Kinetic Theory

Distribution functions

A distribution function $f(\vec{x}, \vec{u}, t)$ is used for the description at a microscopic level. It is a phase space distribution function: $(\vec{x}, \vec{v}) \in \mathbb{R}^6$

Examples:

$f \equiv$ mass density: $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\vec{x}, \vec{u}, t)\, d^3x\, d^3u = M$

$f \equiv$ probability density: $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\vec{x}, \vec{u}, t)\, d^3x\, d^3u = 1$

A particle-by-particle description is not possible, instead we use a statistical description with the distributions.

Definition of $f(\vec{x}, \vec{u}, t)$: Average number of particles contained at time t in a volume element d^3x about $\vec{x}$ and a velocity-space element d^3u about $\vec{u}$ is $f\, d^3x\, d^3u$

Furthermore we demand:

- $f \geq 0$ everywhere
- $u_i \to \infty$, $f \to 0$ sufficiently rapidly so that a finite amount of particles has a finite energy.

Moments

Generally: $Q(\vec{x}, t) = \int_{\mathbb{R}^3} f(\vec{x}, \vec{u}, t)\, q(\vec{u})\, d^3u$

p-th moment: $Q_p(\vec{x}, t) = \int_{\mathbb{R}^3} f(\vec{x}, \vec{u}, t)\, u^p\, d^3u$

0-th moment: $Q_0 = \int_{\mathbb{R}^3} f(\vec{x}, \vec{u}, t)\, d^3u = n(\vec{x}, t)$ number density

1-st moment: $Q_1 = \int_{\mathbb{R}^3} f\, \vec{u}\, d^3u = n(\vec{x}, t) \cdot v(\vec{x}, t)$ average velocity

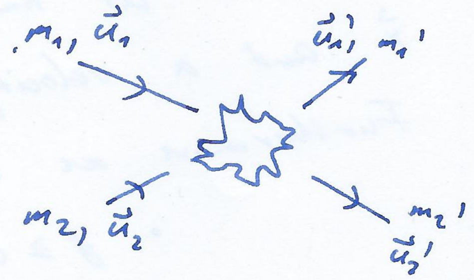
2-nd moment: $E(\vec{x}, t) = \frac{1}{2} m \cdot Q_2$

Binary Collisions

The wave packets of particles are highly localised. To a very high degree of approximation, we can consider the gas to be a collection of classical point particles. We can describe the motion of a (electrically not charged) particle as a sequence of straight lines, each interrupted by a brief collision with another particle. Because the probability of collisions is small, we neglect the possibility of a collision of 3 or more particles simultaneously, and consider only binary collisions.

Binary collisions are collisions between two particles. They conserve energy and are characterised by a collision cross section.

Conserved quantities:



Mass: $M = m_1 + m_2 = m_1' + m_2'$

Momentum: $m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{u}_1' + m_2 \vec{u}_2'$

Energy: $\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$

Now define: $\vec{V} = \frac{1}{M}(m_1 \vec{u}_1 + m_2 \vec{u}_2)$ CoM-velocity

$\vec{v} = \vec{u}_1 - \vec{u}_2$, $\quad \vec{v}' = \vec{u}_1' - \vec{u}_2'$, relative velocity

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

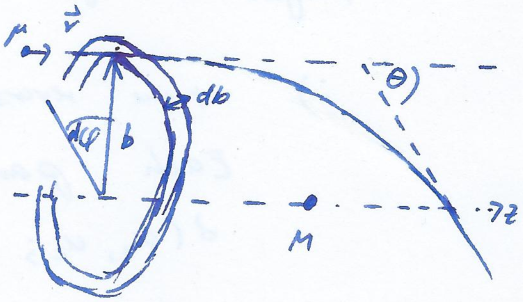
Then: $E = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$

Energy conservation: $E = E' \implies \|v\| = \|v'\|$

$\Rightarrow$ The relative velocity of the two particles only changes the direction after a collision!

Cross sections

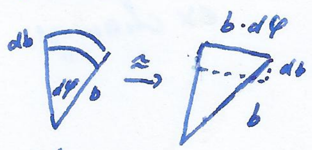
(Differential cross sections)



Choose a particle to act as a collision center and bombard it with a flux of particles.

The rate of collisions R_1 that have the impact parameter in the range $(b, b+db)$ within an increment of $d\varphi$ is $j \, b \, db \, d\varphi$, where j is the incident flux.

