin\chi$ for $k=1$, χ for $k=0$, $\sinh\chi$ for $k=-1$

Then the area is $A = 4\pi S_k^2(\chi)$ with $S_k^2 = r^2$ for $k=0$

$$\Rightarrow \quad \frac{L}{F} = \frac{1}{A(1+z)^2}$$

χ can't be measured directly, so instead attempt to relate the geometry S_k with densities Ω:

For a radially travelling photon, we have $ds^2 = 0 = -dt^2 + a^2 d\chi^2 + \underbrace{a^2 S_k^2 d\ldots}_{\substack{=0 \\ \text{radial}}}$

$$\Rightarrow dt^2 = a^2 d\chi^2 \quad \Rightarrow d\chi = \frac{dt}{a} = \frac{1}{a} dt = \frac{1}{a}\frac{dt}{da} da = \frac{1}{a}\frac{da}{\dot a} = \frac{1}{a}\frac{da}{aH} = \frac{da}{a^2 H}$$

$$\boxed{d\chi = \frac{da}{a^2 H} \quad \Rightarrow \chi = \int_{a_{em}}^{a_{obs}} \frac{da}{a^2 H(a)}}$$

Now using the 1st Friedmann eqn: $H^2 = \frac{8\pi G}{3}\sum_i \rho_i = \frac{8\pi G}{3}\sum_i \rho_{i,0}\, a^{-n_i}$

$$= \frac{8\pi G}{3}\sum_i \rho_{i,0}\,(1+z)^{n_i}$$

with $\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$: $H^2 = H_0^2 \sum_i \Omega_{i,0}\,(1+z)^{n_i}$

$$\equiv H_0^2 E^2(z) \qquad \text{with } E(z) = \sqrt{\sum_i \Omega_{i,0}(1+z)^{n_i}}$$

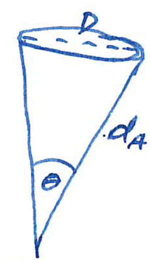
$$\Rightarrow \boxed{\chi(z) = \int_0^z \frac{dz}{H(z)} = \frac{1}{H_0}\int_0^z \frac{dz'}{E(z')}} \qquad \text{with } a = \frac{1}{1+z} \rightarrow z = \frac{1}{a} - 1 \Rightarrow dz = -\frac{1}{a^2} da$$

note the inversed integration limits because of the minus sign!

2) Angular Diameter Distance

The angular diameter distance is the distance we infer from the intrinsic and observed size of the source:

$$\tan(\theta) = \frac{D}{d_A} \approx \theta \quad \text{for } \theta \ll 1 \text{ in Euclidean space.}$$



Generalising for a FRW Universe:

Assume entire observed galaxy is described by the same comoving coordinate.

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right]$$

Looking at an object on the sky: $dt = 0$, $dr = 0$, $d\varphi = 0$, leaving us with

$$ds^2 = a^2 r^2 d\theta^2$$

Then $D = \int_a^b ds = a(t)\, r\, \Delta\theta$ (integrate over whole observed galaxy)

giving us $\boxed{d_A = \frac{D}{\Delta\theta} = ar = (1+z)^{-1} r}$

with the assumption that the whole observed galaxy is one comoving point.

3) Relation between d_L and d_A

With $d_A = ar$ and $d_L = \frac{r}{a}$, we get:

$d_L = (1+z)\, r$, $d_A = \frac{r}{1+z}$

$\Rightarrow \quad d_A = \frac{d_L}{(1+z)^2}$

Age and Lookback Time of the Universe

$$t_0 - t_{em} = \int_{t_{em}}^{t_0} dt = \int_{a_{em}}^{a_0} \frac{da}{aH} = \frac{1}{H_0} \int_{z_0}^{z_{em}} \frac{dz'}{(1+z')\,E(z')}$$

$$\uparrow \frac{da}{a} = -a\,dz = -\frac{dz}{1+z}$$

then $t(z)|_{z=0}$ is the age of the Universe. It is a function of $E(z)$, therefore depends on Ω_i, the components of the Universe. For a given set of cosmological parameters, $t(z)$ can be calculated.

Special cases (analytically solvable):

i) Radiation dominated epoch

$t \ll t_{eq} \Leftrightarrow z \gg z_{eq}$; $\Omega \approx \Omega_{r,0}\,(1+z)^4$

Then $t(z) = \frac{1}{H_0} \int \frac{dz}{(1+z)\sqrt{\sum_i \Omega_i (1+z)^{n_i}}} \approx \frac{1}{H_0\sqrt{\Omega_{r,0}}} \int \frac{dz}{(1+z)\sqrt{(1+z)^4}}$

$= \frac{1}{H_0\sqrt{\Omega_{r,0}}} \int \frac{dz}{(1+z)^3} \approx \left(\frac{1+z}{10^{10}}\right)^{-2} s$

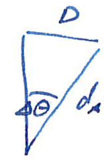
ii) Matter dominated Universe

$t \gg t_{eq} \Leftrightarrow z \ll z_{eq}$, $\Omega \approx \Omega_m \approx 1$

$$t(z) = \frac{1}{H_0}\int \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3}} = \frac{1}{H_0}\int \frac{dz}{(1+z)^{5/2}} = \frac{2}{3H_0}(1+z)^{-3/2}$$

$$\approx \frac{2}{3}(1+z)^{-3/2} \cdot 10^{10}\, h \text{ yr}$$

Other derivation of Luminosity Distance



In a static space: $\tan(\Delta\theta) \approx \Delta\theta = \frac{D}{d_A}$

for angular diameter distance, assuming $\Delta\theta \ll 1$

as well as $F = \frac{L}{4\pi d_L^2}$

In a static space: $d_L = d_A$

Generalisation for FRW-metric:

Assume looking at a static object in the sky $\Rightarrow dt = d\varphi = dr = 0$

then $ds^2 = a^2 r^2 d\theta^2$

$$\Rightarrow D = \int_a^b ds = a(t)\, r\, \Delta\theta$$

Then it follows: $d_A = \frac{D}{\Delta\theta} = a_e r_e$ with $a_e = a(t_{em})$

To get an expression for d_L, consider a proper area $\mathcal{A}$ which is at the origin (position of the observer) and subtends a solid angle β at the object in the sky.

Such a solid angle corresponds to a proper area βd_A^2 at the position of the object by definition of the angular diameter distance. Because of the expansion, the proper area at the origin subtended by a fixed solid angle at a given object is stretched by a factor a^2:

$$\mathcal{A} = \frac{\beta d_A^2}{a(t_{em})^2} \quad \text{with } a(t_{em}) < a(t_0) = 1$$

$$= \beta \frac{a_e^2 r_e^2}{a_e^2} = \beta r_e^2$$

Now consider only monochromatic radiation of frequency ν.

Considering $L \equiv \frac{N h \nu}{A \delta t}$, then the number of photons emitted from the object into the solid angle β within a time interval δt_e is $N = \frac{L \delta t_e \beta}{4\pi h \nu_e}$.

If the same number of photons pass through the area a in the time interval δt_0, we have

$$N = \frac{F \cdot \delta t_0 \, a}{h \nu_0} \quad \Rightarrow \quad \frac{L \delta t_e \beta}{4\pi h \nu_e} = \frac{F \delta t_0 \, a}{h \nu_0}$$

$$\Rightarrow \frac{L \delta t_e \beta}{4\pi h \nu_e} = \frac{F \delta t_0 \beta r_e^2}{h \nu_0}$$

$$\Rightarrow F = \frac{L}{4\pi r_e^2} \frac{\delta t_e}{\delta t_0} \frac{\nu_0}{\nu_e}$$

Finding an expression for $\frac{\delta t_e}{\delta t_0}$:

Consider a light signal propagating through the origin along a radial direction. Photons travel at null geodesics:

$$d\eta = d\chi \qquad [\eta = \text{conformal time}; \ dt = a\, d\eta]$$

A wave crest emitted at t_e reaches the origin at the time

$$\eta(t_0) - \eta(t_e) \overset{c=1}{=} \chi(r_e) - \chi(0) = \chi(r_e)$$

The comoving distance χ between the observer and object doesn't change.

A successive wave crest emitted at a later time $t_e + \delta t_e$ reaches the origin at a time $t_0 + \delta t_0$:

$$\eta(t_0 + \delta t_0) - \eta(t_e + \delta t_e) = \chi(r_e)$$

$$\Rightarrow \eta(t_0 + \delta t_0) - \eta(t_0) = \eta(t_e + \delta t_e) - \eta(t_e)$$

Usually in real applications, $t \ll \delta t$, so expand η around 0:

$$\eta(t) \approx \frac{1}{0!}\eta + \frac{1}{1!}\frac{d\eta}{dt}(\delta t - 0)^1 + \dots \quad \text{to first order}$$

$$\Rightarrow \quad \eta(t_0 + \delta t_0) - \eta(t_0) \approx \eta(t_0) + \frac{d\eta}{dt}(t_0)\,\delta t_0 - \eta(t_0)$$

$$\eta(t_e + \delta t_e) - \eta(t_e) \approx \eta(t_e) + \frac{d\eta}{dt}(t_e)\,\delta t_e - \eta(t_e)$$

with $dt = a\,d\eta$: $\quad \frac{d\eta}{dt} = \frac{1}{a(t)}$

$$\Rightarrow \quad \eta(t_0) + \frac{d\eta}{dt}(t_0)\,\delta t_0 - \eta(t_0) = \eta(t_e) + \frac{d\eta}{dt}(t_e)\,\delta t_e - \eta(t_e)$$

$$\Rightarrow \quad \frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)}$$

$$\Rightarrow \quad \frac{\delta t_e}{\delta t_0} = \frac{a(t_e)}{a(t_0)} \overset{a(t_0)=1}{=} a(t_e)$$

Back to the Luminosity:

$$F = \frac{L}{4\pi\ ^2 d_A^2}\,\frac{\delta t_e}{\delta t_0}\,\frac{\nu_e}{\nu_0} = \frac{L}{4\pi a\, d_A^{\ 2}}\,\frac{\nu_0}{\nu_e}$$

From photon redshift we know: $\nu \propto \frac{1}{a} \Rightarrow \frac{\nu_0}{\nu_e} = \frac{a(t_e)}{a(t_0)} = a(t_e)$

$$\Rightarrow \quad F = \frac{L}{4\pi r_e^{\,2}}\,a_e^2 = \frac{L}{4\pi r_e^{\,2}(1+z)^2} \equiv \frac{L}{4\pi d_L^{\,2}}$$

with $d_L \equiv r_e(1+z)$

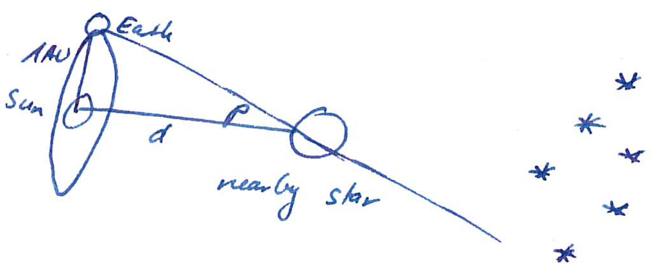
Measuring Distances

A standard candle is a class of astrophysical objects which have known luminosity due to some characteristic quality possessed by the entire class. of objects, such as supernovae or variable stars.

Different techniques and objects are used for different distances:

solar system	nearby stars	Milky Way	nearby galaxy	galaxy clusters	
10^{-4} ly	10^{2} ly	10^{5} ly	10^{7} ly	10^{10} ly	+
radar ranging	parallax	main-sequence fitting	cepheids	(white dwarf supernovae) distant standards, Tully-Fisher relation	+

1) Parallax



$$\sin p \simeq \frac{1AU}{d}$$

Works for $d < 500$ ly

2) Main-sequence fitting and Spectroscopic parallax

If one can plot a star's location on the HR diagram, its absolute magnitude can be read off, making it a "standard candle".

For main sequence stars, that requires the star's apparent magnitude and observed spectrum. For the horizontal branch the spectral type is required; For the vertical branch, the surface gravity can be found through the surface gravity from the broadening of spectral lines.

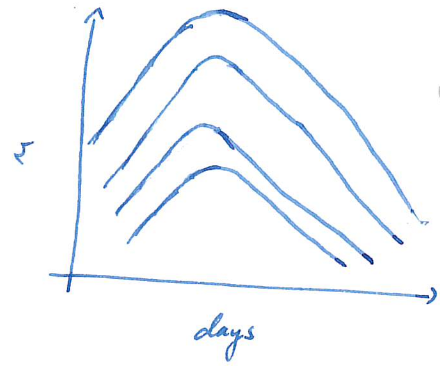
The main sequence fitting can be applied to clusters of stars. They are gravitationally bound, all located at the same distance and formed at the same time from the same cloud of gas and dust.

The HR represents the population of a cluster, because 90% of the cluster stars are on the main sequence.

3) Standard Candles: Type 1a supernovae

Type 1a supernovae come from two normal stars in a binary pair. The more massive star becomes a giant, which spills gas onto the secondary star. The secondary lighter star and the core of the giant star spiral toward within a common envelope, which is ejected away while the stars come closer to each other. The giant becomes a white dwarf. Once the second star ages and becomes a giant, it spills gas onto the white dwarf, which explodes at a critical mass, causing the companion star to be ejected away.

They are excellent standard candles because of their empirical relation of peak luminosity to the decay timescale:



4) Variable stars

A Cepheid variable is a type of star that pulsates radially, varying in both diameter and temperature producing changes in brightness with a well-defined stable period and amplitude. They have a strong direct relationship between a Cepheid variable's luminosity and pulsation period.

5) Hubble's Law

Almost all galaxies appear to move away from us. Their recession velocities increase in direct proportion to their distances to us: $v_r \propto r$

$\rightarrow$ Observational evidence for an expanding Universe.

For a homogeneous and isotropic expansion, the distance between any two comoving objects is $d_{com} = \frac{r}{a} = \frac{r'}{a'} \Rightarrow r = \frac{a}{a'} r' \Rightarrow d = ar$ with $r = r(t_0)$

Then the relative velocity is $v = \dot{d} = \underbrace{a\dot{r}}_{=0:\text{ no peculiar velocity assumed}} + r\dot{a} = \dot{a}r = \dot{a}\frac{d}{a} = Hd$ (primes = today)

$\Rightarrow \boxed{v = Hd}$

Hubble's Law is an approximation for nearby sources, such that the problem of simultaneity can be ignored. It is useful only for sources which define "nearly simultaneous" events.

Thermal History of the Universe

Temperature Relations

- Dust: $k_B T_m \propto \langle v \rangle^2 \propto \frac{1}{a^2}$

$$\Rightarrow \boxed{T_m \propto \frac{1}{a^2}}$$

- Radiation: $\rho_r = \frac{k \sigma T_r^4}{c}$ (blackbody radiation)

$$\rho_r \propto \frac{1}{a^4} \quad \Rightarrow \boxed{T_r \propto \frac{1}{a}}$$

Hot Big Bang Model: Early Universe and History of Matter

Standard Cosmology: The temperature of the Universe was arbitrarily high at the beginning of the Big Bang and has decreased continuously as the expansion progresses. The thermal history of the Universe, together with particle, nuclear and atomic physics, allows a detailed prediction of the matter content of the Universe at each epoch.

Key concepts:

i) At any given time, particles can be created if their rest mass is such that $k_B T \gg mc^2$, where T is the temperature of radiation. $T_r \propto \frac{1}{a}$, so only lighter particles may be produced as time progresses.

ii) The early Universe was filled with radiation and hot plasma. The initial "soup" of elementary particles and radiation is in thermal equilibrium, radiation dominates the expansion.
The thermal equilibrium is maintained by electroweak interactions between particles and radiation.

The interaction rate is described by $\Gamma_i = n_i \langle v_i \sigma_i \rangle$ with n = particle density, v is the velocity and $\sigma = \sigma(v)$ is the interaction cross section.

iii) As the Universe expands, Γ decreases, because n, v decrease. When $\Gamma < H(t)$, the particle species decouples from the photon fluid and the particle density "freezes out" to the value at first decoupling. Subsequently, stable particles' densities are only diluted by the expansion of the Universe.

The extremely high temperatures ($T = 10^{10}$ K at $t = 1s$) guarantee that all elementary particles are relativistic and in thermal equilibrium with radiation.

Units

As this section is concerned with high energy physics, we will use the natural unit system:

$$c = k_B = \hbar = 1$$

$$[c] = \text{cm/s}, \quad [k_B] = \text{g cm}^2 \text{s}^{-1}, \quad [\hbar] = \text{g cm}^2 \text{s}^{-1}$$

These constants being dimensionless implies

$$[\text{energy}] = [\text{mass}] = [\text{temperature}] = [\text{time}]^{-1} = [\text{length}]^{-1}$$

and all physical quantities can be expressed in one unit.

The Chronology of the Hot Big Bang

1) Planck Era $t \sim 10^{-44} s$

The Planck era is defined by the smallest possible physical timescale which we can probe without a quantum theory of gravity.

Planck time: $(\hbar G c^{-5})^{1/2} \sim 5 \cdot 10^{-44} s$, planck energy $\sim 10^{16}$ TeV

Gravity matters at least as much as quantum processes, they appear in the same power in the definition of the timescale. The unified force of the Standard Model is assumed to be unified with gravitation.

2) Grand Unification time to Electroweak epoch: $t \sim 10^{-35} s$ to 10^{-3}

Gravity has separated from the electronuclear force at the end of the Planck era. Then, after the GUT, the strong force separated from the electroweak force.

In this time period, two key processes are believed to have taken place.

i) Inflation

A phase of rapid exponential expansion caused by the decay of a scalar field.

ii) Baryogenesis

Some asymmetric process lead to a much higher abundance of particles than antiparticles. A higher abundance of antibaryons would produce a much higher gamma-ray background than what is observed. Apparently the Universe has a non-zero baryon number.

3) Particle Era

After the electroweak era, all the known elementary particles of the Standard Model have been produced. Bosons can't be created anymore due to decreased thermal energy of the Universe, those created earlier have already decayed. The Universe is made of quarks (not hadrons), leptons and photons in thermal equilibrium.

i) Quark – Hadron transition

$t \sim 10^{-5}\,s$, $T \sim 3 \times 10^{12}\,K$

Quarks get confined in hadrons, energy got low enough.
Once the transition is over, the Universe was filled with a hot plasma consisting of relativistic pions, non-relativistic nucleons and leptons (up to μ) with their associated neutrinos.
Tau-neutrinos have decoupled.

ii) Nucleons era

$t \sim 10^{-4}\,s$, $T \sim 10^{12}\,K$

The pions annihilate and decay. Nucleons remain as only hadrons left. Muons start to annihilate, ν_μ decouple.

iii) Neutron- Proton asymmetry

$T < 10^{11}\,K$

The production of the slightly heavier neutrons is exponentially suppressed by the factor $\exp(-\Delta m/\tau)$. The asymmetry grows until the interaction rates involving protons and neutrons becomes negligible.

iv) ee^+ pair annihilation, neutrinos decoupling

$t \sim 4\,s$, $T \sim 5 \times 10^{9}\,K$

n_{e,e^+} drops, neutrinos decouple. The annihilations heat the photons, but not neutrinos, so the neutrinos expand with a lower temperature. Consequently the n/p ratio freezes out.

v) Nucleosynthesis

$t \sim$ few minutes, $T \sim 10^{9}\,K$

Nucleosynthesis starts, producing D, He and a few others. Due to the high temperature, all these elements are highly ionised.

vi) Recombination era

$t \sim 2 \times 10^{5}$ yrs, $T \sim 4000\,K$

With the cooling, electrons combine with the ions to produce neutral atoms. 50 % of the baryonic matter is in form of neutral atoms. With the number of free electrons decreasing, the Universe becomes transparent to photons. The photons that decouple from matter become the CMB. Most particles are now non-relativistic and the Universe enters the matter dominated era.

Matter-Radiation thermal equilibrium

We have good observational evidence from the perfect blackbody spectrum of the CMB that the early Universe was in local thermal equilibrium.

The number density n, the energy density ρ and pressure P of a given particle species can be written in terms of a distribution function $f = f(\vec{x}, \vec{p}, t) \stackrel{!}{=} f(p, t)$ because the Universe is homogeneous and isotropic.

Then:

$$n(t) = 4\pi \int f(p,t)\, p^2 dp$$

$$\rho(t) = 4\pi \int E(p) f(p,t)\, p^2 dp$$

$$P(t) = 4\pi \int \frac{p^2}{3E(p)} f(p,t)\, p^2 dp \qquad \text{from kinetic theory}$$

with $E^2(p) = m^2 + p^2$

For a particle species in thermal equilibrium:

$$f(\vec{p}, t)\, d^3\vec{p} = \frac{g}{(2\pi)^3} \left[\exp\left(\frac{E(p) - \mu}{T(t)} \right) \pm 1 \right]^{-1} d^3\vec{p}$$

g: spin degeneracy factor ($=1$ neutrinos, $=2$ photons/leptons; $=6$ quarks)

$\left(\frac{1}{2\pi}\right)^3$: Heisenberg uncertainty: No particle can be localised in a phase space volume smaller than $(2\pi\hbar)^3$.

μ: Chemical potential of the species

$\pm$: $+$ for Fermi-Dirac distribution, $-$ for Bose-Einstein

Solutions for special cases:

$$n_{eq} = 4\pi \int_0^\infty f(p)\, p^2\, dp = \frac{4\pi g}{(2\pi)^3} \int \left(\exp\left[\frac{E(p)-\mu}{T}\right] \pm 1\right)^{-1} p^2\, dp$$

using $E^2 = m^2 + p^2$: $dE = \frac{p}{\sqrt{p^2+m^2}}\, dp \Rightarrow dp = \frac{E}{\sqrt{E^2-m^2}}\, dp$

$$\Rightarrow p^2\, dp = \frac{E}{\sqrt{E^2-m^2}}(E^2-m^2)\, dE = E\sqrt{E^2-m^2}\, dE$$

$$\Rightarrow \boxed{n_{eq} = \frac{g}{2\pi^2} \int_m^\infty \frac{E\sqrt{E^2-m^2}}{\exp\left[\frac{E-\mu}{T}\right] \pm 1}\, dE}$$

i) Non-Relativistic particles: $m \gg T$

$$n_{eq} = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(\frac{\mu - m}{T}\right)$$

ii) Relativistic particles: $m \ll T$

$$n_{eq} = \begin{cases} \zeta(3)/\pi^2 \; g T^3 & \text{for bosons} \\ \frac{3}{4}\zeta(3)/\pi^2 \; g T^3 & \text{for fermions} \end{cases}$$

with $\zeta(3) \simeq 1.2$ is the Riemann zeta function of 3.

More massive particles transition earlier to non-relativistic statistics, as the limits are determined by whether $m \gg T$ or $m \ll T$. The mass also determines the abundance of particles: With $T \gg m$ the photons have sufficient energy to create a thermal background number density of particle-antiparticle pairs, while for $T \ll m$ only the exponential tail has sufficient energy for pair-production, causing a similar exponential suppression of their number density. Consequently, particles in thermal equilibrium with the photon gas can only contribute significantly to the energy density and pressure when they are relativistic.
(This is assuming $\mu = 0$)

$$\Rightarrow S_{universe} \approx \sum_i S_{i,\,relativistic}$$

Chemical Potential

The chemical potential μ is an additive quantity which is conserved during a 'chemical' reaction. For a particle species i:

$$\mu_i = \frac{\partial U_i}{\partial N_i}$$

Additive: For a reaction $i + j \rightleftharpoons k + l \Rightarrow \mu_i + \mu_j = \mu_k + \mu_l$

If radiation is in thermodynamical equilibrium with matter, then it is described by blackbody radiation. The equilibrium can only be achieved through interactions between matter and radiation, which includes spontaneous absorption and emission. Thus the number of photons can't be a conserved quantity and $\mu_\gamma = \frac{\partial U_\gamma}{\partial N_\gamma} = 0$.
This furthermore implies that the chemical potential of an antiparticle is exactly the negative of the particle's. Consider for example $e^+ + e^- \rightleftharpoons \gamma + \gamma \Rightarrow \mu_{e+} + \mu_{e-} = 0 + 0 \Rightarrow \mu_{e+} = -\mu_{e-}$

Since the number densities of baryons and leptons are found to be much smaller than the number density of photons, the chemical potential of all species may be set to zero to good approximation in computing the mean energy density and pressure in the early Universe.

Entropy of the Universe

$$dS(V,T) =$$
$$dU = dQ - P dV$$
$$dQ = T dS = dU + P dV$$
$$\Rightarrow dS = \frac{1}{T}\left[d[\rho V] + P(T) dV \right]$$
$$= \frac{1}{T}\left[V \frac{\partial \rho}{\partial T} dT + (P+\rho) dV \right]$$
$$= \frac{\partial S}{\partial V} dV + \frac{\partial S}{\partial T} dT \qquad \Rightarrow \frac{\partial S}{\partial V} = \frac{P+\rho}{T}, \quad \frac{\partial S}{\partial T} = \frac{V}{T}\frac{\partial \rho}{\partial T}$$

Demand integrability: $\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$

$$\Rightarrow \frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right) = -\frac{P+\rho}{T} = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right) = \frac{1}{T}\frac{\partial \rho}{\partial T} \quad \Rightarrow \frac{\partial \rho}{\partial T} = -\frac{P+\rho}{T}$$

$$\Rightarrow dS = \frac{1}{T}\left[-V\left(\frac{\rho+P}{T}\right) dT \right] + \frac{1}{T}(\rho+P) dV = \frac{1}{T}(\rho+P) dV - \frac{V}{T^2}(\rho+P) dT$$

$$\Rightarrow S = \frac{V}{T}(\rho+P) \quad (+\text{const})$$

Define entropy density: $s = \frac{S}{V} = \frac{\rho+P}{T}$

The cosmological principle allows us to apply the entropy on the entire Universe by applying it to a representative volume element dV.

Non-relativistic particles contribute negligibly to the energy density of the early Universe, as they are suppressed by a factor $\exp(-m/T)$. We can sum the entropy densities of the particles that are still relativistic at some time $t \to T(t)$ to get the entropy density of the whole Universe.