We can also assign the process a differential cross section σ defined as the rate at which particles are scattered out of the incident beam into an increment of solid angle $d\Omega$ around some direction $\vec{n}$, specified by the angles (θ, φ). The Rate will be $R_2 = j \, \sigma \, d\Omega$.

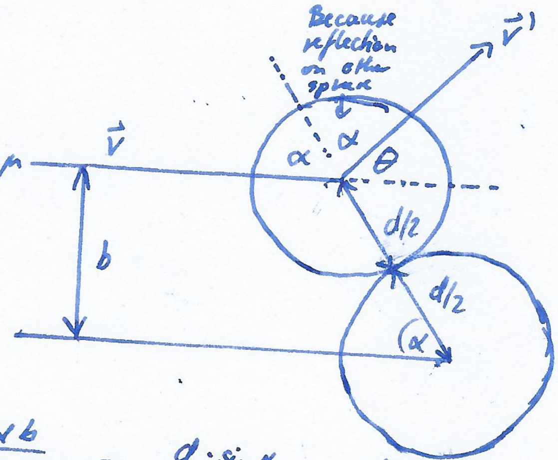


But because such a collision must have an unique solutions, we can relate those two rates: $R_1 = R_2$ defines σ:

With $d\Omega = \sin\theta \, d\theta \, d\varphi$, we get:

$$\boxed{\sigma = \frac{b}{\sin\theta} \frac{db}{d\theta}}$$ differential cross section

For two rigid spheres of diameter d, we can see that $b = 2 \cdot \frac{d}{2} \cdot \sin(\alpha)$
$= d \sin\alpha$



and $\theta = \pi - 2\alpha$.

This gives us: $\sigma = \frac{b}{\sin\theta} \frac{db}{d\theta} = \frac{d \cdot \sin\alpha}{\sin(\pi - 2\alpha)} \frac{db}{d\alpha} \frac{d\alpha}{d\theta} =$

$= \frac{d \cdot \sin\alpha}{2 \sin\alpha \cos\alpha} \cdot d \cdot \cos\alpha \cdot \frac{1}{2} = \boxed{\frac{1}{4} d^2}$

$\sin(\pi - 2\alpha) = \sin(2\alpha) = 2 \sin\alpha \cos\alpha$

Then the total cross section is: $\sigma_{tot} = \int \sigma \, d\Omega = 4\pi\sigma = \boxed{\pi d^2}$

Properties of differential cross sections:

i) time reversal invariance

Each particle must retrace its original trajectory:

$\sigma(u_1, u_2; u_1', u_2') = \sigma(-u_1', -u_2'; -u_1, -u_2)$

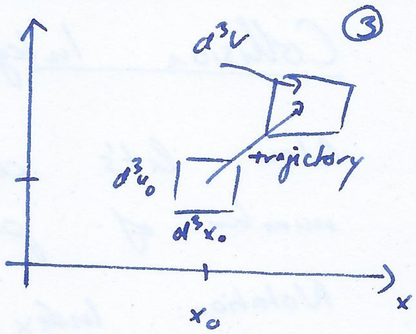
ii) rotation / reflection invariance

The collision only depends on the magnitude and the relative velocities

iii) reverse collision invariance

exchange $u_1, u_2 \leftrightarrow u_1', u_2'$ (time reversal + 180° rotation)

Boltzmann Equation



Vlassov equation

Interpret the timely evolution of a phase-space element as a coordinate transformation:

Neglecting second order terms, we have: $\begin{cases} \vec{x} \simeq \vec{x}_0 + \vec{u}_0\, dt \\ \vec{u} \simeq \vec{u}_0 + \vec{a}\, dt \end{cases}$
(assuming $\frac{\partial a}{\partial u} = 0$). The phase-space element $d^3x_0\, d^3u_0$ "Evolves" to $d^3x\, d^3u$.

The Jacobian of the transformation gives:

$$J = \begin{vmatrix} \partial x/\partial x_0 & \partial x/\partial u_0 \\ \partial u/\partial x_0 & \partial u/\partial u_0 \end{vmatrix} = \begin{vmatrix} 1 & dt \\ \frac{\partial a}{\partial x} dt & 1 \end{vmatrix} = 1 + \frac{\partial a}{\partial x} dt^2 \simeq 1$$

$\Rightarrow$ The volume element is conserved

Let δN_0 be the number of particles in dV_0. Assuming we have no collisions that might remove or add particles, then $\Rightarrow \delta N_0 \doteq \delta N$

$$\Rightarrow \delta N_0 = f(\vec{x}_0, \vec{u}_0, t)\, d^3x_0\, d^3u_0 \overset{!}{=} f(\vec{x}_0 + \vec{u}_0 dt,\ \vec{u}_0 + \vec{a} dt,\ t_0 + dt)\, d^3x\, d^3u$$

Now as shown, $d^3x\, d^3u = d^3x_0\, d^3u_0$

$$\Rightarrow f(\vec{x}_0, \vec{u}, t) = f(\vec{x}_0 + \vec{u}_0 dt,\ \vec{u}_0 + \vec{a} dt,\ t_0 + dt).$$

$\Rightarrow$ The distribution function is the same for all particles everywhere

$\Rightarrow$ Vlasov's equation: Expand to first order in dt:

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{x}} + \vec{a}\frac{\partial f}{\partial \vec{u}} = 0} \boxed{= \frac{Df}{Dt}}$$

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial \vec{x}} \frac{\partial x}{\partial t} dt + \frac{\partial u}{\partial t}\frac{\partial f}{\partial u} dt \quad \text{[First order expansion]}$$

Collision Integral

Now let's consider the case when binary collisions change the number of particles of a phase-space element.

Notation: Index 2: Target particles, Index 1: collider particles

~~Suppose only 1 outward collider particle interacts with the particles in our phase space element dV_2. In that case, the rate of change of the number of particles in~~

The change in particle numbers in the volume element dV_2 can be described as follows: (v: relative velocity)

$$\delta N_2 = \#\text{ particles inside} \cdot [\#\text{ particles incoming} \cdot \text{probab. to collide}]$$

$$= f_2\, d^3v \cdot [\, f_1 \cdot \underbrace{(\sigma\, d\Omega) \cdot v \cdot dt}_{\text{Volume element containing probability of collisions through cross sections}} \,]$$

$$= f_1 f_2\, \sigma v\, d^3v\, dt\, d\Omega$$

$$\Rightarrow \boxed{\frac{\delta N_2}{dt} = f_1 f_2\, \sigma v\, d^3v\, d\Omega}$$

for outgoing collisions.

By demanding the process to be reversible, using the same arguments, we can write for the incoming collisions:

$$\boxed{\frac{\delta N_1}{dt} = f_1' f_2'\, \sigma v\, d\Omega\, d^3v}$$

(also using $v' = v$)

With these two expressions, we find the collision integral:

$$d\left(\frac{Df}{Dt}\right)_{coll} = \text{sources} - \text{sinks} = \frac{\delta N_2}{dt} - \frac{\delta N_1}{dt} = [f_1' f_2' - f_1 f_2]\, \sigma v\, d\Omega\, d^3v$$

$$\Rightarrow \boxed{\left(\frac{Df}{Dt}\right)_{coll} = \iint [f_1' f_2' - f_1 f_2]\, \sigma v\, d\Omega\, d^3v}$$

Collision Invariants

Invariants of the collision integral are also invariants of the Boltzmann equation; Finding them gives us conservation laws.

A moment $Q(\vec{u}_1)$ is an invariant if:

$$I(\vec{x},t) \equiv \iiint Q(\vec{u}_1)\left[(f_1'f_2' - f_1 f_2)\,\sigma v\, d\Omega\, d^3v_2\right] d^3v_1 \overset{!}{=} 0 \Rightarrow Q \text{ invariant}$$

The particles must be interchangeable:

$$\Rightarrow I(\vec{x},t) = \iiint Q(\vec{u}_2)\left[(f_1'f_2' - f_1 f_2)\,\sigma v\, d\Omega\, d^3v_1\right] d^3v_2$$

$$\Rightarrow I = \frac{1}{2}(I+I) = \frac{1}{2}\iiint \left[Q(\vec{u}_1) + Q(\vec{u}_2)\right]\left[(f_1'f_2' - f_1 f_2)\,\sigma v\, d\Omega\, d^3v_1\right] d^3v_2$$

Reverse collisions also have to hold:

$$I = \frac{1}{2}\iiint \left[Q(\vec{u}_1') + Q(\vec{u}_2')\right]\left[f_1 f_2 - f_1'f_2'\right]\sigma v\, d\Omega\, d^3v_1\, d^3v_2$$

$$= \frac{1}{2}(I+I)$$

$$= \frac{1}{4}\iiint \left[Q(\vec{u}_1) + Q(\vec{u}_2) - Q(\vec{u}_1') - Q(\vec{u}_2')\right]\left[f_1'f_2' - f_1 f_2\right]\sigma v\, d\Omega\, d^3v_1\, d^3v_2$$

With $I \equiv 0 \Rightarrow Q(\vec{u}_1) + Q(\vec{u}_2) = Q(\vec{u}_1') + Q(\vec{u}_2')$

Equilibria

LTE: Internal state of a system in which over a timescale of interest no macroscopic flows of matter or energy are present.