For relativistic particles:

$$\rho_{eq} = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{for bosons} \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 & \text{for fermions} \end{cases} \qquad \text{and} \quad P_{eq} = \rho_{eq}/3$$

Then $\boxed{s_{eq,i} = \frac{2\pi^2}{45} g_i T^3}$ and $\boxed{s(T) = \frac{2\pi^2}{45} g_{*,s} T^3}$

with $g_{*,s} = \sum_{i \in \text{Bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i \in \text{Fermions}} \left(\frac{T_i}{T}\right)^3$

The T_i/T allows for the possibility that $T_i \neq T_{eq}$: decoupled particles may still be relativistic. [remember: $\frac{\rho+P}{T} \propto T^3$]

Entropy conservation:

Second Law of thermodynamics: $dU = TdS - PdV$

For a cosmological volume V: $U = \rho V \Rightarrow dU = Vd\rho + \rho dV$

But also: $V \propto a^3 \Rightarrow V = ca^3$

Then $\frac{dV}{dt} = \frac{dV}{da}\frac{da}{dt} = 3ca^2\dot{a} = 3ca^3\frac{\dot{a}}{a} = 3V\frac{\dot{a}}{a} \Rightarrow \frac{\dot{a}}{a} = \frac{1}{3V}\frac{dV}{dt}$

Then using the energy conservation equation:

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}(\rho + p) = -\frac{1}{V}\frac{dV}{dt}(\rho + p)$$

$$\Rightarrow dU = Vd\rho + \rho dV = TdS - PdV$$

$$\Rightarrow V\frac{\partial \rho}{\partial t} + \rho\frac{\partial V}{\partial t} = T\frac{\partial S}{\partial t} - P\frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{\partial S}{\partial t} = \frac{1}{T}\left[V\frac{\partial \rho}{\partial t} + \rho\frac{\partial V}{\partial t} + P\frac{\partial V}{\partial t}\right]$$

$$= \frac{1}{T}\left[-\frac{\partial V}{\partial t}(\rho + P) + \rho\frac{\partial V}{\partial t} + P\frac{\partial V}{\partial t}\right] = 0$$

Also with $S \propto sa^3 \Rightarrow \frac{d(sa^3)}{dt} = 0$

Using $s \sim a^{-3} \Rightarrow sa^3 = \text{const} = g_{*,s}T^3a^3$

Distribution Function for Decoupled Particles

After a particle species decoupled, the mean interaction rate of the particle drops below the expansion rate and the particles basically move freely, i.e. on geodesics.

For a test particle, $v \propto \frac{1}{a} \Rightarrow p \propto \frac{1}{a}$. If it decoupled at t_f, its temperature is approximately equal to the photon temperature at that time, i.e. $T \approx T_f \equiv T_\gamma(t_f)$. The relative momenta are conserved, and the actual distribution at $t > t_f$ can be written as

$$f(\vec{p}, t) = f\left(\vec{p}\,\frac{a(t)}{a(t_f)}, t_f\right) \quad *$$

The form of the distribution is 'frozen in' the moment the particles decouple from the hot plasma.

Depending on the state of the particles at decoupling, we can differentiate two cases:

i) Relativistic decoupling (ν_e): Hot relics

$E \approx p$, $\mu = 0$

$$f(\vec{p}, t)\, d^3\vec{p} = \frac{g}{(2\pi)^3} \left\{ \exp\left[\frac{p}{T_f} \frac{a(t)}{a(t_f)} \right] \pm 1 \right\}^{-1} d^3\vec{p}$$

The distribution function is self-similar to that of a relativistic species in thermal equilibrium, but with a temperature $T = T_f \frac{a(t_f)}{a(t)}$

ii) Non-relativistic decoupling (e^-): Cold relics

$E = m + \frac{p^2}{2m}$, we can ignore the ± 1 term

$$f(\vec{p}, t)\, d^3\vec{p} = \frac{g}{(2\pi)^3} \exp\left[-\frac{m}{T_f}\right] \exp\left[-\frac{p^2}{2m T_f} \frac{a^2(t)}{a^2(t_f)}\right] d^3\vec{p}$$

This has the same form as the Maxwell-Boltzmann distribution for $T = T_f \frac{a^2(t_f)}{a^2(t)}$

* The distribution function remains fixed for the rest of all times, because there are no more collisions by construction. So keep the form by scaling the time dependent momentum back to freeze-out.

Neutrino Decoupling

As shown before: $g_{*s}^{1/3} T \propto a^{-1}$. As long as g_{*s} remains constant, $T \propto a^{-1}$, but as the Universe cools, species become non-relativistic and stop contributing to the entropy density of the Universe.

Neutrinos don't couple directly to the photon field, but are kept in equilibrium through the neutral current weak interaction $e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e$. At a freeze-out temperature of $T_\gamma \sim 1\,MeV$ the interaction rate $\Gamma = n\langle\sigma v\rangle$ drops below the expansion rate H of the Universe and the neutrinos decouple from the photon fluid. Their temperature from this point on strictly decreases as $T_\nu \propto a^{-1}$. The freeze-out of the neutrinos leaves g_{*s} constant, because they still remain relativistic. Consequently, the temperature of the neutrinos remains the same as the photon temperature, despite being decoupled. This changes when $T_\gamma \sim 0.51\,MeV$: the reaction $e^+ + e^- \rightleftharpoons \gamma + \gamma$ gets suppressed, as the photon field doesn't produce enough pairs anymore. The pairs annihilate, heating the photon field, but not the neutrinos. (Also, at this temperature the electrons become non-relativistic and release their entropy to the photons.) The neutrinos conserve their entropy separately, so consequently $T_\gamma > T_\nu$. Their ratio follows from the entropy conservation law:

$$T_{\nu,\,after} = T_{\nu,\,before} = T_{\gamma,\,before}$$

$$\frac{T_{\gamma,\,after}}{T_{\nu,\,after}} = \frac{T_{\gamma,\,after}}{T_{\gamma,\,before}} = \left[\frac{g_{*s}(T_{before})}{g_{*s}(T_{after})}\right]^{1/3}$$

$$\Rightarrow \quad T_{\nu,\,after} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,\,after}$$

It is thus expected that the present-day Universe contains a neutrino background at the temperature $T_{\nu,0} \simeq 0.71 \times 2.73\,K = 1.95\,K$

The Freeze-Out of Stable Particles

Stable particles: Particles with half-time of decay much larger than the age of the Universe.

The evolution of the particle number density of a species i is governed by the Boltzmann equation:

$$\frac{df_i}{dt} = C_i[f]$$

where $C_i[f]$ is the collisional term, which describes the change of f due to interactions with other species.

We can rewrite: $\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \dot{p}\,\frac{\partial f}{\partial p}$

Using $\quad p \propto \frac{1}{a} \quad \rightarrow p \equiv p' a^{-1} \quad \Rightarrow \dot{p} = \frac{dp}{dt} = \frac{dp}{da}\frac{da}{dt} = -\frac{p'}{a^2}\dot{a} = -\frac{p'}{a}\frac{\dot{a}}{a}$

$$= -pH$$

$$\Rightarrow \frac{df_i}{dt} = \frac{\partial f_i}{\partial t} \underbrace{- Hp\,\frac{\partial f_i}{\partial p}}_{\text{Hubble drag term}} = C_i[f].$$

The Hubble drag term describes the dilution of the number density due to the expansion of the Universe.

Integrating over momentum space:

$$\frac{dn_i}{dt} + 3H(t)\,n_i = \int C_i[f]\,d^3p$$

where $\quad \int Hp\,\frac{\partial f_i}{\partial p}\,d^3p = \int H\,\frac{\partial f}{\partial p}\,4\pi\,p^3 d^3p \overset{\text{Int. b. parts}}{=} 4\pi\,Hp^3 f\Big|_0^\infty - 3\int Hf\,4\pi\,p^2 d^3p$

$$= -3H\int f\,4\pi\,p^2 d^3p = -3H\,n_i$$

Single Interaction Case

In general: $C_i[f]$ depends on f_i and on the distribution functions of all other species that interact with i. Other species can still affect the distribution function of i via their contributions to the general expansion of the Universe. Thus the evolution of the matter content of the Universe is described by a coupled set of Boltzmann equations.

Consider a case where species i takes part only in the following two-body interactions: $i + j \rightleftarrows a + b$ (e.g. $e^+ + e^- \rightleftarrows \gamma + \gamma$)

Define α and β:

$\alpha(T)$: Rate per unit volume by which a, b produce i, j

$$a + b \to i + j$$

$\beta(T)$: Rate per unit volume by which i, j annihilate to produce a, b:

$$i + j \to a + b$$

Then the Boltzmann equation can be written as

$$\frac{dn_i}{dt} + 3Hn_i = \alpha(T) n_a n_b - \beta(T) n_i n_j$$

Analogously we can write for species j:

$$\frac{dn_j}{dt} + 3Hn_j = \alpha(T) n_a n_b - \beta(T) n_i n_j$$

$$\Rightarrow \frac{d}{dt}(n_i - n_j) + 3H(n_i - n_j) = 0$$

$$\Rightarrow (n_i - n_j) a^3 = \text{const}$$

Now suppose a and b are in thermal equilibrium with the general hot plasma, so that $T_a = T_b = T$, while the i and j are coupled to the hot plasma through their reactions with a and b. We can then define an equilibrium densities $n_{i,eq}$, $n_{j,eq}$ such that

$$\beta(T) n_{i,eq} n_{j,eq} = \alpha(T) n_a n_b \quad \Leftrightarrow \quad C_{ij}[f] = 0$$

For $\mu = 0$, $n_i = n_j$

$$\Rightarrow \frac{dn_i}{dt} + 3Hn_i = \alpha n_a n_b - \beta n_i n_i = \beta n_{i,eq}^2 - \beta n_i^2 = \beta(n_{i,eq}^2 - n_i^2)$$

β can be determined through the standard model of particle physics, while $n_{i,eq}$ is known beforehand.

With $n_i \propto a^{-3}$ and $s \propto a^{-3}$, it is convenient to define

$$Y_i \equiv \frac{n_i}{s}, \quad Y_{i,eq} \equiv \frac{n_{i,eq}}{s}, \quad \text{and} \quad x_i \equiv \frac{m_i}{T}$$

This removes the dependence on the scale factor.

With $s \propto a^{-3} \leftrightarrow s \equiv s' a^{-3} \rightarrow$ then $\frac{ds}{dt} = -3s'a^{-4}\frac{da}{dt} = -3s'a^{-3}H = -3Hs$

Then $$\frac{dY_i}{dt} = \frac{\partial Y_i}{\partial s}\frac{ds}{dt} + \frac{\partial Y_i}{\partial n_i}\frac{dn_i}{dt} = \frac{n_i}{s^2}3Hs + \frac{1}{s}\left(\beta(n_{i,eq}^2 - n_i^2) - 3Hn_i\right)$$

$$= s(T)\,\beta\,(Y_{i,eq}^2 - Y_i^2)$$

Rewriting with x:

For radiation dominated era, $t \propto a^2 \propto T^{-2} \Rightarrow x \propto a \rightarrow x \equiv x'a$

$$\frac{dY_i}{dt} = \frac{dY_i}{dx}\frac{dx}{da}\frac{da}{dt} = \frac{dY_i}{dx}x'\dot{a} = \frac{dY_i}{dx}x'a\frac{\dot{a}}{a} = \frac{dY_i}{dx}x\,H$$

$$s(T)\beta(Y_{i,eq}^2 - Y_i^2) = -s\beta Y_{i,eq}^2\left(\frac{Y_i^2}{Y_{i,eq}^2} - 1\right) = -\Gamma Y_{i,eq}\left(\frac{Y_i^2}{Y_{i,eq}^2} - 1\right)$$

with $\Gamma = n_{i,eq}\beta = s \cdot Y_{i,eq}\beta$

$$\Rightarrow \frac{dY_i}{dx}x\,H = -\Gamma Y_{i,eq}\left(\frac{Y_i^2}{Y_{i,eq}^2} - 1\right)$$

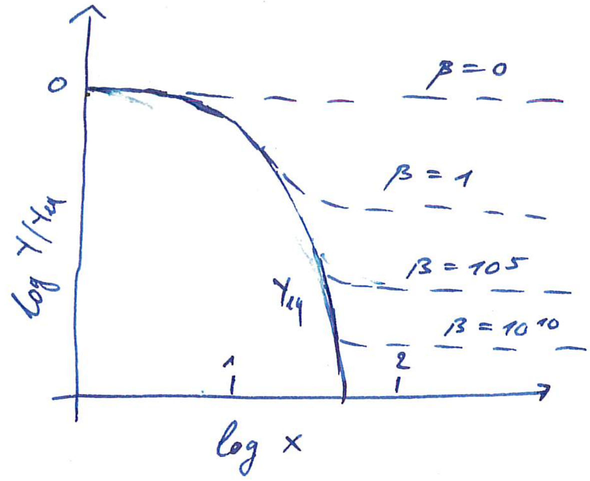
$$\Rightarrow \boxed{\frac{x}{Y_{i,eq}}\frac{dY_i}{dx} = -\frac{\Gamma}{H}Y_{i,eq}\left(\frac{Y_i^2}{Y_{i,eq}^2} - 1\right)}$$

The rate equation

Given m_i and β for a particle species i, the rate equation can be solved for Y_i numerically.

For $x \ll 1$, $m_i \ll T$ (relativistic limit, hot relics)
the solution is given by $Y_i \rightarrow Y_{i,eq}$

The initial conditions follow from $Y_i \xrightarrow[x\to 0]{} Y_{i,eq}$.
A larger interaction cross section ($\hat{=}$ larger β) implies that the species can maintain thermal equilibrium for a longer time.
For a β is such that decoupling occurs in the relativistic regime ($x \ll 1$), the final freeze-out abundance will be comparable to that of photons.



For sufficiently large β, the particles remain in thermal equilibrium well into the non-relativistic regime ($x \gg 1$), causing an exponential suppression of their final freeze-out abundance.

Estimate of the relic abundances:
Assume freeze-out occurs at T_f corresponding to x_f when $\Gamma/H = 1$ and that the relic abundance is simply given by $Y_i(x\to\infty) = Y_{i,eq}(x_f)$
(remember: Y_i does not depend on a!)
We get:

$$Y_i = \begin{cases} \frac{45\,\zeta(3)}{2\pi^4}\left[g_{i,eff}/g_{*S}(x)\right] & x \ll 1 \quad \text{(hot relics)} \\ \frac{90}{(2\pi)^{7/2}}\left[g_i/g_{*S}(x)\right] x^{3/2} e^{-x} & x \gg 1 \quad \text{(cold relics)} \end{cases}$$

where $g_{i,eff} = g_i$ for bosons and $\frac{3}{4} g_i$ for fermions.
The freeze-out temperature follows from $\Gamma(x_f) = n_{i,eq}(x_f)\,\beta(x_f) = H(x_f)$
(This is how β enters the equation)

Particles that are relativistic at freeze-out must have the same order of magnitude of the energy density as photons. (No proof.)
$\Omega_\gamma \sim 10^{-5} h^2$, as measured from the CMB, thus making relativistic dark matter candidates not interesting.

Mass Boundaries for Dark Matter Candidates

We can estimate boundaries for the mass of dark matter candidates using the relic abundance equation:

$$\rho_{i,0} = m_i \left(Y_{i,eq} \, s_0 \right) = m_i \, n_{i,eq}$$

using the present-day value of the entropy density s_0. The candidates are WIMPs: Weakly Interacting Massive Particles, like massive neutrinos and light supersymmetric particles.

i) Hot relics

$$\Omega_{i,0} h^2 \simeq 7.64 \times 10^{-2} \left[\frac{g_i}{g_{*,s}(x_f)} \right] \left(\frac{m_i}{eV} \right)$$

The upper limit for a dark matter candidate is found by demanding $\Omega_{i,0} h^2 \lesssim 1$

$\Rightarrow m_i \lesssim 94\,eV$

ii) Cold relics

$$\Omega_{i,0} h^2 \simeq \mathrm{const} \frac{x_f^{b+1}}{\sqrt{g_{*s}(x_f)}} \left[\frac{m_i}{GeV} \right]^{-2}$$

$$\simeq 1.82 \left(\frac{m_i}{GeV} \right)^{-2} \left[1 + 0.17 \ln \left(\frac{m_i}{GeV} \right) \right]$$

for Dirac-Type neutrinos.

Applying $\Omega_{i,0} h^2 \lesssim 1$ gives us a lower limit:

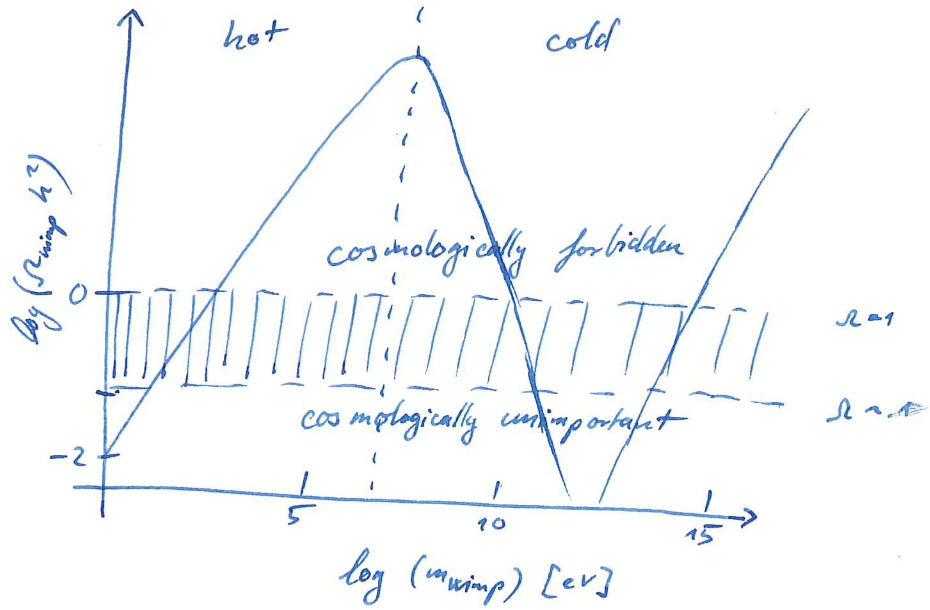
$m_i \gtrsim 1.4\,GeV$

$\Omega_{i,0}$ decreases with increasing particle mass. This reflects the fact that the annihilation cross-section increases with m_i^2 for neutrinos, so that more massive species can stay in thermal equilibrium longer, resulting in a lower freeze out abundance.

$\left[\frac{dY}{dx} \propto \frac{1}{x} \right]$

For particles with $m_i \gg m_Z \simeq 100\,GeV$ (Z boson) the cross section however actually decreases with particle mass as m_i^{-2}, giving the upper limit for the mass:

$m_i \lesssim 3\,TeV$



hot
cold
cosmologically forbidden
cosmologically unimportant
Ω = 1
Ω ~ .1
0
-2
5
10
15
log (Ω_wimp h²)
log (m_wimp) [eV]

Nucleosynthesis

It is possible that all heavier elements are synthesized in Stars. However, the observed mass fraction of helium is roughly a constant everywhere in the Universe, suggesting that most of the helium is in fact primordial/ that an universal process is responsible for it.

Protons and neutrons become non-relativistic at early times ($t \simeq 10^{-6}$ s, $T \simeq 10^{13}$ K). Down to a temperature of ~ 0.8 MeV they maintain thermal equilibrium through weak interactions like

$$p + e^- \rightleftharpoons n + \nu_e , \qquad n + e^+ \rightleftharpoons p + \bar{\nu}_e$$

which is up until the neutrinos decouple.

In thermal equilibrium, their number densities are:

$$n_{n,p} = 2\left(\frac{m_{n,p}T}{2\pi}\right)^{3/2} \exp\left[\frac{-m_{n,p} - \mu_{n,p}}{T}\right] \qquad \text{with } g_{n,p} = 2$$

Let $Q \equiv m_n - m_p = 1.294$ MeV and $\frac{m_n}{m_p} \simeq 1$, then

$$\frac{n_n}{n_p} = \exp\left(-\frac{Q}{T} + \frac{\mu_n - \mu_p}{T}\right) \approx \exp\left(-\frac{Q}{T}\right) \sim 0.2 \quad \text{at } T \sim 0.8\,\text{MeV}$$

where $\mu_n - \mu_p = \mu_e - \mu_\nu \approx 0$.

When the temperature decreases towards ~ 1 MeV, the number density of neutrons starts to drop w.r.t. that of the protons because it is slightly more massive. The ratio will eventually freeze out at $\exp\left(\frac{-1.294}{0.8}\right) \sim 0.2$ because neutrinos decouple then and the rate of the weak interactions is not fast enough to establish thermal equilibrium.

Neutrons however are unstable to beta decay: $n \to p + e + \bar{\nu}_e$ so that even after freeze-out the ratio $\frac{n_n}{n_p}$ continues to decrease.

Let $X_n \equiv \frac{n_n}{n_n + n_p} \quad \Rightarrow \quad X_n \propto \exp\left[\frac{-t}{\tau_n}\right]$ with $\tau_n \approx 900$ s

The mean lifetime of nucleosynthesis is $\tau \sim 200-300$ s. By this time, the ratio $\frac{n_n}{n_p} \sim \frac{1}{7} \lesssim 0.2$, so the neutron decay is negligible during nucleosynthesis. Locking neutrons into nuclei forbids further β-decay due to Pauli's exclusion principle

Why does nucleosynthesis only start at $\sim$ MeV?

The first step of nucleosynthesis is the reaction $p + n \to D + \gamma$. D however has a binding energy $B_D \sim 1$ MeV, so if the Universe is hotter than that, photons will destroy D. Nucleosynthesis needs to wait for the Universe to cool down until the decay $D + \gamma \to p + n$ is inefficient, which is ~ 1 MeV.

This is analogously valid for light nuclei ($A < 7$) which also have a binding energy $\sim$ a few MeV.

At such low temperatures, however, the number densities of protons and neutrons are already much too low to form heavy elements by direct multibody reactions, such as $2n + 2p \to {}^4He$. Nucleosynthesis must proceed through a chain of two-body reactions.

It can only proceed if the first step, the production of deuterium, is sufficiently efficient. Its production serves as a "bottleneck" to get nucleosynthesis started. Even with temperatures $T < B_D$, the high energy photon tail of the planckian can destroy deuterium.

Calculation of Abundances of Nuclei

Nuclei can form in abundant amounts as soon as the temperature of the Universe has cooled down to temperatures corresponding to their binding energy, and the number densities of protons and neutrons are sufficiently high.

For a non-relativistic species with mass number A and charge number Z, the equilibrium number density is given by

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp\left(\frac{-m_A + \mu_A}{T}\right)$$

with $\mu_A = Z\mu_p + (A-Z)\mu_n$

$$\Rightarrow n_A = \frac{g_A A^{3/2}}{2^A} n_p^{\,Z} n_n^{\,A-Z} \left(\frac{m_N T}{2\pi}\right)^{3(1-A)/2} \exp\left(\frac{B_A}{T}\right)$$

with nucleon mass $m_N \equiv \frac{m_A}{A} \approx m_n \approx m_p$ and $B_A = Zm_p + (A-Z)m_n - m_A$

(Intermediate step: $n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right)\left[\exp\left(\frac{\mu_p}{T}\right)\right]^Z \left[\exp\left(\frac{\mu_n}{T}\right)\right]^{A-Z}$)

Let $n_b \equiv n_n + n_p + \sum_i A_i n_{A_i}$ be the number density of baryons in the Universe. Then we can define the 'mass fraction' or 'abundance':

$$X_A \equiv \frac{A n_A}{n_b} \qquad \text{with } \sum_i X_{A_i} = 1$$

giving

$$X_A = \frac{g_A}{2} A^{5/2} \left[\frac{4\zeta(3)}{\sqrt{2\pi}}\right]^{A-1} X_p^{\,Z} X_n^{\,A-Z} \eta^{A-1} \left(\frac{m_N}{T}\right)^{3(1-A)/2} \exp\left(\frac{B_A}{T}\right)$$

with $\eta \equiv \frac{n_b}{n_\gamma} \approx 2.72 \times 10^{-8}\, \Omega_{b,0} h^2$ $\quad$ [$n_\gamma = 2\zeta(3)/\pi^2\, T^3$ and $T_0 = 2.73\,K$

$\Omega_{b,0}$ = present day baryon density

$\Rightarrow$ Species with $A > 1$ can only be produced in appreciable amounts ($X_A \lesssim 1$) once the temperature has dropped to

$$T_A \sim \frac{|B_A|}{(A-1)\left[|\ln \eta| + \frac{3}{2}\ln(m_N/T)\right]} \sim 0.1\,|B_A|$$

Nucleosynthesis can't produce any elements heavier than lithium ($A=7$). This is due to the fact that there are no stable nuclei with $A=5$ or $A=8$. Direct many-body reactions at early epochs are very inefficient in producing heavy elements, so we can conclude that elements heavier than lithium are not produced by primordial nucleosynthesis, but in stars, where the density of helium is so high that a short-lived ^8Be can capture another ^4He to form a stable carbon nucleus, thus allowing further nuclear reactions to proceed.

Observational notes

D, ^3H, ^3He, ^4He, ^6Be, ^7L are nuclei that can be used for cosmological probes to measure $\Omega_{b,0}$ from predicted abundancies. All except ^4He are very rare, but D is a particularly good probe because it is easily destroyed and therefore its production in stars is suppressed.

Hydrogen Recombination

Immediately after primordial nucleosynthesis ($T \sim 0.1\,\mathrm{MeV} \sim 10^9\,\mathrm{K}$) the Universe consists mainly of protons, ^4He nuclei (most neutrons are captured in ^4He), electrons, photons and decoupled neutrinos. With $T \sim 0.1\,\mathrm{MeV} < 0.51\,\mathrm{MeV} = m_e$, baryons and electrons can be considered non-relativistic.

As soon as the temperature of the Universe drops below $B_H = 13.6\,\mathrm{eV}$, electrons and protons start to combine to form hydrogen atoms. At $T \sim 10\,\mathrm{eV}$, $\Gamma_{rec} > H$, $\Rightarrow$ the particles are still in thermal equilibrium. The number densities are given by

$$n_{H,e,p} = g_{H,e,p} \left(\frac{m_{H,e,p} T}{2\pi}\right)^{3/2} \exp\left[\frac{-m_{H,e,p} - \mu_{H,e,p}}{T}\right]$$

with $\mu_H = \mu_e + \mu_p$

Using $\exp\left[\frac{-m_{e,p} - \mu_{e,p}}{T}\right] = \frac{n_{e,p}}{g_{e,p}} \left(\frac{m_{e,p} T}{2\pi}\right)^{-3/2}$ we obtain the Saha equation (= equilibrium density of H):

$$\begin{aligned}
n_{H,eq} &= g_H \left(\frac{m_H T}{2\pi}\right)^{3/2} \exp\left[\frac{-m_H - \mu_H}{T}\right] \\
&= g_H \left(\frac{\overset{m_H \approx m_p}{m_p} T}{2\pi}\right)^{3/2} \exp\left[\frac{-(m_p + m_e) + B_H - \mu_p - \mu_e}{T}\right] \\
&= g_H \left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left[\frac{-m_p - \mu_p}{T}\right] \exp\left[\frac{-m_e - \mu_e}{T}\right] \exp\left[\frac{B_H}{T}\right] \\
&= g_H \left(\frac{m_p T}{2\pi}\right)^{3/2} \frac{n_p}{g_p} \left(\frac{m_p T}{2\pi}\right)^{-3/2} \frac{n_e}{g_e} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left[\frac{B_H}{T}\right] \\
&= \frac{g_H}{g_e g_p} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left(\frac{B_H}{T}\right)
\end{aligned}$$

with $m_H + B_H = m_e + m_p$

Using $n_b = n_p + n_H$ (neglecting all other nuclei), $n_b = \eta n_\gamma$,
with $X_e \equiv \frac{n_e}{n_b} = \frac{n_p}{n_b}$ (ionisation fraction) yields

$$\frac{1 - X_{e,eq}}{X_{e,eq}^2} = \sqrt{\frac{32}{\pi}}\, \zeta(3)\, \eta \left(\frac{m_e}{T}\right)^{-3/2} \exp\left(\frac{B_H}{T}\right)$$

This is the Saha equation for the ionisation fraction in thermal equilibrium. Assuming thermal equilibrium holds, we can compute the temperature T_{rec} and therefore the redshift z_{rec} at which recombination occurs, giving $z_{rec} \approx 1300$ and $T_{rec} \approx 0.3\,eV$. Note $T_{rec} \ll B_H$: tail of blackbody can still keep the majority of the hydrogen atoms ionized

Main reactions of recombination:

i) direct recombination to the ground state
but: A Lyman continuum photon is produced with energy $> 13.6\,eV$ ($\rightarrow$ particles were in motion before binding), which can ionise other hydrogen, so no net recombination can be achieved with this process.

ii) capture to excited state
the electron then cascades down to the ground level. The released photons have $h\nu < 13.6\,eV$, but the process is still ineffective, because these photons can excite hydrogen atoms from their ground states so that multiple absorptions lead to re-ionisation.

iii) Two-photon decay
From meta-stable $H^{*2s} \rightarrow H^{1s} + \gamma + \gamma$ two photons must be emitted to conserve energy and $\underbrace{\text{momentum}}_{\text{angular}}$, and it is possible that the energies of emitted photons fall below the ionisation threshold.
But this process has a slow rate: $\Gamma_{2\gamma} \approx 8.23\,s^{-1}$

iv)

Once redshifted to a lower energy, the Ly α photons produced in the cascade will no longer be able to excite hydrogen atoms from their ground state.

Decoupling and the origin of the CMB

Charged particles and photons interact with each other via Thomson scattering with the rate $\Gamma_{Th} = n_e \sigma_T c$ where $\sigma_T = \frac{8\pi}{3}\left(\frac{q e^2}{m_e c^2}\right)^2 \approx 6\cdot 10$
In the following we neglect scattering of photons on ions because they have a much smaller interaction rate.

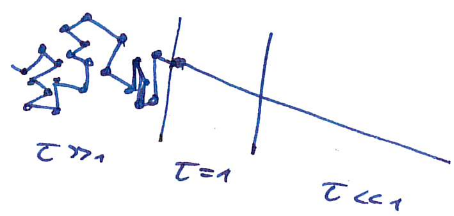
Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. (It is the low-energy limit of Compton scattering).

Using the Saha equation to compute n_e for $X_e \ll 1$, we obtain Γ_{Th} and can compare it to H to obtain T_{dec} and z_{dec} at which decoupling occurs: $T_{dec} \simeq 0.26\,eV$, $z_{dec} \simeq 1100$

Optical Depth

A somewhat more accurate derivation of the redshift of decoupling can be obtained by defining an optical depth τ of Thomson scattering from an observer at $z=0$ to a surface at a redshift z:

$$\ell = \frac{1}{\alpha_s} = \frac{1}{n_e \sigma_{Ts}}$$



ℓ: mean free path
α_s: absorption coefficient

$$\tau = \int_0^s \underbrace{\alpha_s}_{=const} ds' = \alpha_s \int_0^s ds' = \alpha_s s \qquad \left[dI = -\alpha_\nu \, ds \, I = -d\tau \, I \right] \quad = \frac{dE}{dt\, dA\, d\Omega}$$

$$= c \int \underbrace{X_e}_{X_e = \frac{n_e}{n_b} = \frac{n_e}{n_\gamma \eta}} n_\gamma \eta \, \sigma_T \, dt = c \int X_e n_\gamma \eta \sigma_T \frac{dt}{dz} dz \simeq 0.37 \left(\frac{z}{1000}\right)^{14.25}$$

$$\stackrel{!}{=} 1 \quad : \text{surface of last scattering}$$

giving us $z_{LS} = 1067$

The probability that a photon was last scattered in the redshift interval $z \pm dz/2$ can be approximated as $P(z) = e^{-\tau} \frac{d\tau}{dz}$

Compton Scattering

Mean free path: $l_e = \frac{1}{n_e \sigma_T}$, $l_\gamma = \frac{1}{n_\gamma \sigma_T}$ $\Rightarrow \frac{l_e}{l_\gamma} = \frac{n_e}{n_\gamma} = X_e \eta$ $\sim 3\times10^{-8}\,\Omega_{b,0}h^2$

Since $X_e \leq 1 \Rightarrow l_e \ll l_\gamma$: It is much easier for photons to change the energy distribution of the electrons than the other way around.

To compute the redshift at which the matter temperature will decouple from that of the radiation, we compute the rate at which the energy density of matter ε_m changes due to Compton interactions with the radiation field:

$$\frac{d\varepsilon_m}{dt} = n_e n_\gamma \sigma_T c \underbrace{\Delta E}_{\text{per single compton scattering}} = 4 n_e \sigma_T \varepsilon_\gamma \left(\frac{k_B T_e}{m_e c}\right)$$

$\Rightarrow$ This allows us to define the compton rate at which electrons can adjust their energy density to that of photons as

$$\Gamma_{\gamma\to e} = \frac{1}{\varepsilon_m}\frac{d\varepsilon_m}{dt} = 8.9\times10^{-4?}\left(\frac{X_e}{X_e+1}\right)\theta^4 s^{-1}$$

$\theta = T/\text{keV}$

With $\Gamma = H \Rightarrow$ $1+z = 6.8\left(\frac{X_0}{X_e+1}\right)^{-2/5}(\Omega_{m,0}h^2)^{1/5} \approx 150$

This is a much lower redshift than the redshift of decoupling defined by an optical depth of unity for Compton scattering. The electron temperature can remain coupled to that of the photons even if only a tiny fraction (tail of spectrum?) are scattered by the electrons.

Compton scattering can significantly modify the energy distribution of the photon fluid at $z \gtrsim 5\times10^4$ (from $\Gamma_{e\to\gamma}$). Since Compton scattering does not change the number of photons, this process alone can't lead to a Planck blackbody distribution.

Processes that produce photons are: Bremsstrahlung, double Compton emission (which is the dominant process)

Inflation

The standard relativistic cosmology provides a very succesful framework for interpreting observations. There are, however, a number of problems that cannot be solved within the standard framework.

Inflation describes an early exponential phase of growth of the scale factor after $T \sim T_{GUT}$. It is phenomenologically similar to an Universe with a dominant cosmological constant, however inflation needs to end (Λ-dominated phase does not need to). After inflation, the Universe evolves as in the standard Big Bang model. There are multiple inflationary models which share the common aspect that inflation is generated via energy release from a scalar field.

After inflation, the energy density of matter and radiation is nearly zero ($\sim 1/a^3$ with a grown exponentially), but the temperature can be high because of energy released by the inflation (="reheating"). Reheating is necessary because otherwise the temperature is too low for the creation of standard model particles.

i) The Horizon Problem

The observable Universe is homogeneous and isotropic on a large scale: The CMB shows nearly uniform temperature on the entire sky, implying that different regions were causally connected and able to reach thermal equilibrium by exchanging photons.

What is the maximal comoving distance from which an observer can receive photons from $t = t_{em}$? This distance is called the particle horizon and it defines the causally connected region:
$$r_H = \int_{t_{em}}^{t} \frac{c\,dt'}{a(t')}$$

For a radiation or matter dominated Universe, r_H converges ($a \propto \sqrt{t}$ radiation; $a \propto t^{3/2}$ matter), meaning that in encloses some finite region and thus there are emitters outside the horizon, so that not every region is causally connected. r_H diverges for a Λ dominated Universe.

Rough estimate of the size of the particle horizon:

Assume for simplicity $\Omega_{m,0} = 1$ (Einstein-de-Sitter Universe)

At the time of decoupling ($z \sim 1100$),

$$\chi_{h,dec} = \int_0^{a_{dec}} \frac{c\,dt}{a} = \int_0^{a_{dec}} \frac{c\,da}{a\dot{a}} = \frac{c}{H_0}\int_{\infty}^{z_{dec}} \frac{dz}{(1+z)^{3/2}} = 6000\,h^{-1}\,\mathrm{Mpc}\,(1+z)^{-1/2} \sim 180\,h^{-1}\,\mathrm{Mpc}$$

The comoving distance from us to the last scattering surface is

$$D_{LS} = \int_{z_{dec}}^{0} \frac{dz}{(1+z)^{3/2}} = 6000\,h^{-1}\,\mathrm{Mpc}\left[(1+z)^{-1/2}\right]_{z_{dec}}^{0} = 5820\,h^{-1}\,\mathrm{Mpc}$$

So that the particle horizon at decoupling subtended an angle

$$\theta \sim \frac{180}{6000} \sim 0.03 \sim 1.8^\circ \ll 360^\circ$$

This implies that many regions that we observe of the CMB were not in causal contact, yet still have extremely similar temperatures $\rightarrow$ Horizon problem.

ii) Monopole Problem

Grand unified theories predict magnetic monopoles, which are massive particles with net magnetic charge as a result of spontaneous symmetry breaking during the phase transition at the threshold at which the GUT is expected to occur. One expects the formation of magnetic monopoles with a density of about one per horizon volume at that epoch, while their mass is expected to be of the order of the energy scale in consideration, i.e. $m \sim T_{GUT}$. This predicts a present-day energy density in magnetic monopoles of

$$\rho_{mono,0} \sim \left(\frac{T_{GUT}}{10^{11}\,\mathrm{GeV}}\right)^4 \rho_{\gamma,0} \sim 10^{16}\rho_{\gamma,0} \qquad (\rho_{\gamma,0} \cdot$$

from which follows that monopoles are expected to completely dominate the present matter density:

$$\Omega_{mono,0} \sim 10^{16}\,\Omega_{\gamma,0} = 10^{16}\cdot 2.5\cdot 10^{-5}\,h^{-2} \sim 2.5\cdot 10^{11}\,h^{-2}$$

which is in fatal conflict with observations of the flat Universe.

iii) Flatness Problem

In the standard model, the age, density and size of the Universe are assumed to arise as initial conditions at the Planck time, when the Universe emerged from the quantum gravity epoch.
The problem arises if $\Omega = \Omega_m + \Omega_\Lambda + \Omega_r$ differs mildly from unity at the present time, because such a Universe requires extreme fine-tuning of Ω at the Planck time.

From $H^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{a^2} \Rightarrow \Omega^{-1} = \frac{\rho_{crit}}{\rho} = \frac{8\pi G}{H^2 \rho} = 1 - \frac{3kc^2}{8\pi G \rho a^2}$

$\Omega^{-1} - 1 \propto a^2 \propto \frac{1}{T^2}$ in radiation dominated epoch $z > z_{eq}$

$\Omega^{-1} - 1 \propto a \propto \frac{1}{T}$ in matter dominated epoch $z < z_{eq}$

$$\Rightarrow \frac{\Omega_{Pl}^{-1} - 1}{\Omega_0^{-1} - 1} \sim \frac{T_0}{T_{eq}} \left(\frac{T_{eq}}{T_{Pl}}\right)^2 \sim 10^{-60}$$

$\Rightarrow \Omega_{Pl}$ is 60 orders of magnitude closer to unity than the present day value Ω_0.

If for example $\Omega_0 = 0.1$ today, it must have been $1 - 10^{-59}$ at the planck time, which clearly is a fine tuning problem. That's way too exact to be possible.
A 'trivial' way out of this problem is to postulate that $\Omega \equiv 1$, but this has no proper physical explaination.

iv) Structure formation problem

The largest gravitationally bound structures, galaxy clusters, have sizes of $\sim 10\,Mpc$ and gravitational binding energies per unit mass $E/c^2 \sim 10^{-5}$. They are presumed to have grown via gravitational instability from small initial perturbations, where the gravity amplified small perturbations or inhomogeneities in the initial conditions.
Parts of a cluster needed to be in causal contact at some point which can happen only when the horizon grows large enough: $z \lesssim 10^6$
But at this high redshift there is no way to achieve these high binding energies. We need a way to start the perturbations much earlier but inflate them fast so that they acquire a much larger scale very early and still remain causally connected.

The Concept of Inflation

i) Solution of the particle horizon problem

The horizon problem arises because the comoving radius of the particle horizon of a fundamental observer is finite in the standard model:

$$x_h = \int_0^t \frac{c\,dt'}{a(t')} = \int_0^a \frac{da'}{a'}\left[\frac{8\pi G \rho(a') a'^2}{3c^2} - k\right]^{-1/2}$$

To get rid of this problem, x_h must diverge $\Rightarrow \rho \propto a^{-\beta}$ with $\beta < 2$. An example of such an energy component is vacuum energy, where $a \propto e^{Ht}$ and $H = \sqrt{8\pi G\rho/3}$.

Suppose inflation starts at some very early t_i and ends at t_e, making the period of inflation $\Delta t = t_e - t_i$, during which the forward light cone expands exponentially. The past light cone of an observer at the present time t_0 is not affected by the exponential expansion for $t > t_e$.

$\Rightarrow$ If Δt is sufficiently long, the size of the forward light cone can be larger than the size of the past light cone:

$$x_{p,ls} = \int_{t_{ls}}^{t_0} \frac{dt}{a} \simeq 3t_0$$

$$x_{f,ls} \sim \int_{t_i}^{t_{ls}} \frac{dt}{a(t)} = \frac{1}{H}\,\frac{e^{H\Delta t} - 1}{a(t_e)}$$

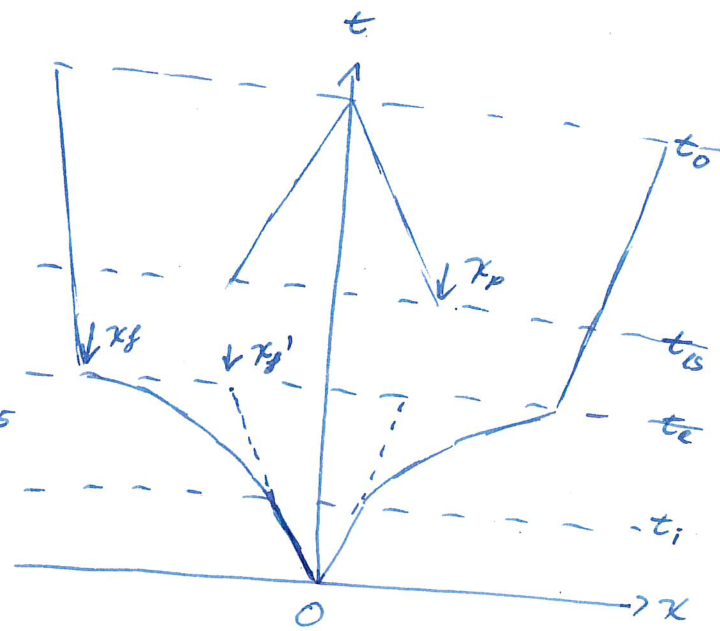
$$x_{p,ls} < x_{f,ls}$$

$$\Rightarrow \frac{e^{H\Delta t}}{H a(t_e)} > 3t_0$$

$$e^{H\Delta t} > 3H\,a(t_e)\,t_0 \sim a(t_e)\frac{t_0}{t_e} \sim 10^{25}$$

In order to solve the horizon problem: $\Delta t \gtrsim \frac{60}{H}$

corresponding to 60 e-folds of the scale factor.



x_f: forward light cone
x_f': forward light cone without inflation
x_p: past light cone

ii) Solution of the Flatness Problem

Using the same concept as for the particle horizon problem:

$$\frac{\Omega^{-1}(t_e)-1}{\Omega^{-1}(t_i)-1} = \left[\frac{a(t_i)}{a(t_e)}\right]^2 \lesssim 10^{-52}$$

Even when $\Omega(t_i)$ deviates substantially from unity, at the end of inflation $\Omega(t_e) \simeq 1$ to very high accuracy. The same number of e-foldings also solves the flatness problem.

Because of inflation the curvature radius (in physical units) increases exponentially, and the observed piece of space in the past light cone looks essentially flat after inflation even if it had a large curvature before. In this sense, inflation predicts that the Universe is spatially flat.

iii) Solution of the monopole problem

If monopoles are produced before inflation, their number density will be reduced by a factor $\sim (e^{H\Delta t})^3 \sim 10^{78}$, making the contribution of monopoles to the cosmic density completely negligible.

Realisation of Inflation

A successful inflation model needs to stop after some time, and it needs to end in a particular way: At the end of inflation, the matter and radiation density of the Universe will be virtually zero ($\rho \propto a^{-n}$, $T \propto 1/a$). $\Rightarrow$ We need a mechanism ("reheating") which at the end of inflation creates matter and radiation.

This can be realised in a natural way with a scalar field. A scalar field can cause/trigger a phase-transition.

At $t \approx t_{GUT}$, the scalar field $\varphi(x)$ is everywhere in the Universe. Its expectation value initially is $\langle \varphi \rangle = 0$. The potential $V(\varphi)$ goes from a finite value to zero for all models (construction requirement). Inflation occurs as $V(\varphi)$ decays from initial $V(\varphi)$ to $V(\varphi)=$
If there is a coupling between the inflation scalar field and the thermal bath, the required reheating is possible, so that $T(t_{end}) \sim T_{EW}$ $\lesssim$ T_{RH} [electroweak]
[This reheating is still an open issue in cosmology and subject of much debate].

The Lagrangian density of a scalar field $\varphi(\vec{x}, t)$ is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi)$$

and

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$

We want to associate $T^{\mu\nu}$ with the energy-momentum tensor of a perfect fluid, because matter-energy is at rest in the comoving frame, so we apply the same for the scalar field. We also can use the flat metric $g_{\mu\nu} = (1, -a^2, -a^2, -a^2)$ because any deviation from the flat metric will be erased very quickly. $T^{\mu\nu}$ has the form of the perfect fluid if the inhomogeneities in φ are small, so we assume φ is homogeneous $\Rightarrow \partial_i \varphi = 0$ $\Rightarrow \mathcal{L} = \frac{1}{2} \dot{\varphi}^2 - V(\varphi)$

$$\Rightarrow T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \left(\frac{1}{2} \dot{\varphi}^2 - V(\varphi) \right)$$

$$= (P + \rho) u^\mu u^\nu - g^{\mu\nu} P \quad \text{for a perfect fluid}$$

(with $u^\mu = (1, 0, 0, 0)$)

$\Rightarrow \quad P = \left(\frac{1}{2}\dot{\varphi} - V(\varphi)\right) \quad ; \qquad p = -\rho$

which is the ~~same~~ Equation of State as for the cosmological constant. This is consistent with an exponential expansion, and exactly what we want for inflation.

For a non-homogeneous φ. (only including non-mixed derivatives as to get exponential growth), then:

$$\rho = \frac{\dot{\varphi}^2}{2} + \frac{(\nabla\varphi)^2}{2a^2} + V(\varphi), \quad p = \frac{\dot{\varphi}^2}{2} - \frac{(\nabla\varphi)^2}{6a^2} - V(\varphi); \quad \rho \neq -p$$

An exponential expansion requires further conditions:

$\ddot{a} > 0 \quad \rightarrow$ Use Friedmann: $\quad \ddot{a} > 0 \Rightarrow \rho + 3p < 0$

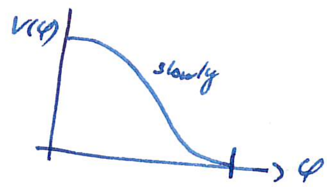
$\Rightarrow \boxed{2[\dot{\varphi}^2 - V(\varphi)] < 0}$ * $\qquad \left(3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p)\right)$

From which follows the slow-roll approximation:

$$\boxed{\dot{\varphi}^2 << V(\varphi)}$$

The field is required to move slowly over the potential well ($\equiv$ small slope)

Inflation has to stop at the zero-point energy minimum well, because then the slow-roll condition is not satisfied any longer. The lost energy is transferred to the matter-energy content. ($\rightarrow$ reheating.)



During Inflation: $\quad H^2 = \frac{8}{3}\pi G\rho - \underbrace{\frac{kc^2}{a^2}}_{=0:\ k \text{ gets flattened, a exponential growth}} = \frac{8\pi G}{3} V(\varphi) \quad$ [V dominates]

$$= \frac{1}{m_{pl}^2}\frac{8\pi V}{3} \qquad \text{with } m_{pl} \equiv \sqrt{\frac{\hbar c}{G}}, \quad \hbar = c = 1$$

* This is the minimal condition to get an acceleration

To translate the slow-roll requirement into a requirement for the shape of the potential $V(\varphi)$, we need to look at the dynamics of a scalar field. The classical equation of motion is obtained by the Euler-Lagrange equation for $S = \int \mathcal{L} \sqrt{-g}\, d^4x$

$$\Rightarrow \ddot{\varphi} + \underbrace{3H\dot{\varphi}} + \frac{dV}{d\varphi} = 0$$ ignoring spatial inhomogeneities.

Hubble drag term: The expansion of the Universe acts as a "friction" on φ

Using dimensional analysis, we estimate:

$$\dot{\varphi} \sim \frac{\varphi}{t} \Rightarrow \ddot{\varphi} \sim \frac{\varphi}{t^2} \sim \frac{\dot{\varphi}^2}{\varphi} \ll \frac{V(\varphi)}{\varphi}$$ with the slow-roll condition

$$\Rightarrow \ddot{\varphi} \ll \frac{dV}{d\varphi}$$

So with the slow-roll approximation: $3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$

The acceleration $\frac{dV}{d\varphi}$ is balanced by the Hubble drag due to the expansion. The gradient of the potential is almost compensated by the expansion.

With this relation, we can get two constraints on the potential:

i) $$\varepsilon \equiv \frac{m_{Pl}^2}{16\pi} \left(\frac{dV/d\varphi}{V} \right)^2 = \frac{m_{Pl}^2}{16\pi} \frac{(3H\dot{\varphi})^2}{V^2} \overset{\dot{\varphi}^2 \ll V}{\ll} m_{Pl}^2 \frac{(3H)^2}{V} \sim 1$$

$\Rightarrow \varepsilon \ll 1$ $\Rightarrow \left| \frac{dV}{d\varphi} \right| \ll 1$ $\Rightarrow$ the first derivative must be very small

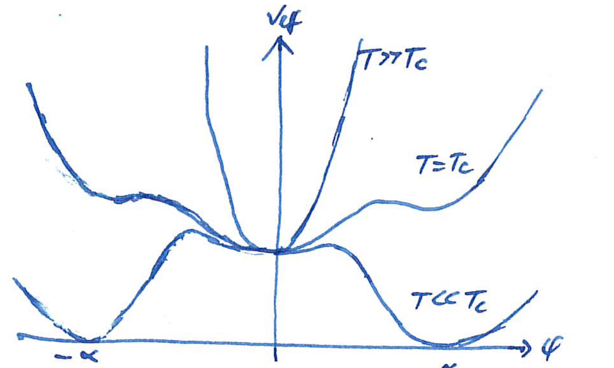
ii) Using $\frac{1}{V} \frac{d^2V}{d\varphi^2} \sim \frac{1}{\varphi V} \frac{dV}{d\varphi} \sim \left(\frac{dV}{d\varphi} \right)^2 \frac{1}{V^2}$:

$$\eta \equiv \frac{m_{Pl}^2}{8\pi} \frac{1}{V} \frac{d^2V}{d\varphi^2} \ll 1$$ $\Rightarrow$ the second derivative must also be very flat.

Models of Inflation

i) Old Inflation

The 'old inflation' model is based on a scalar field which initially gets trapped in a false vacuum at $\varphi=0$ and which at some point undergoes spontaneous symmetry breaking to its true vacuum state via a first order phase transition.



$$V_{eff} = V(\varphi) + \frac{1}{2}\underbrace{\lambda}_{\text{coupling to radiation field}} T^2\varphi^2$$

When the temperature is high, the effective potential has a single minimum at $\varphi=0$. As the temperature decreases, two other minima develop at $T=T_c$. The vacua at $\pm\alpha$ represent two true vacua of the field, while the one at $\varphi=0$ is called a false vacuum state.

When the temperature drops below T_c, the field gets trapped in the false vacuum (expectation value before is $\langle\varphi\rangle=0$). At this point, the slow-roll condition is satisfied and $\rho \sim V(0)$ is dominated by the energy density of the inflaton. The Universe expands exponentially.