Detailled Balance: At equilibrium, each elementary process (= collisions) is balanced by its reverse process.

LTE $\Rightarrow$ Detailled balance, but not the other way around!

In equilibrium: (Global Thermodynamic equilibrium)

$$\frac{\partial f}{\partial t} = 0 \quad \text{static}$$

$$\frac{\partial f}{\partial \vec{x}} = 0 \quad \text{uniform in space}$$

$$\frac{\partial f}{\partial \vec{u}} = 0 \quad \text{no perturbations} \quad (\vec{a} = 0)$$

For local: $\left(\frac{Df}{Dt}\right)_{coll} = 0$ suffices.

Other simplifications:

$\vec{v} = 0$ static

$\vec{a} = 0$ isolated

Maxwell-Boltzmann distribution

The Maxwell-Boltzmann distribution is a distribution for a system in LTE.

Thus demanding detailled balance: $f_1' f_2' \overset{!}{=} f_1 f_2$

$\Rightarrow \ln f_1' + \ln f_2' = \ln f_1 + \ln f_2$

$\Rightarrow$ If $\ln f_i = Q_i$ = an invariant moment, the condition for detailled balance is satisfied and we get conservation laws.

Ansatz: We have 3 collision invariants (m, $m\vec{v}$, $\frac{1}{2}mv^2$) and 3 equations; So f must be a function of those moments so the equations won't be overdetermined

$\Rightarrow \ln f_0 = \alpha + \vec{\beta} \cdot \vec{u} + \gamma \frac{1}{2} u^2$

$$\Rightarrow \boxed{f_0 = \frac{n(\vec{x}, t)}{(2\pi\sigma^2)^{3/2}} e^{-1/2 [\vec{v} - \vec{u}(\vec{x}, t)]^2 / \sigma^2}}$$

With:

$\vec{n}$: particle density

$\vec{v}$: bulk velocity

$\vec{u}$: particle velocity

σ^2: $\frac{k_B T}{m}$ velocity dispersion

statistical dispersion of velocities about the mean velocity.

Properties: $\int f_0 \, dv = 1$

$\int f_0(w) \cdot w \, dw = 0$ f_0 even, w odd, together odd $\Rightarrow 0$

$\int f_0 w^2 dw = \sigma^2$

Collision Time, Mean Free Path

Consider two species of particles. The collision rate between them can be approximated with:

$$C_{12} = \#\text{ species } 1 \cdot \#\text{ species } 2 \cdot \text{prob. to collide } / dt$$

$$= \frac{dN}{dt} = \iint n\, f_1 f_2\, \sigma\, v\, dt\; d^3v_1\, d^3v_2 \,/ dt$$

$$= \frac{n_1 \cdot n_2 \cdot \langle \sigma v \rangle\, dt}{dt} = n_1 n_2 \langle \sigma v \rangle$$

The collision time is defined for a species as the reciprocal of the collision rate:

$$\tau_{coll} = \frac{n_1}{C_{12}} = \frac{1}{n_2 \langle \sigma v \rangle}$$

Mean free path: Average length between two collisions

$$\lambda = \langle v \rangle \cdot \tau_{coll} = \frac{1}{n_2 \sigma} \quad (\text{independent of } T!)$$

For rigid spheres: $\sigma = \sigma_0 = 10^{-15}\,\text{cm}^{-2}$

for $T < 10^3 - 10^4\,K$

$\langle v \rangle \approx \sigma^2$ (velocity dispersion)

in CoM-frame:

$= \sqrt{\frac{k_B T}{\mu}}$

Validity of MB:

Define scale height of parameters $p = n, v, T$: $h(p) \equiv \left| \frac{1}{p} \nabla p \right|$

If the spatial gradients of these parameters is small enough, the system can be considered locally uniform and thus in LTE.