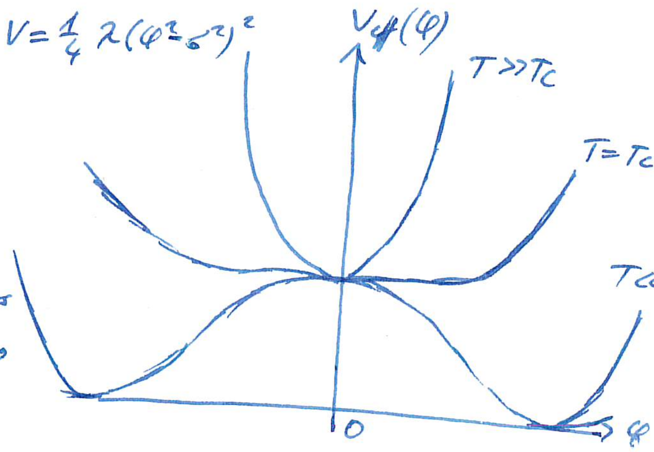
The epoch of inflation only ends when thermal fluctuation or quantum tunneling moves φ across the barrier so that it can proceed towards its true vacuum. This transition is a spontaneous symmetry breaking, and since the field value changes discontinuously, it is of first order. During this transition the energy $V(\varphi=0)$ is liberated and can be used for reheating. If the system stays in the false vacuum sufficiently long, the Universe can be inflated by a sufficiently large number of e-foldings.

$$V(\varphi) = \frac{1}{4}\varphi^4 - \frac{1}{3}(\alpha+\beta)|\varphi|^3 + \frac{1}{2}\alpha\beta\varphi^2 + V_0$$

The 'old inflation' model has a 'graceful exit' problem.
Because the transition is of first order, it proceeds through the nucleation of bubbles of true vacuum in a surrounding sea of false vacuum. These bubbles must grow in a causal way, so their sizes at the end of inflation cannot be larger than the horizon size at that time, which is much smaller than our past light cones.
Additionally, the latent heat needed for reheating is stored in the kinetic energy of the nucleated bubbles, and reheating only occurs when this kinetic energy is thermalized via bubble collisions. $\Rightarrow$ Unless the bubbles can collide and homogenize in the Hubble radius, the model will predict too large inhomogeneities to match the CMB and too little reheating.
Since the space between the bubbles is filled with exponentially expanding false vacuum, while the volume of a bubble expands only with a low power of time, percolation and homogenisation of the bubbles can never happen. Instead, inflation continues indefinitely.
Increasing the volume filling factor of the true vacuum bubbles requires the increase the nucleation rate of true-vacuum bubbles, which requires a high tunneling rate, making the inflation period too short.

ii) New Inflation

$V = \frac{1}{4}\lambda(\varphi^2 - \sigma^2)^2$



At high temperatures, V_{eff} has a single minimum at $\varphi = 0$, but below a critical $T = T_c$, this minimum disappears and becomes a local maximum, while two new minima develop.
The scalar field is again confined to the neighbourhood of $\varphi = 0$ for $T \gg T_c$. As the temperature drops to $T \simeq T_c$, the field configuration evolves towards $\varphi = \pm\sigma$ as the temperature decreases, which is a smooth change, making the spontaneous symmetry breaking occur via a second-order phase transition.

As no maxima in the potential are formed, the new inflation model doesn't require quantum tunneling.

The decay to true vacuum is continuous in all domain, which is much larger than the bubble size in old inflation. The entire Universe is within a single domain, so the horizon problem is solved and the homogeneous state after inflation guaranteed.

Via the decay of φ to photons and other particles, the Universe is reheated to $T \sim T_{\text{before-inflation}}$.

But the new inflation model also has problems:

a) In order to ensure a large enough number of e-foldings, the initial value of φ must satisfy $|\varphi_i| \ll \sigma$, however since the thermal fluctuations of φ at the initial time are expected to be $\sim \sigma$, fine-tuning is needed to get the required initial condition.

b) In order to obtain inflation, we must have $\frac{d^2V}{d\varphi^2} \ll \frac{V}{m_{pl}^2}$ $\Rightarrow \sigma \gg m_{pl}$. Since m_{pl} is the highest energy scale expected in particle physics, this is an unnatural condition.

iii) Chaotic Inflation

No phase transition is involved here, so that no initial thermal bath is required. One starts with a simple potential $V(\varphi) = m\varphi^2/2$, and inflation simply arises because of the slow motion of φ_{init} to the potential minimum. At any given point, the initial field configuration is assumed to be set by some chaotic processes. The values of φ are expected to be the same within regions of a size set by the correlation length. Inflation only occurs in those regions where the conditions needed for inflation are attained; Other regions never inflate. In a region where inflation persists for a sufficiently long period, the boundary of this region can be blown out of the current particle horizon, leaving a universe in which the initial inhomogeneities generated by the chaotic processes have no observable consequences. Our Universe is assumed to have emerged from one of such regions.

Problems: - need $m \ll m_{pl}$ at $t \sim t_{pl}$, when $m \sim m_{pl}$ is expected to solve horizon problem $\Rightarrow$ condition not natural
- monopole problem: If inflation starts too early, it might also finish too early, which might also be an inconsistency.

Cosmological Perturbations

We drop the assumption of large scale isotropy and homogeneity to allow for structure formation that we observe in the Universe today. The dominating force of structure formation is gravity, which competes with the expansion of the Universe. We will focus on the linear regime of the perturbation.

Metric Perturbations

How can we describe the perturbations?
For simplicity, we will consider the flat FRW spacetime.
Let us consider small perturbations $h_{\mu\nu}$ to the FRW metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \qquad \text{with } \bar{g}_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$$
$$\text{and } |h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}| \quad (h \text{ small})$$

It is useful to decompose the metric (perturbation) in a scalar, vectorial and tensorial part.
Recall for 3D-Euclidean space: For any vector field $\vec{v}(\vec{x})$:

$$\vec{v}(\vec{x}) = \vec{v}_{\parallel}(\vec{x}) + \vec{v}_{\perp}(\vec{x})$$

with $\vec{\nabla}\cdot\vec{v}_{\perp} = 0 \Rightarrow \vec{v}_{\parallel} \equiv \vec{\nabla} v$ with v = some scalar

$\vec{\nabla}\times\vec{v}_{\parallel} = 0 \Rightarrow \vec{v}_{\perp} \equiv \vec{\nabla}\times\vec{A}$ with $\vec{A}$ = some vector

(rot(grad) = 0, div(rot) = 0)

v and $\vec{A}$ are not uniquely defined by those relations, we have gauge freedom. Both v and $\vec{A}$ can be added a quantity without influencing $\vec{v}_{\perp}$ or $\vec{v}_{\parallel}$:

$$v \to v + \text{const}$$
$$\vec{A} \to \vec{A} + \vec{\nabla} f \qquad \text{when } f \text{ is a scalar}$$

Similarly, we can decompose the metric perturbation $h_{\mu\nu}(x)$ into

$$h_{00} = -E$$

$$h_{i0} = a\left[\frac{\partial F}{\partial x^i} + G_i\right]$$

$$h_{ij} = a^2\left[A\delta_{ij} + \frac{\partial^2 B}{\partial x_i \partial x_j} + \frac{\partial C_i}{\partial x_j} + \frac{\partial G_j}{\partial x_i} + D_{ij}\right]$$

where A, B, E, F are scalars, C_i, G_i are divergenceless vectors and D_{ij} is a traceless, symmetric and divergenceless tensor. Hence they satisfy:

$$\frac{\partial C_i}{\partial x^i} = \frac{\partial G_i}{\partial x^i} = 0\,; \quad \frac{\partial D_{ij}}{\partial x^i} = 0, \quad D_{ii} = 0, \quad D_{ij} = D_{ji}.$$

$h_{\mu\nu}$, being a symmetric 4×4 tensor, has 10 degrees of freedom. This number remains with the decomposition:

	# components	# restrictions
A, B, E, F	$4\times 1 = 4$	
C_i, G_i	$2\times 3 = 6$	2×1 (div. less)
D_{ij}	9	3 (symmetric) $+ 1$ (traceless) $+3$ (div. less for each j: $\frac{\partial D_{ij}}{\partial x^i} = 0$)
	19	$-\ 9$

$$= \underline{\underline{10}}$$

The decomposition theorem states that scalar, vector and tensor perturbation modes do not couple to first order, i.e. they evolve independently. $\Rightarrow$ Einstein's field equations can be solved for each mode separately.

It can be shown that the amplitude of the vector modes decays as a function of time, while the tensor modes correspond to gravitational waves, which are only important for CMB polarisation. We will only consider scalar modes here.

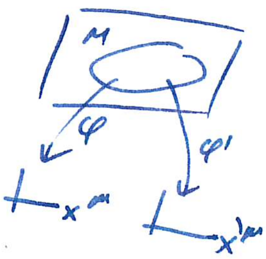
Gauge Transformations

Consider a spacetime coordinate transformation:

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$$

where ε^μ is small. Under such a transformation, the metric transforms as

$$g'_{\mu\nu}(x') = g_{\lambda\kappa}(x) \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\kappa}{\partial x'^\nu}$$



$$= \left(\delta^\lambda_\mu - \frac{\partial \varepsilon^\lambda}{\partial x^\mu}\right)\left(\delta^\kappa_\nu - \frac{\partial \varepsilon^\kappa}{\partial x^\nu}\right) g_{\lambda\kappa}$$

$$\simeq \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) - \frac{\partial \varepsilon^\lambda}{\partial x^\mu} \bar{g}_{\lambda\nu}(x) - \frac{\partial \varepsilon^\kappa}{\partial x^\nu} \bar{g}_{\mu\kappa}(x)$$

to first order in ε, h

Such a coordinate transformation affects the coordinates and unperturbed fields as well as the perturbations to the fields. In order to derive the transformations to the perturbed fields which leave the physics invariant, we consider gauge transformations:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x)$$

For gauge transformations, we keep the coordinate system, but change the metric tensor in a way that leaves the physics invariant. For coordinate transformations, however, the tensor isn't changed, but the coordinates are.

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x+\varepsilon) = g'_{\mu\nu}(x) + \frac{\partial}{\partial x^\lambda} g'_{\mu\nu}(x') \varepsilon^\lambda + O(\varepsilon^2)$$

$$\Rightarrow g'_{\mu\nu}(x) = g'_{\mu\nu}(x') - \frac{\partial g'_{\mu\nu}(x')}{\partial x^\lambda} \varepsilon^\lambda + O(\varepsilon^2)$$

$$= \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) - \frac{\partial \varepsilon^\lambda}{\partial x^\mu} \bar{g}_{\lambda\nu}(x) - \frac{\partial \varepsilon^\kappa}{\partial x^\nu} \bar{g}_{\kappa\mu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\lambda} \varepsilon^\lambda + O(\varepsilon^2)$$

With $g'_{\mu\nu}(x) = \bar{g}'_{\mu\nu}(x) + h'_{\mu\nu}(x) = \bar{g}_{\mu\nu} + h'_{\mu\nu}$:

$\bar{g}_{\mu\nu} = \bar{g}'_{\mu\nu}$ by construction for small perturb.

$$\Delta h_{\mu\nu}(x) \equiv h'_{\mu\nu} - h_{\mu\nu} = -\frac{\partial \varepsilon^\lambda}{\partial x^\mu}\bar{g}_{\lambda\nu} - \frac{\partial \varepsilon^\kappa}{\partial x^\nu}\bar{g}_{\kappa\mu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\lambda}\varepsilon^\lambda$$

$\Rightarrow$ Gauge transformations correspond to changes in the metric perturbations.

transformation of metric perturbations under gauge transform:

$$\Rightarrow \begin{cases} \Delta h_{00} = 2\frac{\partial \varepsilon^0}{\partial t} = -2\frac{\partial \varepsilon_0}{\partial t} \\ \Delta h_{i0} = -\frac{\partial \varepsilon^i}{\partial t}a^2 + \frac{\partial \varepsilon^0}{\partial x^i} = -\frac{\partial \varepsilon_i}{\partial t} + \frac{2}{a}\frac{da}{dt}\varepsilon_i - \frac{\partial \varepsilon_0}{\partial x_i} \\ \Delta h_{ij} = -\frac{\partial \varepsilon^i}{\partial x^j}a^2 - \frac{\partial \varepsilon^j}{\partial x^i}a^2 - 2a\frac{da}{dt}\delta_{ij}\varepsilon^0 = -\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} + 2a\frac{da}{dt}\delta_{ij}\varepsilon_0 \end{cases}$$

Where $\varepsilon_\mu = g_{\mu\nu}\varepsilon^\nu = (\bar{g}_{\mu\nu} + h_{\mu\nu})\varepsilon^\nu \simeq \bar{g}_{\mu\nu}\varepsilon^\nu$ was used.

Now we decompose the spatial part ε^i of ε^μ into the gradient of a scalar plus a divergenceless vector:

$$\varepsilon_i \equiv \frac{\partial \varepsilon^S}{\partial x^i} + \varepsilon_i^V \quad \text{with} \quad \frac{\partial \varepsilon_i^V}{\partial x^i} = 0$$

With this, we get the transformations of metric perturbations under gauge transformations for the scalar modes:

$$\Delta A = \frac{2}{a}\frac{da}{dt}\varepsilon_0$$

$$\Delta B = -\frac{2}{a^2}\varepsilon^S$$

$$\Delta E = 2\frac{d\varepsilon_0}{dt}$$

$$\Delta F = \frac{1}{a}\left(-\varepsilon_0 - \frac{d\varepsilon^S}{dt} + \frac{2}{a}\frac{da}{dt}\varepsilon^S\right)$$

Where the fully written out $\Delta h_{\mu\nu}$ for $\varepsilon_i = \frac{\partial \varepsilon^S}{\partial x^i} + \varepsilon_i^V$ is:

$$\Delta h_{00} = -2\frac{\partial \varepsilon_0}{\partial t} = -E$$

$$\Delta h_{i0} = \frac{\partial}{\partial x^i}\left[-\frac{\partial \varepsilon^S}{\partial t} - \varepsilon_0 + \frac{2}{a}\frac{da}{dt}\varepsilon^S\right] + \frac{\partial}{\partial t}\varepsilon_i^V + \frac{2}{a}\frac{da}{dt}\varepsilon_i^V = a\left[\frac{\partial F}{\partial x^i} + G_i\right]$$

$$\Delta h_{ij} = 2a\frac{da}{dt}\varepsilon_0\delta_{ij} - 2\frac{\partial^2 \varepsilon^S}{\partial x^i \partial x^j} - \frac{\partial \varepsilon_i^V}{\partial x^j} - \frac{\partial \varepsilon_j^V}{\partial x^i} =$$

$$= a^2\left[A\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij}\right]$$

Choice of Gauge

i) Newtonian Gauge

choose ε^s such that $B \stackrel{!}{=} 0$

ε^0 such that $F \stackrel{!}{=} 0$

$E \equiv 2\Psi$

$A \equiv 2\varphi$

$\Rightarrow g_{00} = -1 - 2\Psi, \quad g_{0i} = 0, \quad g_{ij} = a^2 \delta_{ij} [1 + 2\varphi]$

keeping only scalar perturbations.

ii) Synchronous Gauge

choose ε^s such that $F \stackrel{!}{=} 0$

ε^0 such that $E \stackrel{!}{=} 0$

$\Rightarrow g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = a^2 \left[(1+A)\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} \right]$

keeping only the scalar perturbations.

iii) Other possibilities

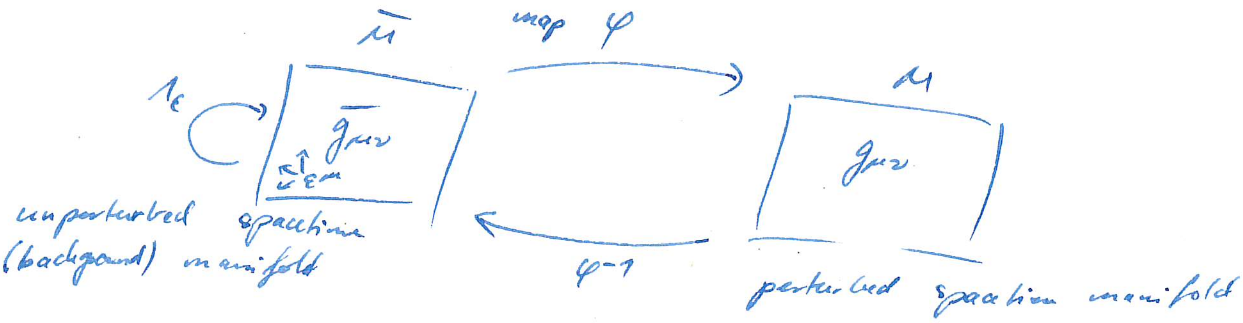
- co-moving gauge

it is also possible to perform cosmological perturbation theory solely in terms of gauge-invariant variables (= "gauge invariant perturbation theory").

We will choose the Newtonian gauge, which is easiest to relate to the Newtonian limit.

Geometrical interpretation of gauge transformations

To define perturbations, we need to compare two manifolds to each other:



Then we can define $h_{\mu\nu} \equiv (\varphi^{-1} g_{\mu\nu}) - \bar{g}_{\mu\nu}$

The choice of φ is not strictly constrained, which is the source that gives rise to gauge freedom. There are many permissible mappings φ so that the perturbation is small.

Consider for example a mapping Λ_ε of $\bar{M}$ onto itself, induced by a vector field ε^μ on $\bar{M}$, i.e. $x^\mu \to x^\mu + \varepsilon^\mu(x)$ in a given coordinate system.

Then $(\varphi \circ \Lambda_\varepsilon)^{-1} = \Lambda_\varepsilon^{-1} \circ \varphi^{-1}$ is a new mapping between M to $\bar{M}$, which induces new metric perturbations:

$$h_{\mu\nu}^{(\varepsilon)} = [(\varphi \circ \Lambda_\varepsilon)^{-1} g]_{\mu\nu} - \bar{g}_{\mu\nu} = [\Lambda_\varepsilon^{-1}(\varphi^{-1} g)]_{\mu\nu} - \bar{g}_{\mu\nu}$$

All possible infinitesimal Λ_ε's correspond to the gauge transformations we considered above.

A coordinate transformation corresponds to relabelling of the coordinates.
A gauge transformation is a choice of mapping between $\bar{M}$ and M.
The problem is that some gauge transformations look like physical perturbations, but aren't. You can think of gauge transform. as redefining the reference potential.

The metric $g_{\mu\nu}$ is a symmetric 4×4 tensor, therefore it has 10 independent components. However with gauge freedom with gauge transformations generated by vector fields ε^μ which have 4 components, only $10-4=6$ physical degrees of freedom are left.

Einstein Equations

In the Newtonian Gauge:

$$g_{00} = -1 - 2\Psi \,, \quad g_{0i} = 0 \,, \qquad g_{ij} = a^2 \delta_{ij} \left[1 + 2\varphi\right]$$

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right]$$

$$\Gamma^{0}_{00} = \partial_0 \Psi$$

$$\Gamma^{0}_{0i} = \Gamma^{0}_{i0} = \partial_i \Psi$$

$$\Gamma^{0}_{ij} = \delta_{ij} a^2 \left[H + 2H [\varphi - \Psi) - \partial_0 \varphi \right]$$

$$\Gamma^{i}_{00} = \frac{1}{a^2} \partial_i \Psi$$

$$\Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \delta_{ij} \left(H + \partial_0 \varphi \right)$$

$$\Gamma^{i}_{jk} = \delta_{ij} \partial_k \varphi + \delta_{ik} \partial_j \varphi - \delta_{jk} \partial_i \varphi$$

$$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha} \Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu} \Gamma^{\beta}_{\mu\alpha}$$

$$R_{00} = -3 \frac{1}{a} \frac{d^2 a}{dt^2} + \frac{1}{a^2} \partial_i^2 \Psi - \partial_0^2 \varphi + 3H \left(\partial_0 \Psi - 2\partial_0 \varphi \right)$$

$$R_{ij} = \delta_{ij} \Big[\Big(2a^2 H^2 + a \frac{d^2 a}{dt^2} \Big) (1 + 2\varphi - 2\Psi) + a^2 H \left(6\partial_0 \varphi - \partial_0 \Psi \right) + a^2 \partial_0^2 \varphi - \partial_i^2 \Psi \Big]$$

$$R = R^{\mu}_{\ \mu} = 6 \Big(H^2 + \frac{1}{a} \frac{d^2 a}{dt^2} \Big) (1 - 2\Psi) - \frac{2}{a^2} \partial_i^2 \Psi + 6\partial_0^2 \varphi - 6H \left(\partial_0 \Psi - 4\partial_0 \varphi \right) - \frac{4}{a^2} \partial_i^2 \varphi$$

The results can be checked by comparing the zeroth order terms to the results of the unperturbed FRW metric.

$\varphi, \Psi = \mathcal{O}(1)$

Then $G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2} g^\mu{}_\nu R$

$$\Rightarrow G^0{}_0 = 3\left[\frac{1}{a}\frac{d^2a}{dt^2} - \frac{1}{a}\left(\frac{da}{dt}\right)^2\right] + 3\left(\frac{da}{dt}\right)^2 - 6\frac{1}{a}\frac{da}{dt}\partial_0\varphi + 6\left(\frac{1}{a}\frac{da}{dt}\right)^2\psi + \frac{2}{a^2}\partial_i{}^2\varphi$$

$$G^i{}_j = A\,\delta_{ij} - \frac{1}{a^2}\left(\partial_i\partial_j\varphi + \partial_i\partial_j\psi\right)$$

where A is some complicated thing which are all proportional to δ_{ij} and therefore only contribute to the trace of $G^i{}_j$. We will only need to consider the longitudinal and traceless part of the Einstein tensor. *

Einstein's field equations:

$$G^\mu{}_\nu = 8\pi G\, T^\mu{}_\nu$$

decomposition

$$\longrightarrow \left(\bar{G}^\mu{}_\nu + \Delta G^\mu{}_\nu\right) = 8\pi G\left(\bar{T}^\mu{}_\nu + \Delta T^\mu{}_\nu\right)$$

→ We need to compute the traceless part of $T^\mu{}_\nu$ from the Boltzmann equations in order to compute the equations governing the evolution of perturbations.

* We only need 2 equations for φ and ψ, so we make our lives easier by not computing the trace because we don't need it.

Perturbed Boltzmann Equation

Non-relativistic Boltzmann Equation

The distribution function $f(x^i, p^i, t)\, d^3x\, d^3p$ describes the number of particles in the phase space volume $d^3x\, d^3p$ at the time t centered on x^i, p^i.

Macroscopic quantities can be obtained by integrating the Boltzmann eqn.:

$$n(\vec{x},t) = \int d^3p\, f(\vec{x},\vec{p},t)$$

$$\rho_m(\vec{x},t) = m \int d^3p\, f(\vec{x},\vec{p},t)$$

$$\vec{v}(\vec{x},t) = \frac{1}{n} \int d^3p\, \frac{\vec{p}}{E} f(\vec{x},\vec{p},t) = \frac{1}{n}\int d^3p\, \frac{\vec{p}c^2}{mc^2} f(\vec{x},\vec{p},t) \text{ for non-relativistic particles}$$

$$\rho_E(\vec{x},t) = \int d^3p\, E f(\vec{x},\vec{p},t)$$

$$P(\vec{x},t) = \int d^3p\, \frac{p^2}{3E} f(\vec{x},\vec{p},t)$$

The time evolution of the collisionless Boltzmann function is

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = 0}$$

In this case, f is constant as one moves along with the fluid.

When collisions are included:

$$\boxed{\frac{df}{dt} = C[f]}$$

Where $C[f]$ is the collisional term:

For a reaction ① + ② → ③ + ④ with $\vec{p}_1 + \vec{p}_2 \to \vec{p}_3 + \vec{p}_4$:

$$C[f] = \int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \times$$ sum over all interactions

$$\times\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \times$$ conserves momentum

$$\times\, \delta(E_1 + E_2 - E_3 - E_4) \times$$ conserves energy

$$\times\, |\mathcal{M}|^2 \times$$ scattering matrix

$$\times\, (f_3 f_4 - f_1 f_2)$$ particles entering/leaving phase space volume

If we are in equilibrium of a spatially homogeneous system the distribution function for non-relativistic particles is Maxwell-Boltzmann:

$$f(\vec{x}, \vec{p}, t) \propto e^{-E/T}$$

More generally:

$$f_{\text{Bose-Einstein}} \propto \frac{1}{e^{E/T} - 1}$$

$$f_{\text{Fermi-Dirac}} \propto \frac{1}{e^{E/T} + 1}$$

$$f_{\text{classical}} \overset{E \gg T}{\propto} e^{-E/T}$$

Relativistic Boltzmann Equation

In GR, the particles are described by x^μ and p^μ where

$$p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = \underbrace{-m^2}$$

We thus have $2 \times 4 - 1$ degrees of freedom.

Consider the distribution function $f = (t, x^i; p^i)$ or $f(t, x^i; p, \hat{p}^i)$

when $p = |\vec{p}|$, $\hat{p}^i$ = unit vector direction.

The moments of the distribution function are:

$$n(x^i; t) = \frac{1}{(2\pi)^3} \int f(t, x^i; p^i) d^3p$$

[integral over smallest possible phase space volume limited by Heisenberg]

$$v^i(x^i; t) = \frac{1}{n} \frac{1}{(2\pi)^3} \int f(t, x^i; p^i) \frac{p^i_{phys}}{E} f(t, x^i; p^i)$$

$$\rho(x^i; t) = \frac{1}{(2\pi)^3} \int d^3p \, E f(t, x^i; p^i)$$

$$P(x^i; t) = \frac{1}{(2\pi)^3} \int d^3p \, \frac{p^2}{3E} f(t, x^i; p^i)$$

$$T^{\mu\nu} = \frac{1}{(2\pi)^3} \frac{1}{E} \int d^3p \, p^\mu p^\nu f(t, x^i; p^i)$$

The time evolution is given by:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p^i} \frac{dp^i}{dt} = C[f]$$

The collision term is given by

$$C[f_1] = \frac{1}{E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \times$$

$$\times (2\pi)^4 \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$ Momentum conservation

$$\times \delta(E_1 + E_2 - E_3 - E_4)$$ Energy conservation

$$\times \underbrace{|\mathcal{M}|^2}_{\text{Scattering amplitude}} \times [f_3 f_4 - f_1 f_2]$$ "Reaction rate", sources - sinks

[$\frac{1}{2E_i}$: ? So that collision integral is Lorentz invariant]

Overview

i) The timely evolution of the distribution function for a species i is given by $\frac{df_i}{dt} = C[f_i]$

ii) The perturbed FRW metric in the Newtonian gauge is given by

$$g_{00} = -(1+2\Psi)$$
$$g_{0i} = 0$$
$$g_{ij} = a^2 \delta_{ij}(1+2\Phi)$$

iii) The Einstein Equations are given by

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

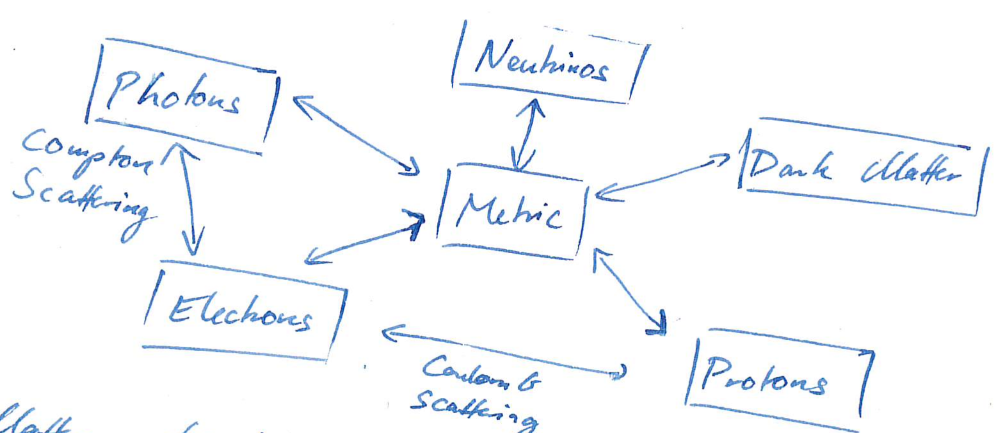
$G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu}$, $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$ for perturbed metric

iv) The stress-energy tensor can be calculated with

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 E} p^\mu p^\nu f(t, x^i, p^i)$$

This set of equations is called the "Einstein-Boltzmann" eqns.