$$\Rightarrow h(p) \equiv \left| \frac{1}{p} \nabla p \right| < 1/\lambda$$

Moments of the Boltzmann equation

Take Boltzmann equation: $\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial x} + \vec{a}\frac{\partial f}{\partial \vec{v}} = \left(\frac{Df}{Dt}\right)_{coll}$

If you multiply both sides with a moment (m, $m\vec{v}$, $\frac{1}{2}mv^2$) and integrate over all space, we can derive for each moment a conservation law. (Because $\int \left(\frac{Df}{Dt}\right)_{coll} \cdot Q \, d^3v = 0$ as shown before; moments are collision invariants)

Remember:
- $t, \vec{x}, \vec{v}$ are independent variables! Interchange the order as you please
- f sinks to 0 more rapidly than any power law (definition of distribution): $\lim_{\alpha \to \infty} \alpha^n f(\alpha) = 0 \;\forall \alpha, n$
- $\vec{a} = \vec{a}(x)$

1) Mass conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \cdot \langle \vec{v} \rangle) = 0}$$

2) Momentum conservation

$$\boxed{\frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}(\rho\vec{u}\otimes\vec{u} + \underline{\underline{P}}) = \rho\vec{a}(\vec{x})}$$

with $(\vec{u}\otimes\vec{u})_{ij} = u_i u_j$

$P_{ij} = \int_{\mathbb{R}^3} m f w_i w_j \, d^3v$ Pressure tensor

3) Energy conservation

$$\boxed{\frac{\partial E}{\partial t} + \vec{\nabla}(E\vec{u} + \underline{\underline{P}}\,\vec{u} + \vec{Q}) = \rho\vec{a}\cdot\vec{u}}$$

with $\vec{Q} = \int \frac{1}{2} m w^2 \vec{w} f \, d^3v$ Heat flux

Euler Equations

The Euler equations are the moments of the Boltzmann equation where $f = f_0 = f_{\text{Maxwell-Boltzmann}}$.

In this case, $\overline{\overline{P}}$ and $\vec{Q}$ simplify:

$$P_{ij} = \int_{\mathbb{R}^3} m w_i w_j f \, d^3v = \int_{\mathbb{R}^3} m w_i w_j n g(w_x) g(w_y) g(w_z) \, d^3v$$

$g(w_i)$: 1D-gauss

For $i \neq j$: Odd power of w_i $\Rightarrow$ Integral $= 0$

$\Rightarrow$ P_{ij} is diagonal $\Rightarrow \overline{\overline{P}} = P\mathbb{1}$

For $i = j$:

$$P_{ii} = \int_{\mathbb{R}^3} m w_i^2 n g(w_x) g(w_y) g(w_z) \, d^3v = 1 \cdot 1 \cdot \rho \cdot \sigma^2$$

And for $\vec{Q}$: $\vec{Q} = \int \frac{1}{2} m w^2 \vec{w} f_0 \, d^3v = 0$ (again even-odd)

This gives us the Euler equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \qquad \frac{D\rho}{Dt} = -\rho \vec{\nabla} \vec{v}$$

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P\mathbb{1}) = \rho \vec{a} \qquad \frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{a}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla}(E + P)\vec{u} = \rho \vec{a} \cdot \vec{u} \qquad \frac{De}{Dt} = -\frac{1}{\rho} P \vec{\nabla} \cdot \vec{u}$$

$$E = \frac{1}{2} \rho u^2 + e$$

$$e = \int \frac{1}{2} m w^2 f \, d^3v = \frac{1}{2} \mathrm{Tr}(P) = (\gamma + 1) P \quad \text{internal energy}$$

Quantum Physics and Kinetic Theory

Typical size of quantum cell: $d^6V = d^3x\, d^3v \approx h^3$

Occupation number N: Number of particles per cell

$$N \equiv f h^3$$

Types of particles:

Fermions: $N \leq 1$ (Pauli exclusion)

Bosons: $N \gg 1$

We need to add a suppression/enhancement factor into the integrand of the collision integral:

depends on the situation of ~~the~~ where the particles will land in!

$$\frac{\delta N_2}{\delta t} = f_1 f_2\, \sigma v\, (1 \pm N_1')(1 \pm N_2')$$

$$\frac{\delta N_1}{\delta t} = f_1' f_2'\, \sigma v\, (1 \pm N_1)(1 \pm N_2)$$

$+$ for Bosons.