We are interested in the anisotropies in the cosmic distribution of photons and inhomogeneities in the matter. To solve for the photons and dark matter, we need to solve for all other species too: they are coupled to each other through interactions ($\hat{=}$ collision integral)



Dark Matter doesn't interact with matter, only with the metric through gravity. The interaction between protons and photons is negligible due to the proton's high mass.

The Collisionless Boltzmann Equation for Photons

Consider first the LHS of the Boltzmann equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt}$$

with the metric $g_{00} = -(1+2\Psi)$, $g_{0i} = 0$, $g_{ij} = a^2\delta_{ij}(1+2\Phi)$

For photons $(m=0)$, the momentum vector is defined as $p^\mu \equiv \frac{dx^\mu}{d\lambda}$ where λ parametrizes the particle's path.

Also:
$$p^\mu p_\mu = 0$$
$$= g_{\mu\nu}p^\mu p^\nu = -(1+2\Psi)(p^0)^2 + \underbrace{g_{ij}p^i p^j}_{\equiv p^2}$$

$$\Rightarrow p^0 = \frac{p}{\sqrt{1+2\Psi}} \approx (1-\Psi)p \quad \text{for } |\Psi| \ll 1$$

Expression for p^2:

$$p^2 = g_{ij}p^i p^j = a^2\delta_{ij}p^i p^j(1+2\Phi) = a^2(1+2\Phi)|\vec{p}|^2$$

$$\Rightarrow \underset{①}{p} = a\sqrt{1+2\Phi}\,\underset{②}{|\vec{p}|} \approx a(1+\Phi)|\vec{p}|$$

Note that p ① is physical, while $|\vec{p}|$ ② is comoving.

Now let's substitute $\vec{p}$ with $|\vec{p}|\hat{p}^i$, where $\delta_{ij}\hat{p}^i\hat{p}^j = 1$ is the unit direction vector. Then it follows:

$$p^i = |\vec{p}|\hat{p}^i = \frac{p}{a(1+\Phi)}\hat{p}^i \approx \frac{p}{a}(1-\Phi)\hat{p}^i$$

$$\Rightarrow \underset{①}{p}\hat{p}^i = a(1+\Phi)\underbrace{\underset{②}{|\vec{p}|}\hat{p}^i}_{\equiv p^i} \equiv p^i_{phys}$$

To insert our finding into the Boltzmann equation, we use:

$$p^\mu = \frac{dx^\mu}{d\lambda} \Rightarrow \frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = p^i\frac{d\lambda}{dx^0} = p^i\left(\frac{dx^0}{d\lambda}\right)^{-1} = \frac{1}{p^0}p^i$$

$$= \frac{1}{(1-\Psi)p}\frac{p}{a}(1-\Phi)\hat{p}^i \approx \frac{1}{a}(1+\Psi-\Phi)\hat{p}^i$$

$[\Psi\Phi = \mathcal{O}(2)]$

This gives us for the second term of the Boltzmann equ:

$$\frac{\partial f}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial f}{\partial x^i}\frac{1}{a}(1+\Psi-\Phi)\hat{p}^i$$

But: For an isotropic Universe in zeroth order, the gradient of the distribution function is $\frac{\partial f}{\partial x^i}=0$. Therefore, $\frac{\partial f}{\partial x^i}$ must be first order, leaving the second term up to first order with:

$$\frac{\partial f}{\partial x^i}\frac{dx^i}{dt} = \frac{1}{a}\frac{\partial f}{\partial x^i}\hat{p}^i$$

The third term of the Boltzmann equation is

$$\frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} \simeq \frac{\partial f}{\partial p}\frac{dp}{dt} \quad \text{to first order}$$

because both $\frac{\partial f}{\partial \hat{p}^i}$ and $\frac{d\hat{p}^i}{dt}$ are first order:

$\frac{\partial f}{\partial \hat{p}^i}$ must be first order because zeroth order is isotropic, so if non-zero, it must be first order. Same goes for the direction $\frac{d\hat{p}^i}{dt}$: The direction of photons in a homogeneous, isotropic Universe is constant.

To compute $\frac{dp}{dt}$, consider the geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \Rightarrow \frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta = 0$$

$$\Rightarrow \frac{dp^0}{d\lambda} = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta = \frac{dp^0}{dt}\frac{dt}{d\lambda} = \frac{dp^0}{dt}\frac{dx^0}{d\lambda} = \frac{dp^0}{dt}p^0$$

$$= \frac{d}{dt}[p(1-\Psi)]\,p(1-\Psi)$$

$$\Rightarrow \frac{d}{dt}[p(1-\Psi)] = -\Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+\Psi)$$

$$= \frac{dp}{dt}(1-\Psi) - p\frac{d\Psi}{dt} \qquad \Big|\cdot(1+\Psi)$$

$$\Rightarrow \frac{dp}{dt}(1-\Psi) = p\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+\Psi)$$

$$\frac{dp}{dt}(1-\Psi)(1+\Psi) = \frac{dp}{dt}(1-\Psi^2) = \frac{dp}{dt}$$

$$\frac{dp}{dt} = p(1+\Psi)\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+\Psi)^2$$

$$= p(1+\Psi)\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+2\Psi)$$

Now using $\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x^i}\frac{dx^i}{dt}$ $\quad [\psi = \psi(\vec{x}, t)]$

$$= \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x^i}\frac{dx^i}{d\lambda}\frac{d\lambda}{dt}$$

$$= \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x^i} p^i \cdot \frac{1}{p^0} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x^i}\frac{1}{(1-\psi)p}\frac{p}{a}(1-\varphi)\hat{p}^i$$

$$= \frac{\partial \psi}{\partial t} + \underbrace{\frac{\partial \psi}{\partial x^i}}_{1^{st}\text{ Order}}\frac{\hat{p}^i}{a}(1 + \underbrace{\psi - \varphi}_{1^{st}\text{ order}})$$

$$\approx \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x^i}\frac{\hat{p}^i}{a}$$

$$\Rightarrow \frac{dp}{dt} = p(1+\psi)\frac{d\psi}{dt} - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+2\psi)$$

$$\approx p\frac{d\psi}{dt} - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+2\psi)$$

$$= p\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x^i}\frac{\hat{p}^i}{a}\right) - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+2\psi)$$

Now evaluate the sum over Christoffel symbols:

$$\Gamma^0_{00} = \frac{\partial \psi}{\partial t}$$

$$\Gamma^0_{0i} = \Gamma^0_{i0} = \frac{\partial \psi}{\partial x^i}$$

$$\Gamma^0_{ij} = \delta_{ij}\, a^2\left[H + 2H(\varphi - \psi) + \frac{\partial \varphi}{\partial t}\right]$$

$$\Rightarrow \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p} = \Gamma^0_{00}(p^0)^2 + 2\Gamma^0_{0i}p^0p^i + \Gamma^0_{ij}p^ip^j$$

$$= \frac{\partial \psi}{\partial t}\underbrace{(p^0)^2}_{=p^2(1-\psi)^2 \approx p^2} + 2\frac{\partial \psi}{\partial x^i}\underbrace{p^0p^i}_{p(1-\psi)\frac{p}{a}\hat{p}^i(1-\varphi) \approx p^2\frac{\hat{p}^i}{a}} + a^2\left[H + 2H(\varphi-\psi) + \frac{\partial \varphi}{\partial t}\right]\cdot\underbrace{\delta_{ij}p^ip^j}_{|\vec{p}|^2 = \frac{p^2}{a^2}(1-2\varphi)}$$

$$\approx \frac{\partial \psi}{\partial t}p^2 + 2\frac{\partial \psi}{\partial x^i}\frac{p^2\hat{p}^i}{a} + \frac{p^2}{a^2}(1-2\varphi)a^2\left[H + 2H(\varphi-\psi) + \frac{\partial \varphi}{\partial t}\right]$$

$$= \frac{\partial \psi}{\partial t}p^2 + 2\frac{\partial \psi}{\partial x^i}\frac{p^2\hat{p}^i}{a} + p^2\left[H - 2\varphi H + 2H\varphi - 2H\psi + (1-2\varphi)\frac{\partial \varphi}{\partial t}\right]$$

$$= \frac{\partial \psi}{\partial t}p^2 + 2\frac{\partial \psi}{\partial x^i}\frac{p^2\hat{p}^i}{a} + p^2\left[H(1-2\psi) + \underbrace{(1-2\varphi)}_{\approx 1 \text{ to first order}}\frac{\partial \varphi}{\partial t}\right]$$

$$\Rightarrow \Gamma^0_{\alpha\beta} \frac{p^\alpha p^\beta}{p}(1+2\Psi) = \frac{\partial \Psi}{\partial t} p + 2 \frac{\partial \Psi}{\partial x^i} \frac{p^i}{a} + p\left[H(1-2\Psi) + \frac{\partial \Phi}{\partial t}\right](1+2\Psi)$$

$$= p\left[\frac{\partial \Psi}{\partial t} + 2\frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a} + \left(H(1-4\Psi^2) + \frac{\partial \Phi}{\partial t}\right)\right]$$

$$= p\left[\frac{\partial \Psi}{\partial t} + 2\frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a} + H + \frac{\partial \Phi}{\partial t}\right]$$

$$= p\left[2\frac{\partial \Psi}{\partial t} + 2\frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a} + H\right]$$

$$\Rightarrow \frac{dp}{dt} = p\left(\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right) - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+2\Psi)$$

$$= p\left(\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right) - p\left[\frac{\partial \Psi}{\partial t} + 2\frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a} + H + \frac{\partial \Phi}{\partial t}\right]$$

$$= -p\left[\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a} + H\right].$$

$$\Rightarrow \frac{\partial f}{\partial p}\frac{dp}{dt} = -p\frac{\partial f}{\partial p}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$$

$$\Rightarrow \boxed{\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{①} + \underbrace{\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i}}_{②} - p\frac{\partial f}{\partial p}\left[\underbrace{H}_{③} + \underbrace{\frac{\partial \Phi}{\partial t}}_{④} + \underbrace{\frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a}}_{⑤}\right]}$$

- Terms ① and ③ are zeroth order.
- When integrated, terms ① and ② yield the Euler and Continuity equations.
- Term ③ $\hat{=}$ photons loose energy as the Universe expands.
- Term ④ and ⑤ encode the effects of over-/underdense regions.

To proceed, we need to assume a form for f. We expand f about its zero-order Bose-Einstein value:

$$f(\vec{x}, p, \hat{p}, t) = \frac{1}{\exp\left\{\frac{p}{T(t)[1+\Theta(\vec{x}, \hat{p}, t)]}\right\} - 1} = f^{(0)}(p, T(1+\Theta))$$

The zero-order temperature T is a function of time only. The perturbation is characterized by $\Theta = \frac{\delta T}{T}$ which depends on $\vec{x}$, $\hat{p}$ and t, but not p. This follows directly from the fact that the magnitude of the photon momentum is virtually unchanged during a compton scatter.

The perturbation Θ is small, so we expand:

$$\begin{aligned} f(\vec{x}, p, \hat{p}, t) &= f^{(0)}(p, T) + \frac{\partial f^{(0)}}{\partial T} T\Theta + \mathcal{O}(\Theta^2) \\ &= f^{(0)}(p, T) - p\frac{\partial f^{(0)}}{\partial T}\Theta \end{aligned}$$

When we used that

$$\frac{d}{dT}\left(\frac{1}{e^{p/T}-1}\right) = \frac{+1}{(e^{p/T}-1)^2}\frac{p}{T^2}e^{p/T}$$

$$\frac{d}{dp}\left(\frac{1}{e^{p/T}-1}\right) = \frac{-1}{(e^{p/T}-1)^2}\frac{1}{T}e^{p/T}$$

$$\Rightarrow \quad p\frac{\partial f}{\partial p} = -T\frac{\partial f}{\partial T}$$

Using this ansatz, we can derive the full form of $\frac{df}{dt}$ and separate the terms by order:

$$\left.\frac{df}{dt}\right|_{0^{th}} = \frac{\partial f^{(0)}}{\partial t} - p\frac{\partial f^{(0)}}{\partial p} H = 0$$

$$\left.\frac{df}{dt}\right|_{1^{st}} = -p\frac{\partial f^{(0)}}{\partial p}\Bigg[\underbrace{\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}}_{\text{"Free streaming": anisotropies on increasingly small scales as Universe expands}} + \underbrace{\frac{\partial\phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}}_{\text{Effects of gravity}}\Bigg]$$

Note that every time x appears in the equations, it is accompanied by a, because that is the physical distance.

Also note that $\frac{df}{dt}\Big|_{\text{zeroth}} = 0$. There is no zeroth order collision term, because the zero-order distribution function is set precisely by the requirement that the collision vanishes, as an equilibrium state is demanded.

Now let's have a look at the LHS of the Boltzmann eqn, the collision term.
For photons, the dominant scattering process is Compton scattering

$$e^-(\vec{q}) + \gamma(\vec{p}) \rightleftharpoons e^-(\vec{q}\,') + \gamma(\vec{p}\,')$$

with $\vec{q}, \vec{p}$ being the momenta of the particles.

The Collision term is given by:

$$C[f] = \frac{1}{E(p)} \int \frac{d^3q}{(2\pi)^3 2E_e(q)} \int \frac{d^3q'}{(2\pi)^3 2E_e(q')} \int \frac{d^3p'}{(2\pi)^3 2E(p')} \times$$
$$\times (2\pi)^4 \delta^3(\vec{p} + \vec{q} - \vec{p}\,' - \vec{q}\,') \, \delta(E(p) + E(q) - E(p') - E(q')) \times$$
$$\times |\mathcal{M}|^2 \times \left[f_e(q') f(p') - f_e(q) f(p) \right]$$

The time of interest for us is $T \sim 100\,eV \ll m_e \simeq 511\,keV$, so the energies of the particles are given by

$$E(p) = p$$
$$E(q) = \sqrt{q^2 + m_e^2} \approx m_e + \frac{q^2}{2m_e}$$

With $T \ll m_e$, we can also approximate to be in the Thomson scattering regime where $|\mathcal{M}|^2 = 8\pi \sigma_T m_e^2 \overset{!}{=} \text{const}$.
This is wrong for two reasons:

i) The amplitude squared has an angle dependence.

ii) The amplitude squared has a polarisation dependence.

Using these facts and solving the integrals, we get

$$C[f(p)] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b]$$

with $\Theta_0 = \frac{1}{4\pi} \int d\Omega' \, \Theta(\vec{p}\,', \vec{x}, t)$ = monopole part of perturbation

and $\vec{v}_b$ = velocity of baryons (electrons)

Finally, we can put the whole Boltzmann equation together:

$$-p \frac{\partial f^{(0)}}{\partial p} \left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i} + \frac{\partial \varphi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b]$$

$$\Rightarrow \frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i} + \frac{\partial \varphi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b]$$

Expressed with the conformal time $d\eta = dt/a$:

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\varphi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a [\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b]$$

Now Fourier transform the equation:

$$\tilde{Q}(\vec{k}) = \int d^3x \, Q(\vec{x}) \, e^{-i\vec{k}\vec{x}}$$

so that $\frac{\partial}{\partial x^i} \to ik_i$

Furthermore, let $\mu \equiv \frac{\vec{k} \cdot \hat{p}}{k}$ and assume $\vec{v}_b = v_b \hat{k}$ and use the optical depth $\tau(\eta) = \int_\eta^{\eta_0} d\eta' \, a \frac{1}{\ell} = \int d\eta \, n_e \sigma_T a$

$$\Rightarrow \frac{d\tau}{d\eta} = \dot{\tau} = -n_e \sigma_T a$$

$$\Rightarrow \boxed{\dot{\tilde{\Theta}} + ik\mu \tilde{\Theta} + \dot{\tilde{\varphi}} + ik\mu \tilde{\Psi} = -\dot{\tau} [\tilde{\Theta}_0 - \tilde{\Theta} + \mu \tilde{v}_b]}$$

Boltzmann Equation for photons

The Boltzmann Equation for Cold Dark Matter

The dark matter always behaves like a fluid so it can always be described completely by $T_{\mu\nu}$.

- "Dark": It doesn't interact with any of the other constituents of the Universe. $\Rightarrow$ No collision term.
- "Cold": = Non-relativistic; $m \gg p$

Difference from photon case: Not massless particles anymore,

$$g_{\mu\nu} p^\mu p^\nu = -m^2 \quad , \quad p^\mu = \left(E(1-\Psi), \, p\hat{p}^i \frac{1-\varphi}{a}\right)$$

Using E instead of p as an independent variable gives:

$$\frac{df_{dm}}{dt} = \frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{dm}}{\partial E}\frac{dE}{dt} + \frac{\partial f_{dm}}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}$$

Analogously to the photon case, we get:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\left[\frac{da/dt}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \varphi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0$$

The main difference is the presence of the factors of $\frac{p}{E}$, or velocity. For dark matter particles, these velocity factors suppress any free streaming.

Using the fact that $n_{dm} = \int \frac{d^3p}{(2\pi)^3} f_{dm}$ and $v^i \equiv \frac{1}{n_{dm}} \int \frac{d^3p}{(2\pi)^3} f_{dm} \frac{p\hat{p}^i}{E}$, we can compute the zeroth order moment of the Boltzmann eqn:

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial(n_{dm} v^i)}{\partial x^i} + 3\left[\frac{da/dt}{a} + \frac{\partial \varphi}{\partial t}\right] n_{dm} = 0$$

The zero-order terms are:

$$\boxed{\frac{\partial n_{dm}^{(0)}}{\partial t} + 3\,\frac{da/dt}{a}\,n_{dm}^{(0)} = 0}$$

$\Rightarrow \; n_{dm} \propto a^{-3}$

For the first-order terms we need to expand $n_{dm} = n_{dm}^{(0)}\left[1+\delta(\vec{x},t)\right]$ where δn_{dm} is the first order part.

We get:

$$\boxed{\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0}$$

Euler equation for expanding fluids

Remember that we set $m \gg p$, $\Rightarrow v \sim \frac{p}{m} \ll 1 \Rightarrow v$ is first order

We have introduced two new perturbation variables for dark matter, δ and $\vec{v}$. So we need a second equation for them, which we get from the first moment of the Boltzmann equation:

$$\frac{\partial (n_{dm} v^j)}{\partial t} + 4\,\frac{da/dt}{a}\,n_{dm} v^j + \frac{n_{dm}}{a}\frac{\partial \Psi}{\partial x^j} = 0$$

which is completely first order. With $n_{dm} \propto a^{-3}$, we get

$$\frac{\partial v^j}{\partial t} + \frac{da/dt}{a} v^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = 0$$

Expressed with conformal time $d\eta = dt/a$ and Fourier transformed, we get

$$\boxed{\begin{aligned} &\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0 \\ &\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0 \end{aligned}}$$

The Boltzmann Equation for Baryons

Even though electrons are leptons, we consider them here as baryons.

Electrons and protons are coupled by Coulomb scattering. The Coulomb scattering rate is much larger than the expansion rate at all epochs of interest. This tight coupling forces the electron and proton overdensities to a common value:

$$\delta_b \equiv \frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}}$$

and similarly: $\vec{v}_b \equiv \vec{v}_e \simeq \vec{v}_p$

Starting point: Unintegrated equations for electrons and protons.

Notation: $p, p' \to \gamma$; $q, q' \to e^-$; $Q, Q' \to p^+$

Then

$$\frac{df_e}{dt}(\vec{x}, \vec{q}, t) = \langle C_{ep} \rangle_{QQ'q'} + \langle C_{e\gamma} \rangle_{pp'q'}$$

$$\frac{df_p}{dt}(\vec{x}, \vec{Q}, t) = \langle C_{ep} \rangle_{qq'Q'}$$

The angular brackets denote integration over all momenta in the subscripts.

$$C_{e\gamma} = (2\pi)^4 \delta^4(p + q - p' - q') \frac{|M|^2}{8\, E(p)E(p')E(q)E(q')} \{f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\}$$

Again we obtain the equations by taking the zeroth and first moment of the Boltzmann eqn up to first order:

$$\boxed{\begin{aligned} &\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0 \\ &\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = \dot{\tau}\,\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right] \end{aligned}}$$

with $\Theta_1 = i\int_{-1}^{1} \frac{d\mu}{2}\, \mu\, \Theta(\mu)$ and $\mu = \frac{\vec{p}}{p}\cdot\hat{k}$

more generally: $\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1} \frac{d\mu}{2}\, P_\ell(\mu)\, \Theta(\mu)$ with $P_\ell(\mu)$ = Legendre Polynomials

Einstein Equations

For the perturbed FRW model in the Newtonian gauge we have for the Einstein tensor, only first order:

$$G^0{}_0 = \frac{3}{a^2}\left(\frac{da}{dt}\right)^2 - 6H\frac{\partial\varphi}{\partial t} + 6H^2\psi + \frac{2}{a^2}\partial_i^2\varphi$$

$$G^i{}_j = A\delta^i{}_j - \frac{1}{a^2}\left(\partial_i\partial_j\varphi + \partial_i\partial_j\psi\right)$$

The relationship between $G_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ is given by the Einstein field Equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

The stress-energy tensor of the Universe is described by

$$T^{\mu\nu} = \sum_{s\in\text{species}} T^{\mu\nu}_{(s)}$$

$$T^{\mu\nu}_{(s)} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{E_s}p^\mu p^\nu f_s(\vec{x},\vec{p},t)$$

Let's look at the $T^0{}_0$ components.

$$T^0{}_0 = g_{00}T^{00} = -(1+2\psi)T^{00} = -(1+2\psi)\sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_s}(p^0)^2 f_s$$

with $(p^0)^2 = (E_s(1-\psi))^2 \approx E_s^2(1-2\psi)$

$$\Rightarrow T^0{}_0 = -\sum_{(s)}\int\frac{d^3p}{(2\pi)^3}E_s f_s(\vec{x},\vec{p},t)$$

For the species:

i) Dark Matter: $E_s \approx m_{DM}$

$$(T_{DM})^0{}_0 = -\int\frac{d^3p}{(2\pi)^3}m_{DM}f_{DM} = -m_{DM}\int\frac{d^3p}{(2\pi)^3}f_{DM} = -m_{DM}n_{DM}$$

$$= -m_{DM}n^{(0)}_{DM}(1+\delta) = \boxed{-\rho^{(0)}_{DM}(1+\delta)}$$

ii) Baryons

$E_b \approx m_b$

$$\Rightarrow (T_b)^0{}_0 = \boxed{-\rho_b(1+\delta_b)}$$

iii) Photons

$$f = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$(T_\gamma)^0{}_0 = -\int\frac{d^3p}{(2\pi)^3}p\left(f^{(0)} - \frac{\partial f^{(0)}}{\partial p}p\Theta\right) = -\int\frac{d^3p}{(2\pi)^3}p f^{(0)} + \int\frac{d^3p}{(2\pi)^3}p^2\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$= -\rho_\gamma + \int\frac{dp}{(2\pi)^3}\, \underset{\substack{\uparrow \\ d^3p = p^2 dp\, d\Omega}}{p^4}\, \frac{\partial f^{(0)}}{\partial p}\underbrace{\int d\Omega\, \Theta}_{\equiv 4\pi\Theta_0}$$

Integrating by parts:

$$\int\frac{dp}{(2\pi)^3}p^4\frac{\partial f^{(0)}}{\partial p} = \underbrace{\frac{1}{(2\pi)^3}fp^4\Big|_{-\infty}^{\infty}}_{=0} - \int\frac{4}{(2\pi)^3}fp^3dp = -\int\frac{4}{(2\pi)^3}fp^3dp$$

$$= -\int\frac{4}{4\pi(2\pi)^3}d^3p\, fp$$

$$\Rightarrow \int\frac{d^3p}{(2\pi)^3}p^2\frac{\partial f^{(0)}}{\partial p}\Theta = -4\pi\Theta_0\int\frac{4\, d^3p}{(4\pi)(2\pi)^3}f^{(0)}p = -4\Theta_0\rho_\gamma$$

$$\Rightarrow (T_\gamma)^0{}_0 = \boxed{-\rho_\gamma(1+4\Theta_0)}$$

This gives us for the Einstein equations:

$$-6H\partial_0\varphi + 6H^2\psi + \frac{2}{a^2}\partial_i^2\varphi = -8\pi G\left[\rho_{DM}\delta + \rho_B\delta_B + \rho_\gamma 4\Theta_0\right]$$

for the first order components. Fourier transformated and substituted time for conformal time, we get

$$\boxed{k^2\tilde{\varphi} + 3\frac{\dot{a}}{a}\left(\dot{\tilde{\varphi}} - \frac{\dot{a}}{a}\tilde{\psi}\right) = 4\pi G a^2\left[\rho_{DM}\tilde{\delta} + \rho_B\tilde{\delta}_B + \rho_\gamma 4\tilde{\Theta}_0\right]}$$

Note that in the absence of expansion ($\equiv a=1$), we recover the Poisson equation: $k^2\tilde{\varphi} = -4\pi G\tilde{\delta\rho}$

To obtain a second evolution equation, we turn to the $^i{}_j$ components of the Einstein equation.

After some algebra we find:

$$k^2(\hat{\Phi} + \hat{\Psi}) = -32\pi G a^2 [S_\gamma \hat{\Theta}_2]$$

The two gravitational potentials are equal and opposite unless the photons (and neutrinos, which we have neglected) have appreciable quadrupole moments.

To avoid dealing with the complicated term A, we only considered the longitudinal, traceless part of $G^i{}_j$ which can be extracted by contracting $G^i{}_j$ with $\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^j_i$, a projection operator. It kills all terms proportional to δ_{ij}.