These terms give us the statistics:

Bose-Einstein: $\boxed{f_{BE} = \frac{1}{h^3} \frac{1}{e^{(E-\mu)/k_B T} - 1}}$

Fermi-Dirac: $\boxed{f_{FD} = \frac{1}{h^3} \frac{1}{1 + e^{(E-\mu)/k_B T}}}$

Equations of State

EoS relate state quantities such as pressure, density and temperature.

Isothermal EoS: $P = a^2 \rho$

$$a^2 = \sigma^2 = \frac{k_B T_0}{m} \quad \text{(ideal gas equ)}$$

adiabatic stuff: $\Gamma = \frac{c_p}{c_v} = \frac{f+2}{f}$ (ideal gas)

$$P = (\gamma - 1)\rho\varepsilon = (\gamma - 1)e \qquad e = \frac{U}{V} = \frac{N k_B T}{V}$$

$$E = \rho\varepsilon + \frac{1}{2}\rho v^2$$

Polytropic EoS: $P = P_0 \left(\frac{\rho}{\rho_0}\right)^\Gamma$

Degenerate (for very high densities, where QM is important again)

For low temperatures, depends only on ρ (pauli pushes)

Derivation: Isotropic stress tensor, assume radial symmetry

Approximate $E \approx \frac{p^2}{2m}$, $v \approx \frac{p}{m}$

Take fermi distribution: $f = \frac{g}{h^3} \frac{1}{1 + e^{(E-\mu)/k_B T}}$

with $\mu = E_F + m_e c^2$

Degenerate Equations of State

fermi-dirac distribution: $f(E) = \frac{g}{h^3} \frac{1}{e^{(E-E_F)/k_BT}+1}$

number density: $n = \int f \, d^3p = 4\pi \int f p^2 dp$

Degeneracy: $f(E > E_F) \approx 0$ For $k_B T \ll E_F$

Pressure for isotropic gas:

$$P_{zz} = \int m f v_z v_z \, d^3p = \int v_z p_z f \, d^3p =$$

$$= \int v\cos\theta \, p\cos\theta \, f p^2 \sin\theta \, d\theta d\varphi dp$$

$$= \int v p f p^2 \sin\theta \cos^2\theta \, d\theta d\varphi dp$$

Now use: $\int \cos^2\theta \sin\theta d\theta = [-\cos\theta\cos^2\theta]_0^\pi - 2\int \cos^2\theta \sin\theta d\theta$

$\Rightarrow 3\int \cos^2\theta \sin\theta d\theta = [+1+1] = 2$

$\Rightarrow \int \cos^2\theta \sin\theta d\theta = 2/3$

$\int u'v = uv - \int uv'$
$u' = \sin \quad v = \cos^2$
$u = -\cos \quad v' = 2\cos\sin$

$$= \frac{2}{3} \cdot 2\pi \int v p f p^2 dp = \boxed{\frac{1}{3} \int v p f \, 4\pi p^2 dp}$$

1) Relativistic Regime

$$E = \sqrt{m^2c^2 + p^2c^4} \approx pc$$

$$\hookrightarrow f(E) = f(p) = \frac{g}{h^3} \frac{1}{e^{(p-p_F)c/k_BT}+1}$$

a) Cold case: $k_B T \ll E_F$

for $p < p_F$: $e^{(p-p_F)c/k_BT} \approx 0$

$$\hookrightarrow f(p) \approx \frac{g}{h^3}$$

$$\Rightarrow n = \int_0^\infty f d^3p = 4\pi \int_0^{p_F} \frac{g}{h^3} p^2 dp = \boxed{\frac{4\pi}{3} \frac{g}{h^3} p_F^3}$$

$$P = \frac{4\pi}{3}\int_0^\infty f \cdot v \cdot p^3 dp \overset{v \approx c}{=} \frac{4\pi}{3}\int_0^{p_F} \frac{4\pi g}{h^3} \cdot c \cdot p^3 dp$$

$$\boxed{= \frac{4\pi^2}{3}\frac{g}{h^3} c p_F^4}$$

2) Non-Relativistic Regime

Same arguments: $f \approx \frac{g}{h^3}$

$\rightarrow \quad n = \int_0^{p_F} 4\pi f p^2 dp = \frac{4\pi}{3}\frac{g}{h^3} p_F^3$

$$P = \frac{4\pi}{3}\int f p v p^2 dp \overset{v = p/m}{=}$$

$$= \frac{4\pi}{3}\int_0^{p_F} \frac{g}{h^3} \frac{p^4}{m} dp$$

$$\boxed{= \frac{4\pi}{15}\frac{g}{h^3}\frac{1}{m} p_F^5}$$

What if non-degenerate? $E - E_F \gg k_B T$?