Summary

The Einstein - Boltzmann equations are a coupled set of differential equations which are linear, homogeneous and non-autonomous. (= coefficients may be time dependent)

For the various particle species, they are: (here considering neutrinos)

i) Photons

$$\dot{\tilde{\Theta}} + ik\mu\,\tilde{\Theta} = -\dot{\tilde{\Phi}} - ik\mu\,\tilde{\Psi} - \dot{\tilde{\tau}}\left(\Theta_0 - \Theta - \mu v_b \tfrac{1}{2} P_2(\mu)\Pi\right)$$

$$\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0}$$

$$\dot{\Theta}_P + ik\mu\,\Theta_P = -\dot{\tau}\left(-\Theta_P + \tfrac{1}{2}(1 - P_2(\mu))\Pi\right)$$

ii) Dark Matter

$$\dot{\tilde{\delta}} + ik\tilde{v} = -3\dot{\tilde{\Phi}}$$

$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} = -ik\Psi$$

iii) Baryons

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b = -3\dot{\tilde{\Phi}}$$

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b = -ik\tilde{\Psi} + \frac{\dot{\tau}}{R}(v_b + 3i\Theta_1) \qquad R^{-1} = \frac{4\rho_\gamma}{3\rho_b}$$

iii) Massless Neutrinos

$$\dot{W} + ik\mu W = -\dot{\Phi} - ik\mu\Psi. \qquad \Theta_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1} d\mu\, P_\ell(\mu)\Theta$$

Einstein Equations:

$$k^2\tilde{\Phi} + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \frac{\dot{a}}{a}\Psi\right) = 4\pi G a^2\left(\rho_{DM}\tilde{\delta} + \rho_b\tilde{\delta}_b + 4\rho_\gamma\tilde{\Theta}_0 + 4\rho_\nu\tilde{W}_0\right)$$

$$k^2(\Phi + \Psi) = -32\pi G a^2(\rho_\gamma\Theta_0 + \rho_\nu W_2)$$

Inhomogeneities

We want to calculate the inhomogeneities and anisotropies in the Universe.

We will focus on dark matter, thus neglect baryons ($\Omega_b \ll \Omega_{DM}$) and neutrinos. Furthermore, we will neglect all $\ell > 2$ moments for photons and polarisation. This is an ok approximation for early times $a \ll a_{eq}$, but not strictly for late times $a \gg a_{eq}$. But at these times, matter dominates and the radiation has little impact on the metric, which is its only connection to dark matter, so it's negligible when considering dark matter.

Using these simplifying assumptions, the Boltzmann equations simplify:

$$\dot{\tau} \propto n_e \propto n_b \approx 0 \qquad \text{baryons neglected}$$

$$\dot{\Theta} + ik\mu\Theta = -\dot{\phi} - ik\mu\psi$$

$$\dot{\delta} + ikv = -3\dot{\phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = -ik\psi \simeq ik\phi$$

[~ for Fourier transforms are left out. The equations are still in Fourier space]

Taking the moments of the first equation and separating by order:

0th order: $$\dot{\Theta}_0 + k\Theta_1 = -\dot{\phi}$$

1st order: $$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\phi$$

The full set of the simplified Einstein-Boltzmann equations is:

$$\dot{\Theta}_0 - k\Theta_1 = -\dot{\varphi}$$

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\varphi$$

$$\dot{\delta} + ikv = -3\dot{\varphi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\varphi \qquad [\varphi \simeq \psi]$$

$$k^2\varphi + 3\frac{\dot{a}}{a}\left(\dot{\varphi} + \frac{\dot{a}}{a}\varphi\right) = 4\pi G a^2 (\rho_{DM}\delta + 4\rho_r\Theta_0)$$

or $$k^2\varphi = 4\pi G a^2\left(\rho_{DM}\delta + 4\rho_r\Theta_0 + \frac{3aH}{k}(i\rho_{DM}v + 4\rho_r\Theta_1)\right)$$

Initial Conditions

Inflation was introduced to solve various problems, like the flatness, ~~horizon~~, monopole and the seed of structures problem: it provides a mechanism to produce primordial inhomogeneities, when microscopic quantum fluctuations are promoted to macroscopic fluctuations by inflation.

The perturbations are best described in terms of the Fourier modes. The mean of a given Fourier mode, for example for the gravitational potential is zero: $\langle\Phi(\vec{k})\rangle = 0$

Further, the perturbations to one Fourier mode are uncorrelated with those to another. However, a given mode has nonzero variance, so

$$\langle\Phi(\vec{k})\,\Phi^*(\vec{k}')\rangle = (2\pi)^3 P_\Phi(k)\,\delta^3(\vec{k}-\vec{k}')$$

For scalar perturbations, $P_\Phi \propto k^{n-4}$, $n \simeq 1$

Primordial fluctuations, density variations in the early Universe, are quantified by a power spectrum which gives the power of the variation as a spatial scale.

Inflation also predicts primordial tensor modes ($\hat{=}$ gravitational waves)

Solving Simplified Einstein-Boltzmann Equations

There are no known general analytic solutions, so our options are to integrate the equations numerically or to consider limit cases.

To find the limits, we first need to identify the scale of the problem:

Look at the horizon: It gives us the first scale.

$$ds^2 = -dt^2 + a^2(d\chi^2 + r^2(\chi)\, d\Omega^2) = a^2(-d\eta^2 + d\chi^2 + r^2(\chi)\, d\Omega^2)$$

Then the maximal distance a photon can travel in radial direction from $t = 0$ to $t = t_0$ is

$$0 = ds^2 = a^2(d\chi^2 - d\eta^2) \quad \Rightarrow \quad d\eta = d\chi = \frac{dt}{a}$$

$$\chi - \chi_0 = \int_{\chi_0}^{\chi} d\chi' = \int_0^{\eta} d\eta' = \eta = \int_0^t dt' \frac{1}{a(t')}$$

$\Rightarrow$ η is the maximum distance travelled by a photon and corresponds to the comoving size of causally connected regions: "comoving horizon"

With $H = \frac{1}{a}\frac{da}{dt} \quad \Rightarrow \quad dt = \frac{1}{a}\frac{da}{H}$

$$\Rightarrow \eta = \int_0^{\eta} d\eta' = \int_0^t \frac{dt'}{a} = \int_0^a \frac{da'}{a'} \underbrace{\frac{1}{a' H(a')}}_{\text{Comoving Hubble radius}}$$

The second scale is given by matter-radiation domination equality.

At $a = a_{eq}$, $\rho_r \doteq \rho_m$ by definition.

Using that $\rho_r \propto a^{-4}$ and $\rho_m \propto a^{-3}$, we can calculate

$$\rho_r = \rho_{crit} \Omega_{r,0} a_{eq}^{-4} \doteq \rho_{crit} \Omega_{m,0} a_{eq}^{-3} \quad \Rightarrow \quad a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{4.15 \cdot 10^{-5}}{0.27} \approx 1.5 \cdot 10^{-4}$$

The comoving horizon at the time of equality is

$$\eta_{eq} = \int_0^{a_{eq}} \frac{da'}{a'} \frac{1}{a' H(a')} \approx 30\, h^{-1}\, \text{Mpc} \quad \text{for } \Omega_m = 1,\ \Omega_\Lambda = 0$$

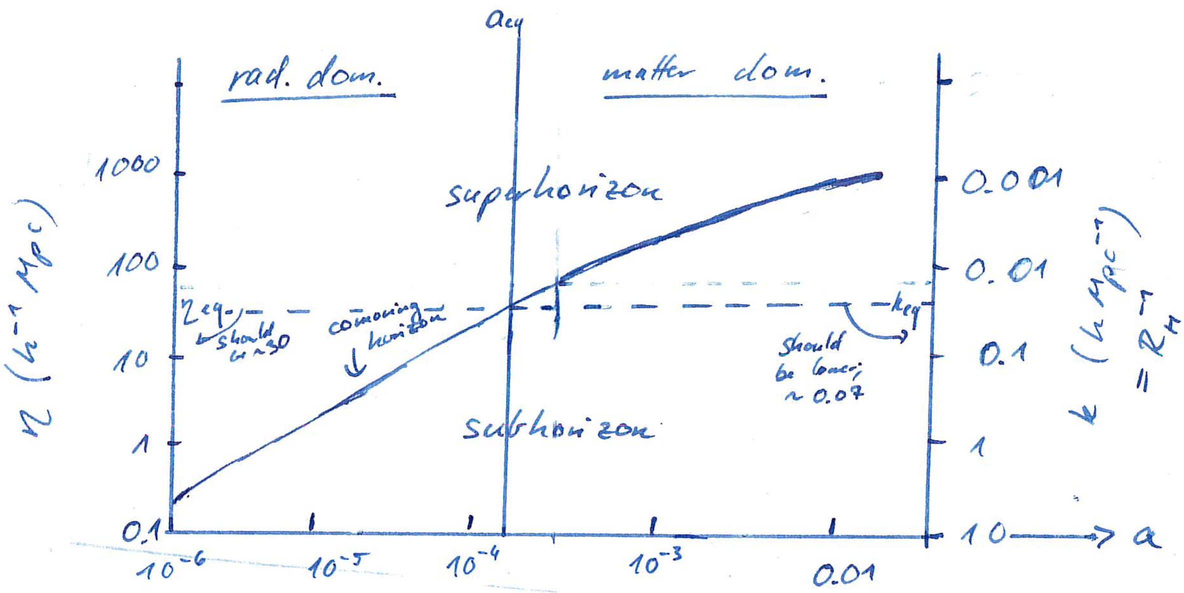
The comoving hubble radius at equality is

$$R_{H_{eq}} = [a_{eq} H(a_{eq})]^{-1} \approx 13.7\, \text{Mpc} / (\Omega_m h^2)$$

We can translate it to the wave number at equality.

$$k_{eq} = R_{H,eq}^{-1} \approx 0.073\ Mpc^{-1}\ \Omega_m h^2$$

These define the important scales in the problem, so we can define different regimes:



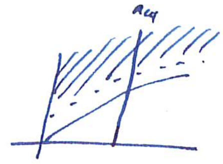
Note that k decreases upwards in this plot.

$k \ll \eta$ modes (superhorizon) are not subject to causal physics, while $k \gg \eta$ (subhorizon) modes are.

Small scale modes ($k \gg k_{eq}$) cross the horizon in the radiation era ($a_{cross} < a_{eq}$).

Large scale modes ($k \ll k_{eq}$) cross the horizon in the matter dominated era ($a_{cross} > a_{eq}$)

Large Scales



Large scales are superhorizon scales; i.e. $k\eta \ll 1$
In this limit, the Einstein-Boltzmann equation drops all terms that contain k:

$$\dot{\Theta}_0 = -\dot{\Phi}$$
$$\dot{\delta} = -3\dot{\Phi}$$
$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 \left[\rho_{dm}\delta + 4\rho_r \Theta_0 \right]$$

We can go further with

$$\dot{\delta} = -3\dot{\Phi} = 3\dot{\Theta}_0 \implies \frac{d}{d\eta}(\delta - 3\Theta_0) = 0 \implies \delta - 3\Theta_0 = \text{const}$$

Note that $\dot{\Theta}_1$ and v, the velocities, decouple from the evolution equations. This reduces the number of equations from 5 to 3.

The constant for $\delta - 3\Theta_0 = \text{const}$ is given by initial conditions:

- const $= 0$: adiabatic perturbations
- const $\neq 0$: isocurvature perturbations

We will assume adiabatic perturbations, because isocurvature perturbations are not successful models. Using $n_{dm} = n_{dm}^{(0)}(1+\delta)$ and $n_\gamma = n_\gamma^{(0)}(1+3\Theta_0)$, we find that for

With $\delta = 3\Theta_0$, we have

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 \left[\rho_{dm}\delta + 4\rho_r \Theta_0 \right]$$
$$= 4\pi G a^2 \rho_{dm}\delta \left[1 + \frac{4}{3y} \right]$$

where $y \equiv \dfrac{\rho_m}{\rho_r} = \dfrac{\rho_{dm}}{\rho_r} = \dfrac{\Omega_{m,0}\,\rho_{crit}\,a^{-3}}{\Omega_{r,0}\,\rho_{crit}\,a^{-4}} = \dfrac{\Omega_{m,0}}{\Omega_{r,0}}\,a$

with $\dfrac{\rho_{dm}(a_{eq})}{\rho_r(a_{eq})} = \dfrac{\rho_m(a_{eq})}{\rho_r(a_{eq})} = \dfrac{\Omega_{m,0}}{\Omega_{r,0}}\,a_{eq} = 1 \implies \dfrac{\Omega_{m,0}}{\Omega_{r,0}} = \dfrac{1}{a_{eq}}$

$$\implies \boxed{y \equiv \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}}$$

Strategy: Turn the two first-order equations for δ and ϕ into one second-order equation. Substitute $\frac{d}{d\eta} = \frac{dy}{d\eta}\frac{d}{dy} = aHy\frac{d}{dy}$

$$\left(= \frac{dy}{dt}\frac{dt}{d\eta}\frac{d}{dy} = \frac{\dot a}{a_{eq}}\cdot a\cdot\frac{d}{dy}\right)$$

The Einstein equations become

$$y\frac{d\phi}{dy} + \phi = \frac{4\pi G\rho_{dm}}{3H^2}\left(1+\frac{4}{3y}\right)$$

Using the Friedmann equation $H^2 = \frac{8\pi G\rho}{3} \approx \frac{8\pi G}{3}(\rho_{dm}+\rho_r) = \frac{8\pi G}{3}\rho_{dm}\left(1+\frac{1}{y}\right)$ we get

$$\delta = \frac{6(y+1)}{3y+4}\left(y\frac{d\phi}{dy}+\phi\right)$$

For δ we know: $\dot\delta = -3\dot\phi \Rightarrow \frac{d\delta}{dy} = -3\frac{d\phi}{dy}$

$$\Rightarrow -3\frac{d\phi}{dy} = \frac{d\delta}{dy} = \frac{d}{dy}\left[\frac{6(y+1)}{3y+4}\left(y\frac{d\phi}{dy}+\phi\right)\right]$$

There is an analytic solution for this equation:

$$\phi = \frac{\phi(0)}{10}\frac{1}{y^3}\left[16\sqrt{1+y}+9y^3+2y^2-8y-16\right]$$

Now consider limits for y:

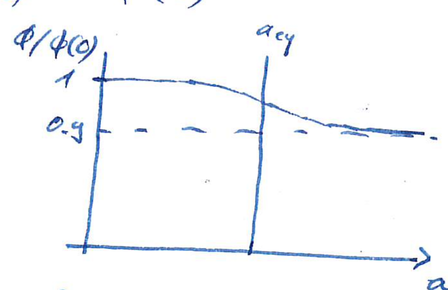
i) $y\to 0$ (early times)

Use Taylor expansion for $\sqrt{1+y} = 1+\frac{1}{2}y-\frac{1}{8}y^2+\frac{1}{16}y^3-\frac{5}{128}y^4+\mathcal{O}(y^5)$

$$\Rightarrow \phi = \frac{\phi(0)}{10}\frac{1}{y^3}\left[(16+8y-2y^2+y^3-\frac{5}{8}y^4+9y^3)+2y^2+9y^3-8y-16\right]$$

$$= \frac{\phi(0)}{10}\frac{1}{y^3}\left[10y^3+\mathcal{O}(y^4)\right] \xrightarrow{y\to 0} \frac{10}{10}\phi(0) = \phi(0)$$

$$\Rightarrow \boxed{\text{For } y\to 0 \quad \phi\to\phi(0)}$$



ii) $y\to\infty$ (late times)

$$\phi = \frac{\phi(0)}{10}\left(16\frac{\sqrt{1+y}}{y^3}+\frac{2}{y}-\frac{8}{y^2}-\frac{16}{y^3}+9\right) \xrightarrow{y\to\infty} \frac{9}{10}\phi(0)$$

$\Rightarrow$ For large scale modes ($k\eta \gg 1$) ϕ is constant in radiation and matter dominated eras and drops by 10% slowly in the radiation-matter transition.

Derivation Einstein equ with y

$$\frac{d}{d\eta} = \frac{dy}{d\eta}\frac{d}{dy} = \frac{dy}{dt}\frac{dt}{d\eta}\frac{d}{dy} = \frac{\dot{a}}{a_{eq}}\cdot a\frac{d}{dy} = aHy\frac{d}{dy}$$

with $y \equiv \frac{a}{a_{eq}}$

Then $3\frac{\dot{a}}{a}\left(\dot{\phi} + \frac{\dot{a}}{a}\phi\right) = 4\pi G a^2 \rho_{dm}\,\delta\left[1+\frac{4}{3y}\right]$

Using $\frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{d\eta} = \frac{1}{a}\frac{da}{dt}\frac{dt}{d\eta} = \frac{a}{a}\frac{da}{dt} = aH$

$$\Rightarrow 3aH\left(aHy\frac{d\phi}{dy} + aH\phi\right) = 4\pi G a^2\rho_{dm}\,\delta\left[1+\frac{4}{3y}\right]$$

$$= 3a^2H^2\left(y\frac{d\phi}{dy} + \phi\right)$$

$$\Rightarrow y\frac{d\phi}{dy} + \phi = \frac{4\pi G\rho_{dm}}{3H^2}\,\delta\left[1+\frac{4}{3y}\right]$$

Now use Friedmann equ $H^2 = \frac{8\pi G\rho}{3} \approx \frac{8\pi G}{3}(\rho_{dm}+\rho_r)$

$$\approx \frac{8\pi G}{3}\rho_{dm}\left(1+\frac{1}{y}\right)$$

$$\Rightarrow y\frac{d\phi}{dy} + \phi = \frac{4\pi G\rho_{dm}}{3\frac{8\pi G}{3}\rho_{dm}}\,\frac{\delta\left[1+\frac{4}{3y}\right]}{1+1/y}$$

$$= \frac{1}{2}\delta\left(1+\frac{4}{3y}\right)\left(\frac{y}{y+1}\right) \quad (*)$$

$$= \frac{1}{2}\delta\,\frac{3y+4}{3(y+1)} = \frac{3y+4}{6(y+1)}\delta$$

$$(*)\quad \left(1+\frac{4}{3y}\right)\left(\frac{y}{y+1}\right) = \frac{y}{y+1} + \frac{4y}{3y(y+1)} = \frac{3y^2+4y}{3y(y+1)} = \frac{3y+4}{3(y+1)}$$

From σ we know: $\dot{\sigma} = -3\dot{\phi} \Rightarrow \frac{d\sigma}{dy} = -3\frac{d\phi}{dy}$

$$\Rightarrow -3\frac{d\phi}{dy} = \frac{d\sigma}{dy} = \frac{d}{dy}\left[\frac{6(y+1)}{3y+4}\left(y\frac{d\phi}{dy}+\phi\right)\right] =$$

$$= \frac{d}{dy}\left(\frac{6(y+1)}{3y+4}\right)\left(y\frac{d\phi}{dy}+\phi\right) + \frac{6(y+1)}{3y+4}\frac{d}{dy}\left(y\frac{d\phi}{dy}+\phi\right)$$

$$= \frac{6(3y+4)-6(y+1)\cdot 3}{(3y+4)^2}\left(y\frac{d\phi}{dy}+\phi\right) +$$

$$+ \frac{6(y+1)}{3y+4}\left(\frac{d\phi}{dy}+y\frac{d^2\phi}{dy^2}+\frac{d\phi}{dy}\right)$$

$$= \frac{1}{(3y+4)^2}\left[(18y+24-18y-18)\left(y\frac{d\phi}{dy}+\phi\right) + \right.$$

$$\left. + (18y^2+18y+24y+24)\left(2\frac{d\phi}{dy}+y\frac{d^2\phi}{dy^2}\right)\right]$$

$$= \frac{1}{(3y+4)^2}\left[6y\frac{d\phi}{dy}+6\phi + (18y^2+42y+24)\left(2\frac{d\phi}{dy}+y\frac{d^2\phi}{dy^2}\right)\right]$$

$$= \frac{18y^3+42y^2+24y}{(3y+4)^2}\frac{d^2\phi}{dy^2} + \frac{6y+36y^2+84y+48}{(3y+4)^2}\frac{d\phi}{dy} + \frac{6}{(3y+4)^2}\phi$$

$$= \frac{18y^3+42y^2+24y}{(3y+4)^2}\frac{d^2\phi}{dy^2} + \frac{36y^2+90y+48}{(3y+4)^2}\frac{d\phi}{dy} + \frac{6}{(3y+4)^2}\phi$$

Use that $\frac{18y^3+42y^2+24y}{(3y+4)^2} = \frac{6y(y+1)(3y+4)}{(3y+4)^2} = \frac{6y(y+1)}{(3y+4)}$

$$\Rightarrow -\frac{d^2\phi}{dy^2} = \frac{36y^2+90y+48}{6y(y+1)(3y+4)}\frac{d\phi}{dy} + \frac{6}{6y(y+1)(3y+4)}\phi$$

$$= \underbrace{\frac{6y^2+15y+8}{y(y+1)(3y+4)}}\frac{d\phi}{dy} + \frac{1}{y(y+1)(3y+4)}\phi$$

some coefficients in the dividend are wrong, but I can't find the mistake...

Note on limits and horizons

We have seen that η is the comoving horizon, the maximal comoving distance a photon could travel since the beginning of the Universe.

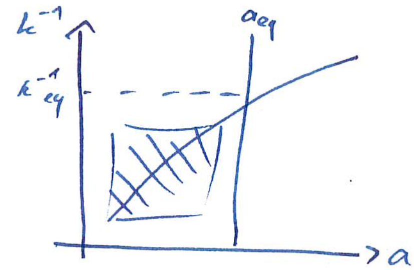
The wavenumber k of a perturbation is roughly equal to the inverse of the comoving wavelength of the mode in question: $\lambda \sim \frac{2\pi}{k}$

$$\Rightarrow k\eta = \frac{\text{max distance}}{\text{wavelength}}$$ of the perturbation.

$\Rightarrow k\eta \ll 1$: the mode in question has a wavelength so large that no causal physics could have affected it.

However, the horizon grows over time, but comoving wavelengths remain constant. Eventually these cosmological modes enter the horizon, after which causal physics begin to operate on them.

Small Scales at Early Times



Now let's look at modes with $k \ll k_{eq}$ and $a \ll a_{eq}$ ($\equiv$ radiation era)

When the Universe is radiation dominated, the potential is determined by perturbations to the radiation. The dark matter perturbations are influenced by the potential, but do not themselves influence the potential.

Since we are interested in the dark matter perturbations, we first must solve equations for Θ_0, Θ_1 and Φ and only then solve the equation for matter evolution using the potential as an external driving force.

The limiting case simplifies the equations: (DM terms drop out)

$$\dot{\Theta}_0 + k\,\Theta_1 = -\dot{\Phi}$$

$$\dot{\Theta}_1 - \frac{k}{3}\dot{\Theta}_0 = -\frac{k}{3}\Phi$$

$$k^2\Phi = 4\pi G a^2\left[4\rho_r\Theta_0 + \frac{3aH}{k}(4\rho_r\Theta_1)\right]$$

Using the Friedmann equation in the radiation era:

$$H^2 = \frac{8\pi G}{3}\rho_r \propto a^{-4} \quad \Rightarrow \quad H \propto a^{-2}$$

with $d\eta = \frac{dt}{a} = \frac{dt}{da}\frac{da}{a} = \frac{da}{a^2H} \Rightarrow \eta = \int\frac{da}{a^2H} \propto \int\frac{da}{a^2a^{-2}} = \int da \propto a$

In fact, we have $\eta = 1/aH$

$$\longrightarrow \quad \Phi = \frac{6a^2H^2}{k^2}\left[\Theta_0 + \frac{3aH}{k}\Theta_1\right]$$

Using the equations for $\dot{\Theta}_0$ and $\dot{\Theta}_1$ with the Einstein equation and differentiating with respect to η gives

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0$$

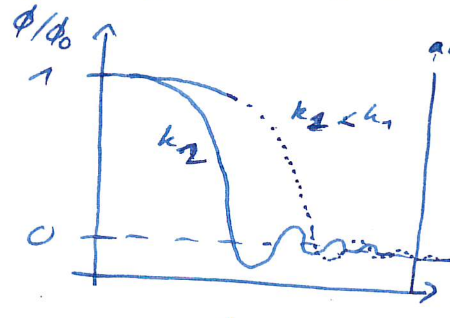
This equation can be solved analytically with a clever substitution we get the spherical Bessel equation of order 1.

We assume initial conditions to be $\Phi \to \Phi(0)$ for $\eta \to 0$

This gives

$$\Phi(\eta) = 3\,\Phi(0)\,\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}$$

As soon as a mode enters the horizon ($k\eta = 1$), its potential starts to decay (factor $(k\eta)^{-3}$). After decaying, the potential oscillates. Smaller k enter the horizon later ($k \sim 1/\eta$)



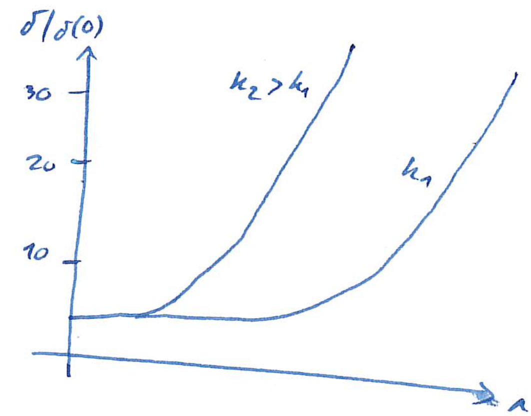
We can now determine the evolution of the matter perturbations. Differentiating and combining the two matter evolution equations, we get:

$$\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = \underbrace{-3\ddot{\Phi} + k^2\Phi - \frac{3}{\eta}\dot{\Phi}}_{\text{"Source term"}} \equiv S(k,\eta)$$

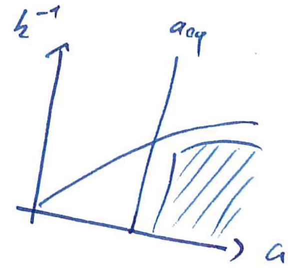
This needs to be solved using the solution for $\Phi(\eta)$, or in other words go fuck yourself.