$\rightarrow$ exponential term of FD-distribution is dominant:

$$f \approx \frac{g}{h^3} e^{-(E-\mu)/k_B T} \overset{E = mc^2 + p^2/2m}{=} \frac{g}{h^3} e^{-(mc^2 + p^2/2m - \mu)/k_B T}$$

$$= \frac{g}{h^3} e^{-mc^2 + \mu} e^{-p^2/2mk_B T}$$

$$= \frac{g}{h^3} e^{-mc^2} e^{-\frac{v^2}{2 k_B T/m}}$$

$\rightarrow$ distribution approximates Maxwell-Boltzmann distribution again (non-relativistic)

Boltzmann's H-Theorem

A gaskinetic formulation of the second law of thermodynamics. (Entropy.) H can only decrease over time.

Consider an isolated ($\vec{a}=0$) uniform ($\frac{\partial}{\partial x}(\,)=0$) medium.
Then define $H = \int_{\mathbb{R}^3} f(\vec{v},t)\ln(f(\vec{v},t))\,d^3v$

$$\hookrightarrow \frac{dH}{dt} = \int_{\mathbb{R}^3} \frac{d}{dt}\left[f\ln(f)\right] d^3u_1 =$$

$$= \int d^3u_1 \left[\frac{df}{dt}\ln(f) + f\cdot\frac{d\ln(f)}{dt}\right] d^3u_1$$

$$= \int d^3u_1 \left[\Big(\frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial x}\cdot\vec{v} + \frac{\partial f}{\partial v}\vec{a}}_{=0}\Big)\ln f + f\cdot\frac{1}{f}\Big(\frac{\partial f}{\partial t}+0+0\Big)\right] d^3u_1$$

$$= \int d^3u_1\,(1+\ln f)\,\frac{\partial f}{\partial t} \qquad [1]$$

On the other hand, consider the Boltzmann equation in this case:

$$\frac{\partial f}{\partial t} + \underbrace{\vec{v}\frac{\partial f}{\partial \vec{x}} + \vec{a}\frac{\partial f}{\partial \vec{v}}}_{=0} = \frac{\partial f}{\partial t} = \left(\frac{Df}{Dt}\right)_{coll} = \iint (f_1'f_2' - f_1f_2)\,u\,\sigma\,d\Omega\,d^3u_2$$

$$\Rightarrow \frac{\partial H}{\partial t} = \iiint \underbrace{[\ln f + 1]}_{Q}(f_1'f_2' - f_1f_2)\,v\,\sigma\,d\Omega\,d^3u_1\,d^3u_2$$

We see that we can ~~interpret~~ treat $\ln f + 1$ as a collision invariant Q

By demanding that collisions can be inversed in time ($Q(u_1)=Q(u_2)$, then $\frac{\partial H}{\partial t} = \frac{1}{2}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial t}_{reversed}\right)$ and then applying the demand for inverse collisions ($\cdot\,-1$) we obtain:

$$\frac{\partial H}{\partial t} = -\frac{1}{4}\iiint\left[\ln(f_1'f_2') - \ln(f_1f_2)\right](f_1'f_2' - f_1f_2)\,v\,\sigma\,d\Omega\,d^3u_1\,d^3u_2$$

or: $$\frac{dH}{dt} = -\frac{1}{4}\iiint \underbrace{\ln\left(\frac{f_1' f_2'}{f_1 f_2}\right)}_{a} \underbrace{(f_1' f_2' - f_1 f_2)}_{b} v\sigma \, d\Omega \, d^3u_1 \, d^3u_2$$

Now if $(f_1 f_2) < (f_1' f_2')$: $a > 0,\ b > 0 \Rightarrow \frac{\partial H}{dt} \leq 0$

$(f_1 f_2) > (f_1' f_2')$: $a < 0,\ b < 0 \Rightarrow \frac{dH}{dt} \leq 0$

$\Rightarrow \frac{dH}{dt} \leq 0.$

Now look at eq [1]. In an LTE, we must have $\frac{\partial f}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0.$

That means that in LTE, we have $f_1' f_2' = f_1 f_2$

↳ We get MB distributions.