We expect after the mode has entered the horizon:

$$\delta \propto \text{const} + \ln(k\eta) \qquad \text{for } a \gg a_{cross}$$



Late times and subhorizon case



$\Rightarrow$ $k\eta \gg 1$, $a \gg a_{eq}$

We assume that all modes are inside the horizon in the matter dominated era, giving us the simplified Boltzmann equations: (neglecting Θ)

$$\dot{\delta} + ikv = -3\dot{\phi}$$

$$\dot{v} + \frac{\dot{a}}{a} v = ik\phi$$

$$k^2\phi = 4\pi G a^2 \Big(\rho_{dm}\delta + \underbrace{\frac{3aH}{k}}_{\approx 0:\ k \gg \eta^{-1}} i\rho_{dm} v\Big)$$

With some algebra, we get

$$\frac{d^2\delta}{da^2} + \left(\frac{d\ln H}{da} + \frac{3}{a}\right)\frac{d\delta}{da} - \frac{3\Omega_m H_0^2}{2a^5H^2}\delta = 0$$

Solving it gives $\boxed{\delta \propto H}$

Since we're in the matter dominated era, we can do:

let $\alpha \equiv \frac{\delta}{H}$

with $\frac{H}{H_0} = (\Omega_m a^{-3} + \Omega_\Lambda)^{-1/2}$

$$\Rightarrow \frac{d^2\alpha}{da^2} + 3\left(\frac{d\ln H}{da} + \frac{1}{a}\right)\frac{d\alpha}{da} = 0$$

$$\Rightarrow \frac{d\alpha}{da} \propto (aH)^{-3}$$

$$\Rightarrow \alpha \propto \int (a'H)^{-3}\, da' \Rightarrow \delta = H\alpha = H(a)\int_0^a (a'H(a'))^{-3} da'$$

with $H \propto a^{-3/2}$ in matter dominated era:

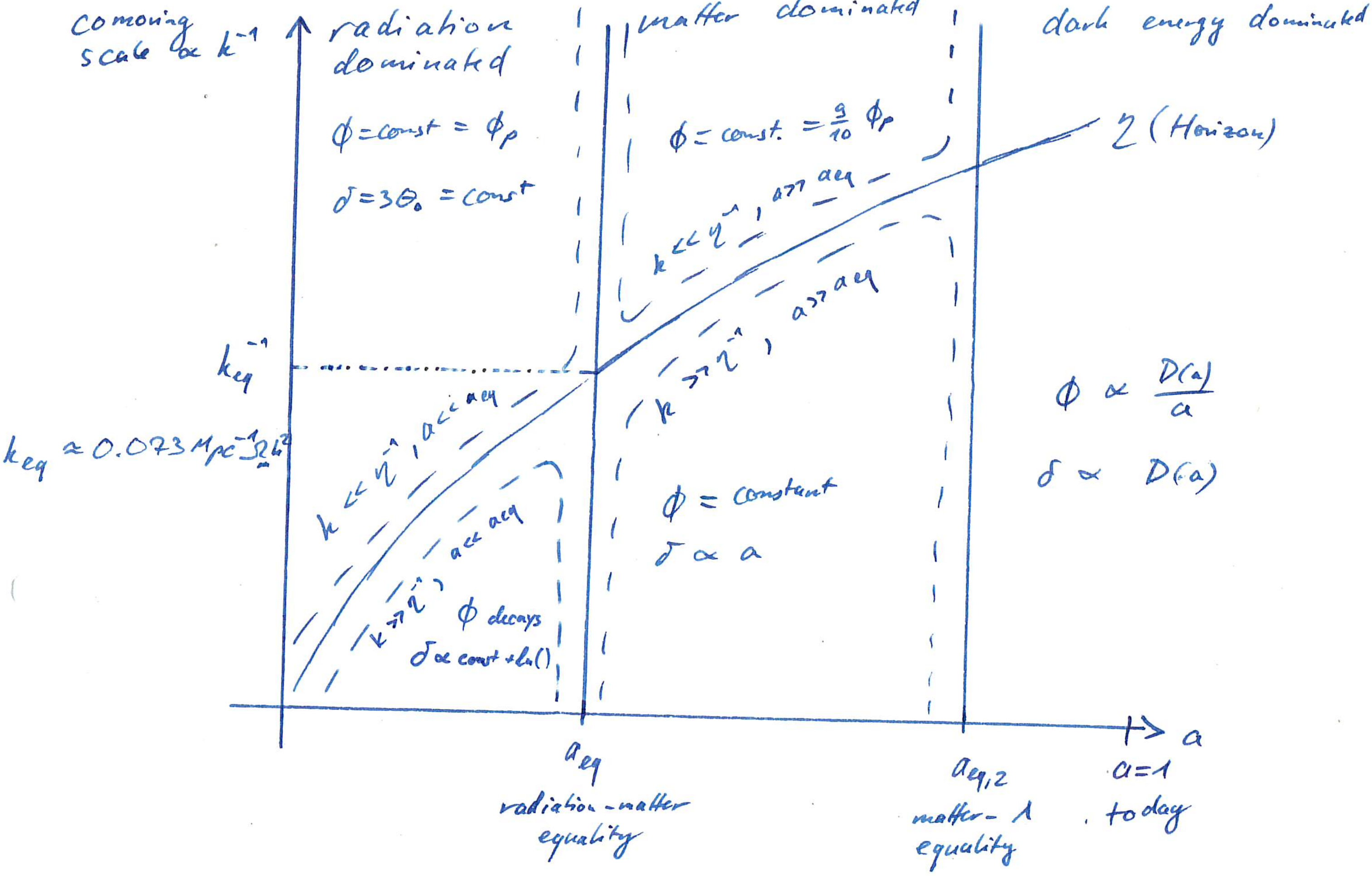
$$\delta \propto a^{-3/2}\int a'^{1+3/2}\, da' \propto a$$

$$\Rightarrow \boxed{\delta \propto a}$$

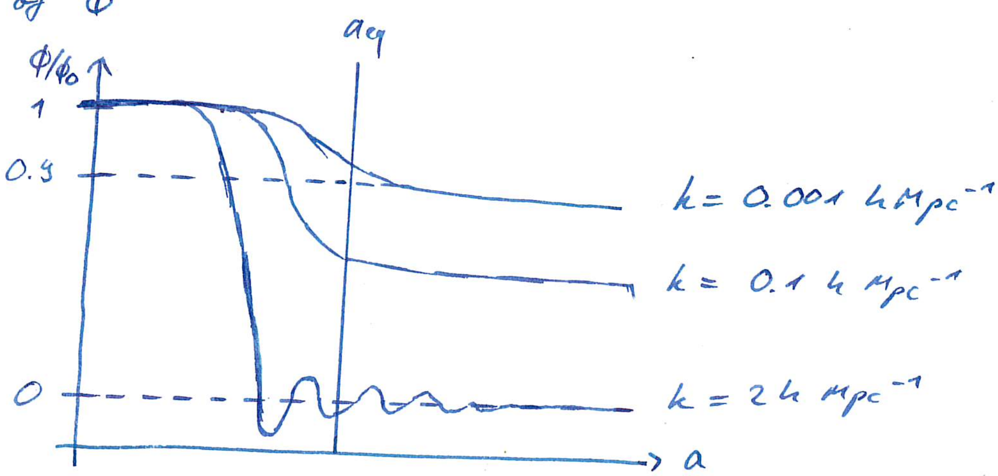
Using $\phi = \frac{3 H_0 \Omega_m}{2 h^2 a} \delta$: $\Rightarrow \phi \propto \frac{\delta}{a}$

$$\Rightarrow \boxed{\phi \propto \frac{\delta}{a} \propto \text{const}}$$

Summary



Evolution of ϕ

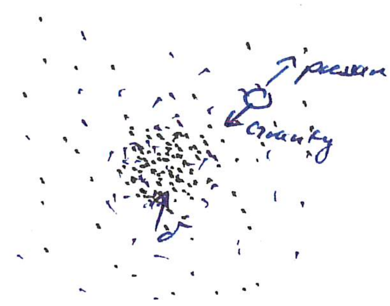


- large k (=small scale) modes enter horizon first and in radiation
  → they decay more and have smaller amplitudes at later times
- ϕ = const. for $a \ll a_{eq}$ and $a \gg a_{eq}$. ϕ only evolves during the matter-radiation transition era and in the dark energy dominated era.

Gravitational instability is most likely responsible for the structure in our Universe. As time evolves, matter accumulates in initially overdense regions. It doesn't matter how small the initial overdensity was, eventually enough matter will be attracted to the region to form structure.

The $F = ma$ of gravitational instability is the equation governing overdensities δ. Schematically, it reads

$$\boxed{\ddot{\delta} + [\text{Pressure} - \text{Gravity}]\,\delta = 0.}$$



Gravity increases the overdensities, but since there are more particles in an overdense region, random thermal motion causes a net loss of mass in an overdense region.

$\Rightarrow$ If pressure is strong, inhomogeneities do not grow.
For strong pressures, δ will oscillate. For weak pressures, δ will grow exponentially.

The evolution of cosmological perturbations breaks up naturally into 3 stages (see plot for ϕ on previous page.)

- Early times: All modes are outside the horizon, $k\eta \ll 1$, potential is constant.
- Intermediate times: The wavelengths fall within the horizon and the Universe evolves from radiation domination to matter domination at $a = a_{eq}$. Whether the mode enters the horizon before or after a_{eq} greatly affects the potential.
- Late times: All the modes evolve identically (are constant for $\Omega_m = 1$)

We are able to observe the distribution of matter predominantly at late epochs, in the third stage of evolution, when all modes are evolving identically. To relate the potential during these times to the primordial potential ϕ_p set up during inflation, we write schematically

$$\boxed{\phi(\vec{k}, a) = \phi_p(\vec{k}) \times \{\text{Transfer Function}\,(k)\} \times \{\text{Growth Function}\,(a)\}}$$

The transfer function describes the evolution of perturbations through the epochs of horizon crossing and radiation/matter transition. The growth factor describes the wavelength-independent growth at late times.

$$\Rightarrow \boxed{\Phi(\vec{k}, \vec{a}) = \frac{9}{10}\, \Phi_p(k)\, \underbrace{T(k)}_{\text{Transfer function}}\, \underbrace{\frac{D(a)}{a}}_{\text{Growth function}}} \quad \text{for } a > a_{late}$$

The factor $\frac{1}{a}$ for the growth function such that in the matter dominated era (with $\Omega_m = 1$) $\frac{D(a)}{a} = 1$ and the potential remains constant.

The transition function is defined as

$$T(k) \equiv \frac{\Phi(k, a_{late})}{\Phi_{large\text{-}scale}(k, a_{late})} = \frac{\Phi(k, a_{late})}{9/10\, \Phi_p(k)} \quad \text{with } a_{late} > a_{eq}$$

So the factor $\frac{9}{10}$ in the potential evolution equation above "evens it out": $T(k) \to 1$ for $k \to 0$.

At late times, when the potential is constant and all modes are within the horizon, the overdensity $\delta \propto a$ (hence "growth function").

For $a \gg a_{eq}$ in the large-k no-radiation limit, the Einstein equation becomes

$$\Phi = \frac{4\pi G \rho_m a^2 \delta}{k^2} = \frac{3 H_0^2 \Omega_{m,0}\, \delta}{2 k^2 a}$$

$$\Rightarrow \delta = \frac{k^2 a}{3/2\, \Omega_{m,0} H_0^2}\, \Phi(\vec{k}, a) = \frac{3}{5} \frac{k^2}{\Omega_{m,0} H_0^2}\, \Phi_p(\vec{k})\, T(k)\, D(a)$$

Recall the definition of the power spectrum:
In the context of inflation, $\phi_p(\vec{k})$ is drawn from a Gaussian distribution with mean zero and variance

$$\langle \phi(\vec{k})\, \phi^*(\vec{k}') \rangle = (2\pi)^3 P_\phi(k)\, \delta^3(\vec{k}-\vec{k}')$$

We can define similarly a power spectrum of the perturbation:

$$\langle \delta(\vec{k},a)\, \delta^*(\vec{k}',a) \rangle = (2\pi)^3 P_\delta(k,a)\, \delta(\vec{k}-\vec{k}')$$

The power spectrum at late times is

$$\boxed{P(k,a) = 2\pi^2 \delta_H^2 \cdot \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D(a)}{D(a=1)}\right)^2}$$

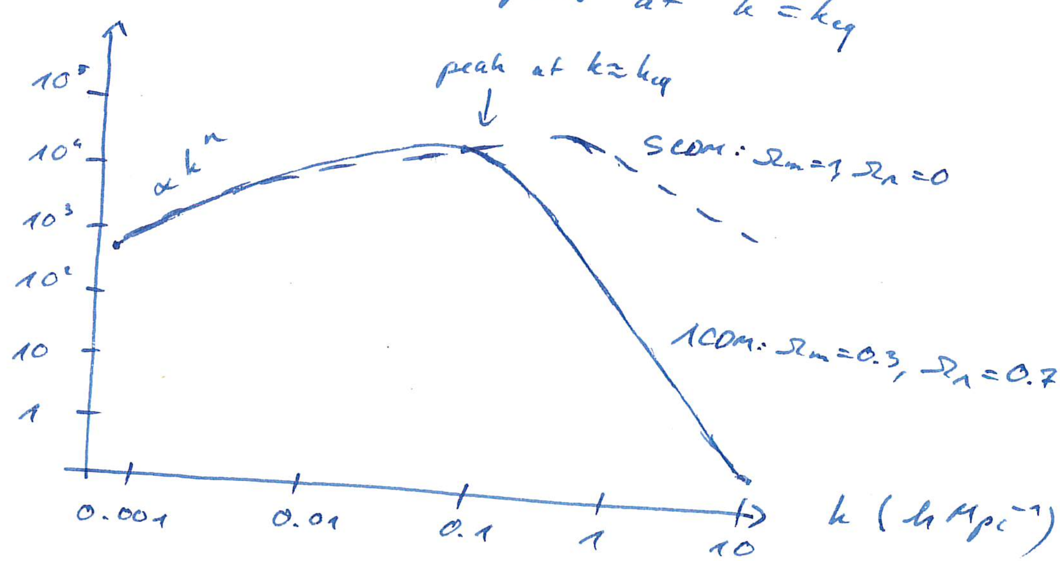
Where δ_H^2 is a normalisation factor, typically fixed by the CMB, and n is

For the transfer function we have

$$T(k) = \begin{cases} = 1, & k \ll k_{eq} \\ \propto \frac{12 k_{eq}^2}{k^2} \ln\left(\frac{k}{8k_{eq}}\right), & k \gg k_{eq} \end{cases}$$

from which follows

$$P_\delta(k) \propto k^n T^2(k) \propto \begin{cases} \propto k^n & k \ll k_{eq} \\ \propto k^{n-4} \ln^2\left(\frac{k}{8k_{eq}}\right) & k \gg k_{eq} \\ \text{peaks at } k = k_{eq} \end{cases}$$



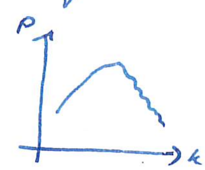
Matter power spectrum

The matter power spectrum $P(k,a)$ can be probed using spectroscopic galaxy surveys, weak gravitational lensing, Lyα forest and many other "Large Scale structure" probes.

The transfer function is obtained through numerical codes.

Eventually modes become non-linear for $k \gtrsim k_{NL} \approx 0.1\,\mathrm{Mpc}^{-1}$
This allows the formation of clusters. The power spectrum P_{NL} can be obtained through N-body simulations and fitting functions.

Baryons also have a small effect on the power spectrum. They cause small oscillations in the photon-baryon fluid before decoupling (Baryon Acoustic Oscillations). These oscillations are also prominent in the spectrum of the radiation perturbations (see next chapter).



Massive neutrinos also effect the power spectrum. Neutrinos are fast and stream out of high density regions thus suppressing perturbations on scales smaller than the free-streaming scale.

Effects of dark energy:

i) With $\Omega = 1$ (flat Universe) and $\Omega_\Lambda \sim 0.7$, $\Rightarrow \Omega_m < 1$, which leads to a turn over of the power spectrum at a much larger scale than predicted by SCDM. This was one of the pieces of evidence for dark energy.

ii) Different models of dark energy predict different growth factors.

Anisotropies

We now want to study the perturbations in radiation. They can be observed through the CMB. That can again be done either numerically or analytically by considering limits.

Large Scales

The Einstein-Boltzmann equations for superhorizon modes are

$$\dot{\Theta}_0 = -\dot{\Phi}$$

$$\dot{\delta} = -3\dot{\Phi}$$

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 \left[\rho_{dm}\,\delta + 4\rho_r\,\Theta_0 \right]$$

$$\Rightarrow \quad \Theta_0 + \Phi = \text{const}$$

Adiabatic initial conditions give us $\Theta_0 = \Phi/2$, so that

$$\boxed{\delta(\eta_p) = 3\Theta_0(\eta_p) = \frac{3}{2}\Phi \equiv \Phi_p}$$

$$\Rightarrow \quad \Theta_0(\eta > \eta_p) = \underbrace{-\Phi + \text{const}}_{\text{from EB-eqn}} = -\Phi + \frac{3\Phi_p}{2} \quad \left(\text{so that } \Theta_0(\eta_p) = -\Phi_p + \frac{3}{2}\Phi_p = \frac{1}{2}\Phi_p\right)$$

At the time of recombination η_*, the Universe in the ΛCDM model is after equality and thus in the matter dominated era, where we have found that $\Phi(\eta > \eta_{eq}) = \frac{9}{10}\Phi_p$

$$\Rightarrow \quad \Theta_0(k, \eta_*) = -\Phi + \frac{3}{2}\,\frac{10}{9}\Phi = -\Phi + \frac{5}{3}\Phi = \frac{2}{3}\Phi(k, \eta_*)$$

Using $\Phi \approx -\Psi$ at early times, we get

$$\boxed{(\Theta_0 + \Psi)(k, \eta_*) = \frac{1}{3}\Psi(k, \eta_*)}$$

The second equation is

$$\boxed{\delta(\eta_*) = 2\Phi(\eta_*)}$$

Together, they give $(\Theta_0 + \Psi)(k, \eta_*) = -\frac{1}{6}\delta(\eta_*)$ for the observed anisotropy at recomb

The observed anisotropy of an overdense region will be negative.
For large scale perturbations, overdense regions contain hotter photons at recombination than do underdense regions. However, to get to us today, they first must travel out of their potential wells. In doing so they lose energy, and this energy loss more than compensates for the fact that the photons were initially hotter than average.
$\rightarrow$ When we observe hot spots on the sky today, we are actually observing regions that were underdense at the time of recombination.

It also allows us to relate: $\frac{\delta T}{T} \propto -\frac{1}{6}\frac{\delta \rho}{\rho}$
Roughly speaking, an anisotropy of order 10^{-5} corresponds to an overdensity of 6×10^{-5}.

Tightly Coupled Limit of the Boltzmann Equation

Before η_*, when all electrons were ionized, the mean free path for a photon was much smaller than the horizon of the Universe. Compton scattering caused the electron-proton fluid to be tightly coupled with the photons.
The tightly coupled limit corresponds to the scattering rate $\tau \gg 1$ with $\tau \equiv \int_{\eta}^{\eta_0} n_e \sigma_T a(\eta')\, d\eta'$, $n_e = n_e(\eta')$

Taking the moment of the Boltzmann equation and using the recurrence relation for Legendre polynomials, we get

$$\dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} + \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} = \dot{\tau}\,\Theta_\ell$$

For the case when $\tau \gg 1$, the tight coupling limit, we can neglect high multipoles $\ell \geq 2$.
We get:

$$\boxed{\ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F(k,\eta) = -\frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi}}$$

where $c_s^2 = \frac{1}{3(1+R)}$ and $R^{-1} = \frac{4}{3}\frac{\rho_\gamma^{(0)}}{\rho_b^{(0)}}$ photon-baryon ratio
speed of sound

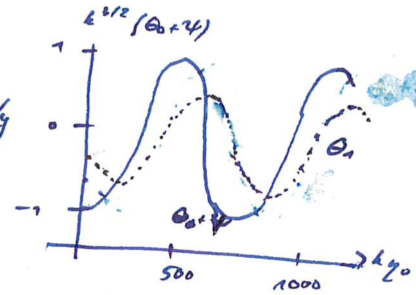
This is the equation of a damped harmonic oscillator driven by dark matter at $a \gg a_{eq}$ that exerts force $F(k,\eta)$ on the photon (- baryon) fluid.
The sound speed depends on the baryon density in the Universe. The presence of baryons makes the fluid heavier, thereby lowering the sound speed.

Neglecting the damping as a first step, we obtain oscillating solutions:

$$S_1 = \sin(k r_s), \qquad S_2 = \cos(k r_s)$$

$$\text{with } r_s = r_s(\eta) = \int_0^\eta d\eta' c_s(\eta') \equiv \text{sound horizon}$$

Clearly, peaks for Θ_0 occur at $k_{peak} = n\frac{\pi}{r_s}$
The dipole can also be calculated. It is completely out of phase with the monopole: $\Theta_0 \propto \cos$, $\Theta_1 \propto \sin$

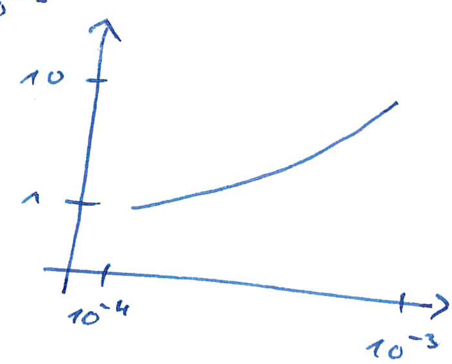


Diffusion Damping

Diffusion is characterised by a small but nonnegligible quadrupole. With diffusion considered, modes with small k are damped because the photons can diffuse out of the potential wells and dilute the latter.
When calculated we see that

$$\Theta_0, \Theta_1 \propto \exp[-k^2/k_D^2] \quad \text{with} \quad k_D^{-1} \sim \sqrt{\frac{\eta}{n_e \sigma_T a}}$$

For ΛCDM, $k_D^{-1} \approx 10\,\text{Mpc}$ at $a = 10^{-3}$



Inhomogeneities to Anisotropies

The goal is to relate inhomogeneities at recombination ($a = a_*$) to the observed anisotropies today ($a = a_0$).

We massage the full EB-equation

$$\dot\Theta + ik\mu\Theta = -\dot\Phi - ik\mu\Psi - \dot\tau\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

with $\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0}$

Θ: Photon perturbation variable

$\Theta_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\,\Theta(\mu)$: Photon multipole moment

Θ_P : Polarisation strength

we get:

$$\boxed{\Theta_\ell(k,\mu,\eta_0) = \int_0^{\eta_0} d\eta\; S(k,\eta)\, j_\ell[k(\eta_0-\eta)]}$$

where

$$S(k,\eta) \equiv e^{-\tau}\left[-\dot\Phi - \dot\tau\left(\Theta_0 + \frac{1}{4}\Pi\right)\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{i v_b \dot\tau}{k}\right)\right] - \frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\dot\tau\Pi\right]$$

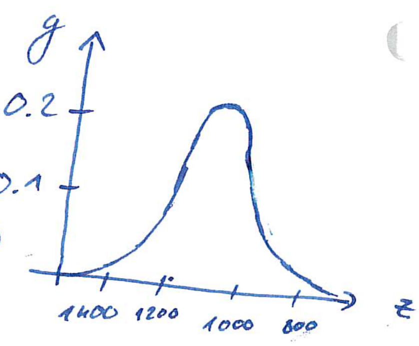
is the source function and j_ℓ is the spherical Bessel function.

It is useful to define the visibility function

$$g(\eta) \equiv -\dot\tau e^{-\tau}$$

It is the probability that the photon last scattered at η.

It is normalised: $\int_0^{\eta_0} d\eta\, g(\eta) = 1$.



τ is large in early times and suppresses the visibility function, just like $\dot\tau$ at late times, which is the scattering rate, $\Rightarrow g(\eta) \approx \delta(\eta - \eta_*)$

With the visibility function, we can express:

$$\begin{aligned}\Theta_\ell(k,\mu,\eta_0) =& \int_0^{\eta_0} d\eta\; g(\eta)\left[\Theta_0(k,\eta) + \Psi(k,\eta)\right] j_\ell[k(\eta_0-\eta)] && \text{monopole}\\ &- \int_0^{\eta_0} d\eta\; g(\eta)\,\frac{i v_b(k,\eta)}{k}\,\frac{d}{d\eta} j_\ell[k(\eta_0-\eta)] && \text{dipole}\\ &+ \int_0^{\eta_0} d\eta\; e^{-\tau}\left[\dot\Psi(k,\eta) - \dot\Phi(k,\eta)\right] j_\ell[k(\eta_0-\eta)] && \text{integrated Sachs-Wolfe}\end{aligned}$$

Integrated Sachs-Wolfe term: Small correction if potentials are time-dependent.

Power Spectrum of the CMB

The goal is to relate the Θ_ℓ to the observed anisotropy. Recall that we wrote the temperature field as

$$T(\vec{x}, \hat{p}, \eta) = T(\eta)\left[1 + \Theta(\vec{x}, \hat{p}, \eta)\right]$$

We decompose the field Θ in terms of spherical harmonics.

$$\Theta(\vec{x}, \hat{p}, \eta) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\vec{x}, \eta)\, Y_{\ell m}(\hat{p})$$

All the information contained in the temperature field T is also contained in the space-time dependent amplitude $a_{\ell m}$.

$$a_{\ell m} = \int d\Omega\; \Theta(\vec{x}, \hat{p}, \eta)\, Y^*_{\ell m}(\hat{p})$$

The mean value of $a_{\ell m}$ is zero, but they will have nonzero variance:

$$\langle a_{\ell m} \rangle = 0, \qquad \langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell$$

The variance is called C_ℓ and does not depend on m.
With some massive black magic we can relate the C_ℓ to the matter power spectrum:

$$C_\ell = \frac{2}{\pi} \int_0^\infty dk\, k^2 P(k) \left| \frac{\Theta_\ell(k)}{\delta(k)} \right|^2$$

Large Scales

For large scale (large-angle: $\theta \sim \frac{1}{\ell}$), the perturbations responsible for these anisotropies were on scales far larger than could be connected via causal processes. They are not affected by any microphysics. Only the monopole contributes to the anisotropy, giving us

$$C_\ell^{SW} = \frac{\Omega_m^2 H_0^2}{2\pi D(a=1)} \int_0^\infty \frac{dk}{k^2}\, j_\ell^2\left[k(\eta - \eta_0)\right] P(k, \eta_0)$$

Further using only very large scales:

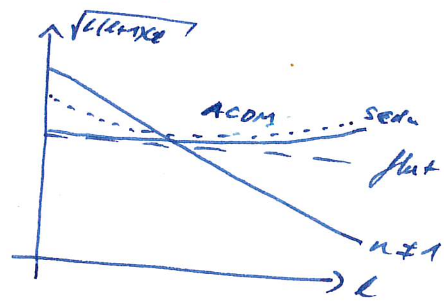
$$C_\ell^{SW} \simeq \pi H_0^{1-n} \left(\frac{\Omega_m}{D(a=1)}\right)^2 \delta_H^2 \int_0^\infty \frac{dk}{k^{2-n}} j_\ell^2[k(\eta_0-\eta)]$$

where the superscript SW is for 'Sachs Wolfe' and n is the tilt. For $n=1$, we get

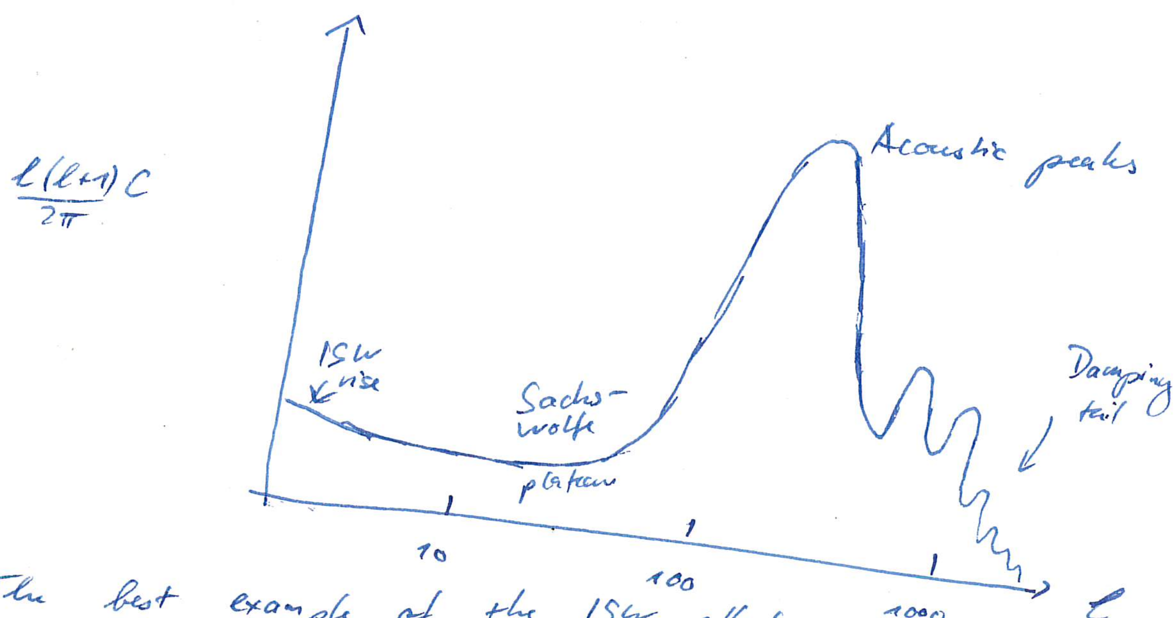
$$\boxed{\ell(\ell+1)C_\ell^{SW} = \frac{\pi}{2}\left(\frac{\Omega_m}{D(a=1)}\right)^2 \delta_H^2 = \text{const}}$$

$\Rightarrow$ At low ℓ, we expect a plateau. But the true spectrum is not completely flat, because $n \neq 1$ in general and at recombination the dipole contributes slightly, when we have only taken the monopole into account. as well as the Sachs-Wolfe terms for ΛCDM,

At intermediate scales, we have accoustic oscillations with peaks at $k_{pn} = n\frac{\pi}{r_s(\eta_*)}$, $n=1,2,3\ldots$ through the tight coupling. This starts at $\ell \simeq 200$ for $n=1$.



At small scales, we have diffusion damping, starting at $\ell \sim$ a few thousand.



The best example of the ISW effect is that due to residual radiation at recombination, when $\Phi = -\Psi$ isn't strictly true.

Cosmological Parameters

The CMB power spectrum depends on several cosmological parameters:

- Curvature density $\qquad \Omega_k \equiv 1 - \Omega_m - \Omega_\Lambda$
- Normalisation $\qquad \delta_H = C_{\ell=10}$
- Primordial tilt $\qquad n$
- Amplitude of tensor modes $\qquad r$
- Reionisation, parametrised by optical depth $\qquad \tau$
- Baryon Density $\qquad \Omega_b h^2$
- Matter density $\qquad \Omega_m h^2$
- Cosmological constant $\qquad \Omega_\Lambda$

i) Curvature

Curvature causes the largest shift in the location of the peaks. The physical scale with maximal anisotropy (= first peak) gets projected onto a much smaller angular scale in an open Universe $\Rightarrow$ ℓ peak is at higher ℓ

ii) Degenerate parameters: δ_H, τ, r, n

Move spectrum up and down, hardly change its shape.

iii) Distinct imprints: $\Omega_m, \Omega_\Lambda, \Omega_b$

Each of these parameters induces a small shift in the location of the peaks and throughs in the spectrum

Initial Perturbations from Inflation

Inflation was introduced as a scheme to solve the horizon, flatness, monopole and primordial perturbation of initial conditions problem.

The principle of inflation is that in an early Universe $t_0 < t < t_e$ there is a period during which the Hubble constant was actually approximately a constant:

$$H = \frac{1}{a}\frac{da}{dt} \approx \text{const} \quad \Rightarrow \quad a(t) = a_e e^{H(t-t_0)}$$

As a result, the comoving Hubble radius $\frac{1}{aH} \propto a^{-1}$ decreased. To solve the horizon problem with an initial time t_0 corresponding to $T_0 \approx 10^{15}$ GeV (GUT) we need $a(t)$ to increase by about 60 e-foldings $(H(t_e - t_0) \approx 60)$

The hubble radius must decrease: $R = \frac{1}{aH}$ with $H \approx$ const, a growing

Modes of interest were first in the horizon (in causal contact), then left it in the inflation era and re-entered it after inflation. To have such an accelerated expansion, we need the Universe to be dominated by a component with negative pressure $P < -\frac{\varepsilon}{3}$. So we need an era that is dominated by something other than radiation, matter or dark matter and has negative pressure. A solution is to introduce the scalar field "inflaton":

$$\phi(\vec{x}, t): \text{ inflaton} \qquad (\text{not potential!})$$

The energy-momentum tensor for a scalar field is given by

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - g^{\alpha}{}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right]$$

where $V(\phi)$ is the potential.

$V(\phi)$ needs to be specified, and there are various models.

Scalar Field Perturbations

Let's decompose the scalar field into a zero-order homogeneous part and a perturbation:

$$\phi(\vec{x},t) = \phi^{(0)}(t) + \delta\phi(\vec{x},t)$$

To zeroth order, we have

$$\rho = -T^{(0)0}{}_0 = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi^{(0)})$$

$$P = T^{(0)i}{}_i = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi^{(0)})$$

Because we need inflation to last for some time, we introduce the 'slow roll' condition:

$$\left(\frac{d\phi^{(0)}}{dt}\right) \ll V(\phi^{(0)})$$

Which gives us an accelerated expansion with $\rho \approx V(\phi)$:

$$H = \sqrt{\frac{8\pi G\rho}{3}} \approx \sqrt{\frac{8\pi G V(\phi^{(0)})}{3}} \approx \text{const}$$

From the space-space component of the Einstein equations for the zeroth order background we get the equation of motion for inflation:

$$\frac{d^2\phi^{(0)}}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + \frac{dV(\phi^{(0)})}{d\phi^{(0)}} = 0$$

$$\Rightarrow \ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2V' = 0 \qquad \text{with } \dot{} = \frac{d}{d\eta}$$

To quantify the slow roll, usually two variables are defined which vanish in the limit that ϕ remains constant:

$$\varepsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = \frac{-\dot{H}}{aH^2} \qquad \geq 0 \text{ because } H \text{ always decreases}$$

$$\delta \equiv \frac{1}{H}\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} = \frac{-1}{aH\dot{\phi}^{(0)}}\left[3aH\dot{\phi}^{(0)} + a^2V'\right]$$

$\hat{=}$ how slowly the field is rolling [nothing with overdensities]

The slow-roll condition is given by

$$\varepsilon \ll 1 \quad \text{and} \quad \varepsilon - \delta \ll 1$$

Review Quantum Harmonic Oscillator

In order to compute the quantum fluctuations in the metric, we need to quantize the field. → Rewrite the problem so that it looks like a simple harmonic oscillator.

In classical physics, the equation of motion for a harmonic oscillator is given by

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

For a quantum system, we replace the variable $x \to$ operator $\hat{x}$.
In the Schrödinger picture, $\hat{x} = \text{const}$, but the state $|\psi\rangle$ is time dependent. In the Heisenberg picture, $\hat{x}(t)$ is time dependent, but the state $|\psi\rangle$ is not. We will use the Heisenberg picture, as it is more convenient.

Using the raising and lowering operators $\hat{a}, \hat{a}^\dagger$: with

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$$

$\hat{a}|0\rangle = 0$: $\hat{a}$ annihilates vacuum

$\langle 0|a^\dagger = 0$

we can define

$$\hat{x} = v(\omega, t)\hat{a} + v^*(\omega, t)\hat{a}^\dagger \quad \text{and} \quad \hat{p} = \frac{d\hat{x}}{dt} = \frac{dv}{dt}\hat{a} + \frac{dv^*}{dt}\hat{a}^\dagger$$

Such that the commutation relation $[\hat{x}, \hat{p}] = i$ is valid, as long as $v(\omega, t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}$ is normalized.

$v(\omega, t)$ must be the solution of the classical equation of motion.

This allows us to compute the quantum fluctuations of the operator $\hat{x}$ in the ground state $|0\rangle$:

$$\langle |\hat{x}|^2 \rangle = \langle 0|\hat{x}^\dagger\hat{x}|0\rangle = \langle 0|(v^* a^\dagger + v\hat{a})(v\hat{a} + v^*\hat{a}^\dagger)|0\rangle =$$

$$= \langle 0|v^* v\hat{a}^\dagger\hat{a} + v^* v^* \hat{a}^\dagger\hat{a}^\dagger + vv\hat{a}\hat{a} + vv^*\hat{a}\hat{a}^\dagger|0\rangle$$

$$= \langle 0|v^* v\,\hat{a}\hat{a}^\dagger|0\rangle + |v|^2 \underbrace{\langle 0|\hat{a}^\dagger}_{=0}\underbrace{\hat{a}|0\rangle}_{=0} + |v^*|^2 \underbrace{\langle 0|\hat{a}^\dagger}_{=0}\hat{a}^\dagger|0\rangle + |v^2|\langle 0|\hat{a}\underbrace{\hat{a}|0\rangle}_{=0}$$

$$= |v^2|\langle 0|\hat{a}\hat{a}^\dagger|0\rangle$$

$$= |v|^2 \langle 0|\hat{a}\hat{a}^+|0\rangle = |v|^2 \langle 0|\underbrace{[\hat{a},\hat{a}^+]}_{=1} + \underbrace{\hat{a}^+\hat{a}|0\rangle}_{=0}$$

$$= |v|^2 = \frac{1}{2\omega}$$ fluctuation of ground state (variance of $\hat{x}$: $Var(x) = \langle x^2\rangle - \underbrace{\langle x\rangle^2}_{=0}$)

Tensor Modes

Recall the metric perturbations:

$$h_{00} = -E$$

$$h_{i0} = a\frac{\partial F}{\partial x^i} + G_i$$

$$h_{ij} = a^2\left[A\,\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij}\right]$$

with A, B, E, F scalars, F, G divergenceless vectors and D_{ij} traceless, divergenceless, symmetric tensor.

The decomposition theorem tells us that the different modes evolve independently, so now let's look only at tensor modes: Set $h_{00} = h_{i0} = 0$, $h_{ij} = a^2 D_{ij}$

with $D_{ij} = D_{ji}$; $D_{ii} = 0$, $\frac{\partial D_{ij}}{\partial x^i} = 0$

This gives us $\underset{\text{symmetric}}{9-3} \underset{\text{traceless}}{-1} \underset{\text{div. less}}{-3} = 2$ degrees of freedom.

Consider the Fourier transform of D_{ij}: $D_{ij}(\vec{x},\tau) \to \tilde{D}_{ij}(\vec{k},\tau)$ and furthermore let's choose a coordinate system so that $\vec{k} = k\hat{e}_z$. This allows us to make the Ansatz:

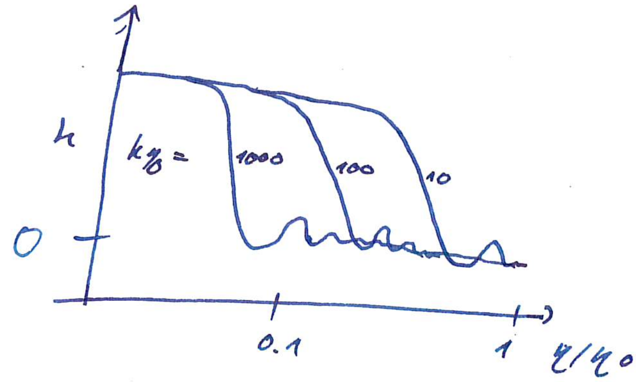
$$\tilde{h}_{ij}(\vec{k},\tau) = a^2 \tilde{D}_{ij}(\vec{k},\tau) = a^2 \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

After computing $\Gamma^\alpha_{\mu\nu}$, $R_{\mu\nu}$, R and $G_{\mu\nu}$ we find that the tensor modes are governed by

$$\boxed{\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0}$$ with $\alpha = +, \times$

(where we neglected a small contribution from radiation terms)

$\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0$ is a wave equation, corresponding solutions are called gravity waves. It is a generalised equation to an expanding Universe. The expansion produces a friction term $\propto \dot{h}_\alpha$.
The solutions for the tensor modes are constant for superhorizon modes $k\eta \gg 1$ and oscillatory and damped for subhorizon modes.



$\Rightarrow$ Only the large-scale anisotropies are impacted by gravity waves. At decoupling, only modes with $k\eta_0 \lesssim 100$ persist.

Generation of Tensor Modes during Inflation

Tensor perturbations obey $\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0$.
We want to relate this equation to the harmonic oscillator.

Let $\tilde{h} \equiv \frac{a h}{\sqrt{16\pi G}}$ where $\frac{1}{\sqrt{16\pi G}}$ is a normalisation such that we get the right units for a scalar field.

Then it follows: $\frac{d\tilde{h}}{d\eta} = \dot{\tilde{h}} = \frac{a\dot{h}}{\sqrt{16\pi G}} + \frac{h\dot{a}}{\sqrt{16\pi G}}$

$$\Rightarrow \frac{\dot{h}}{\sqrt{16\pi G}} = \frac{\dot{\tilde{h}}}{a} - \frac{h\dot{a}}{a\sqrt{16\pi G}} = \frac{\dot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\tilde{h}$$

Inserting this into the evolution equation gives

$$\ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h} = 0$$

which is the equation for a harmonic oscillator.

It has no damping term so we can quantize this for each tensor mode in the Heisenberg picture with

$$\hat{\hat{h}}(\vec{k}, \eta) = v(k, \eta)\hat{a}_{\vec{k}} + v^*(\vec{k}, \eta)a^+_{\vec{k}}$$

The coefficients again must satisfy $\ddot{v} + (k^2 - \frac{\ddot{a}}{a})v = 0$.

Using the harmonic oscillator analogy, we can write the variance of perturbation in the $\hat{h}$ field as

$$\langle \hat{h}^+(k)\hat{h}(\vec{k}', \eta)\rangle = |v(\vec{k}, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$\Rightarrow \langle \hat{h}^+(\vec{k}, \eta)\, \hat{h}(\vec{k}', \eta)\rangle = \frac{16\pi G}{a^2} |v(k, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$= (2\pi)^3 P_h(k)\, \delta^3(\vec{k} - \vec{k}')$$

with P_h being the power spectrum of h.

$$\Rightarrow \quad P_h = 16\pi G \frac{|v(\vec{k}, \eta)|^2}{a^2}$$

In order to evaluate $|\vec{v}(\vec{k}, \eta)|^2$: Evaluate $\frac{\ddot{a}}{a}$ during inflation

During early inflation: $\eta = \int_{a_0}^{a} \frac{da'}{a'^2} \frac{1}{H(a')} \approx \underbrace{\frac{1}{H}}_{\approx const} \int_{a_0}^{a} \frac{da'}{a'^2} \approx -\frac{1}{aH}$

+ bla bla bla

gives solution $|v|^2 = \frac{1}{2k} \, k - \frac{i}{k\eta}|^2 \approx \frac{1}{2k} |\frac{i}{k\eta}|^2$ for $k\eta \ll 1$,

so for modes well within the horizon, giving us

↓ becomes small because horizon shrinks during inflation

$$\boxed{P_h = \frac{16\pi G}{a^2} \frac{1}{2k^3\eta^2} = \frac{8\pi G H^2}{k^3}}$$

the primordial power spectrum of gravitational waves.

Since $H = \sqrt{\frac{8\pi G \varepsilon}{3}} \approx \sqrt{\frac{8\pi G V}{3}}$ during inflation, the detection of gravitational waves amounts to the measurement of V of the inflation, because it would measure H during inflation.

Generation of Scalar Perturbations

The goal is to find the perturbation spectrum of $\Psi \simeq \Phi$ emerging from inflation. The complication is the presence of perturbations in the scalar field ϕ driving inflation, perturbations which are coupled to Ψ. Note inflation field $\phi \neq$ potential Φ

Approach: first consider perturbations for ϕ and ignore perturbations for Φ and Ψ (no space-time perturbations) and only then consider the coupling between ϕ and Ψ.

Inflation Perturbation

Decompose the inflation scalar field: $\phi(\vec{x},t) = \underbrace{\phi^{(0)}(t)}_{\text{homogeneous}} + \delta\phi(\vec{x},t)$

We want to derive an equation of motion for $\delta\phi$.

For that, consider the energy conservation of the stress energy tensor:

$$T^{\mu}{}_{\nu;\mu} = \frac{\partial T^{\mu}{}_{\nu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\alpha\mu} T^{\alpha}{}_{\nu} - \Gamma^{\alpha}_{\nu\mu} T^{\mu}{}_{\alpha} = 0$$

With $T^{\alpha}{}_{\beta} = g^{\alpha\nu} \frac{\partial\phi}{\partial x^{\nu}} \frac{\partial\phi}{\partial x^{\beta}} - g^{\alpha\beta}\left[\frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^{\mu}} \frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right]$

For the $\nu = 0$ (time) component, we get

$$0 = \frac{\partial\delta T^{0}{}_{0}}{\partial t} + ik_i \delta T^{i}{}_{0} + 3H\delta T^{0}{}_{0} - H\delta T^{i}{}_{i}$$

With some algebra we get

$$\ddot{\delta\phi} + 2aH\dot{\delta\phi} + k^2\delta\phi = 0$$

when we neglected a small term in V for the small roll condition.

This is identical to the tensor perturbation equation to the metric.

Thus we can finally copy the power spectrum of fluctuations:

$$P_{\delta\phi} = \frac{H^2}{2k^3}$$

$\delta\phi$ is already a scalar field

The factor $16\pi G$ is missing here though because we don't need normalisation

Again, H should be evaluated at horizon crossing.

Coupling with Metric Perturbation

We found $P_{\delta\phi} = \frac{H^2}{2k^2}$ neglecting metric perturbations.

One way to deal with the coupling between the metric perturbations and those to the energy is to define

$$\xi \equiv - \frac{i k_i \delta T^0{}_i H}{k^2(\rho + P)} - \Psi = \text{Const}$$

which is conserved at large scales across horizon crossing.

At early times for subhorizon modes and those that just left the horizon, $\delta\Psi$ is negligible and $\delta T^0{}_i$ is dominated by inflaton, giving us

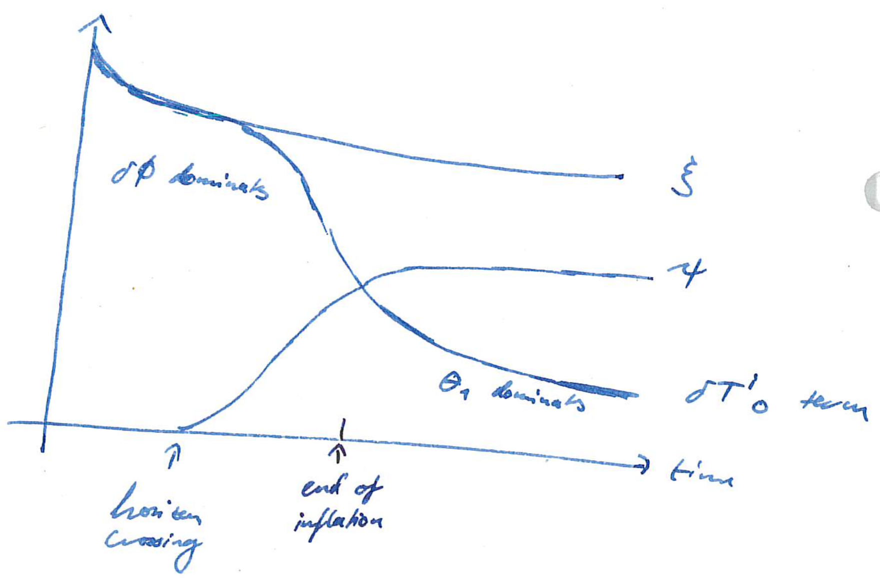
$$\xi = - a H \frac{\delta\phi}{\dot\phi^{(0)}}$$

After inflation, $\delta\Psi$ dominates with a contribution from radiation (dipole Θ_1 in $\delta T^0{}_i$)

We get:

$$\xi = - \frac{3aH\Theta_1}{k} - \Psi \simeq - \frac{3}{2}\Psi$$

Schematical plot:



Perturbations start off as $\delta\phi$ dominated up to horizon crossing and end up Ψ & Θ_1 dominated at the end of inflation.

Matching early and late times by $\xi = \text{const}$ gives

$$P_\Psi |_{after} = P_\Phi |_{after} = \frac{8\pi G}{9k^3} \frac{H^2}{\varepsilon}\Big|_{aH=k}$$ with ε = slow roll parameter

P_Φ gets enhanced by $\frac{1}{\varepsilon}$ factor compared to tensor power spectrum.

Summary

Inflation predicts that quantum-mechanical perturbations in the very early Universe are first produced when the relevant scales are causally connected. Then these scales are whisked outside the horizon by inflation, only to reenter much later to serve as initial conditions for the growth of structure and anisotropy in the Universe.

The perturbations are best described in terms of Fourier modes. The mean of a given Fourier mode is zero: $\langle \Phi(\vec{k}) \rangle = 0$

However a given mode has nonzero variance, so

$$\langle \Phi(k) \Phi^*(k') \rangle = (2\pi)^3 P_\Phi(k) \delta^3(\vec{k} - \vec{k}')$$

the Dirac δ function enforces the independence of the different modes.

Inflation produces both scalar and tensor perturbations with following spectra:

$$P_\Phi(k) = \frac{8\pi G}{9k^3} \frac{H^2}{\varepsilon}\Big|_{k=aH} \quad \text{for scalar perturbations}$$

$$P_h(k) = \frac{8\pi G}{k^3} H^2 \Big|_{k=aH} \quad \text{for tensor perturbations}$$

The scalar perturbations spectrum depends on the slow roll parameter $\varepsilon \propto \frac{dH}{dt}$, with $\varepsilon \ll 1$ because $H \approx$ const. during inflation.

A spectrum in which $k^3 P_\Phi$ is constant ($\equiv$ does not depend on k) is called 'scale invariant' or 'scale-free'.

To quantify the deviations from scale invariance, it is conventional to write the primordial power spectra as

$$P_\phi(k) = \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n-1} \delta_H^{\,2}\left(\frac{\Omega_m}{D(a=1)}\right)^2$$

$$P_h = A_T\, k^{n_T - 3}$$

with δ_H, A_T amplitudes and n, n_T spectral indices.
With this convention, scale-free modes correspond to $n = 1$ and $n_T = 0$.
It can be shown that

$$n = 1 - 4\varepsilon - 2\delta$$

$$n_T = -2\varepsilon$$

As we have seen before, $r \equiv \frac{P_h}{P_\phi} \propto \varepsilon \implies n_T \propto \frac{P_h}{P_\phi}$
The ratio is proportional to the spectral indices, which is a generic prediction of inflation.

$\implies$ Extracting data for ε, δ and $\frac{P_h}{P_\phi}$ is tantamount to probing the potential of the field driving inflation, which is thought to be at 10^{15} GeV scales!

Thomson vs Compton Scattering

Compton scattering is the inelastic scattering of a photon by a charged particle, usually an electron.

Compton wavelength: $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta) \overset{max}{=} \frac{2h}{m_e c}$

Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. It is the low-energy limit of Compton scattering: the particle kinetic energy and photon frequency do not change as a result of the scattering.

Free Streaming

A free streaming particle, often a photon, is one that propagates through a medium without scattering.

Hubble Radius

$R_H \equiv \frac{c}{H}$, distance from the observer at which the recession velocity of a galaxy would equal the speed of light.

$\hat{=}$ Radius of the observable Universe.

Excursus: Energy Conditions

An Energy Condition is an imposed restriction on the stress-energy tensor conjectured to hold for all physically reasonable matter.

→ It's just "a guess at the precise definition of 'physical reasonableness'"

Basically, they're generalisations of the statement that "energy density cannot be negative" to the whole stress tensor.

Null Energy Condition:

For every future-pointing null vector field $\vec{k}$:

$$\rho = T_{ab} k^a k^b \geq 0$$

Weak Energy Condition:

For every timelike vector field $\vec{X}$, the matter density observed by the corresponding observers is always non-negative:

$$\rho = T_{ab} X^a X^b \geq 0$$

Dominant Energy Condition:

The Weak Energy Condition holds true. In addition, for every future-pointing causal vector field (= time like or null) $\vec{Y}$, the vector field $-T^a{}_b Y^b$ must be a future pointing causal vector. ⇒ Mass-energy can never be observed to be flowing faster than light

Strong Energy Condition

For every future-pointing timelike vector field $\vec{X}$, the trace of the tidal tensor measured by the corresponding observers is always non-negative:

$$(T_{ab} - \frac{1}{2} T g_{ab}) X^a X^b \geq 0$$

$\overset{!}{\circ} \rightarrow$ there are many matter configurations which violate the strong energy condition, at least from a mathematical perspective (it is not clear whether these violations are physically possible in a classical regime). It is also violated in any cosmological inflationary process.

Energy Conditions for Perfect Fluids

In a comoving frame, $T^a{}_b = \mathrm{diag}(\rho, p, p, p)$
The energy conditions can then be reformulated in terms of these Eigenvalues:

Null Energy Condition: $p + \rho \geq 0$

Weak Energy Condition: $\rho \geq 0, \; p + \rho \geq 0$

Dominant Energy Condition: $\rho \geq |p|$

Strong Energy Condition: $\rho + p \geq 0 \; ; \; \rho + 3p \geq 0$

Assuming an equation of state $p = w\rho$, the dominant energy condition demands $|w| \leq 1$