H must have a lower boundary because systems must have finite energy.

$\Rightarrow$ If the system is in LTE, we have detailled balance.

Chapman - Enskog theory

= a perturbative approach for dealing with systems in non-LTE conditions.

Ansatz: $f = f_{MB} + \delta f$, $\quad |\delta f| \ll |f_{MB}|$

Collision time approximation: $\frac{\partial f}{\partial t} + \vec{u}\,\vec{\nabla} f \approx -\frac{\delta f}{\tau_{coll}}$

Results: Viscosity (= non-diagonal elements of pressure tensor) and heat flux re-enter the equations

Heat conduction coefficient:

Assume $\rho = \rho_0$, $\vec{v} = 0$

Then $\frac{\partial E}{\partial t} = -\nabla(\kappa \vec{\nabla} T) = -\vec{\nabla}\vec{Q} = -\kappa \Delta T$

with $E = \frac{f}{2} N k_B T$: $\quad \frac{\partial T}{\partial t} = \frac{-\kappa}{\frac{f}{2} n k_B}\Delta T = w \Delta T$

Dimensional analysis:

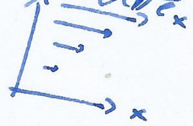
$[w] = m^2/s \longrightarrow$ Ansatz $w = \sigma \cdot \lambda \qquad \sigma$: vel. disp.

with $\lambda = \sigma \tau_{coll} \rightarrow w = \sigma^2 \tau_{coll} = -\frac{\kappa}{\frac{f}{2} n k_B}$

$\Rightarrow \kappa = -\frac{f}{2} n k_B \sigma^2 \tau_{coll} = -\frac{f}{2}\frac{\rho}{m} k_B \frac{T}{T} \sigma^2 \tau_{coll} = \boxed{-\frac{f}{2}\rho \frac{\sigma^4}{T}\tau_{coll}}$

Viscosity coefficient

Viscosity tensor: $D_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\delta_{ij}$



Gives us: $\frac{\partial \vec{v}}{\partial t} = \frac{\nu}{\rho}\frac{\partial^2 u}{\partial \cancel{x}} = w \frac{\partial^2 u}{\partial \cancel{x}}$

Dimensional analysis: $[w] = \frac{m^2}{s} \rightarrow w = \sigma^2 \tau_{coll}$ $(= \sigma\lambda)$

$w = \sigma^2 \tau_{coll} = \frac{\nu}{\rho} \Rightarrow \boxed{\nu = \rho \sigma^2 \tau_{coll}}$

$\kappa \propto T^{1/2} m^{-1/2}$: small particles dominate

$\nu \propto T^{1/2} m^{1/2}$: large particles dominate

Coulomb gas

Problem: $\frac{1}{r}$ potential $\neq 0$ for very large distances.

→ Add cut-off when you consider plasma to be neutral again.

Get impact parameter for 90° deflection: (= must stop at center completely) $\Rightarrow \frac{1}{2} m v^2 = \frac{e^2}{b_{90}} \Rightarrow b_{90} = \frac{2e^2}{m v^2}$

Cross section rigid spheres:

$$\sigma_{CG} = \pi b_{90}^2 = \frac{4e^4}{m^2 v^4} = \frac{4e^4}{(k_B T/m)^2} \times \underbrace{\ln \Lambda}_{\text{coulomb logarithm for cut off}}$$

Virial Theorem

Scalar moment of inertia: $I(t) = \int_{V(t)} \rho(x,t)\, x^2 d^3x$

Reynold's transport theorem: With $I = \int_{V(t)} \alpha\, d^3x$

$$\dot{I} = \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \vec{\nabla}(\alpha \cdot \vec{v}) \right] d^3x$$

Virial theorem:

$$\frac{1}{2}\ddot{I}(t) = \int \rho \vec{v}^2 d^3x + \int 3P d^3x + \int \rho \vec{x}\vec{g}\, d^3x - \int_S P\vec{x}\cdot\vec{n}\, dS$$

$$= 2K + V - S$$

For isolated objects: $S = 0$

Simple solutions:

Poisson equ: $\Delta \phi = 4\pi G \rho$; $\vec{\nabla}\phi = -\vec{g}$

$$\rightarrow \vec{g}(r) = -\frac{GM(r)}{r}\vec{e}_r$$

Static case: $\vec{u} = 0$

$$K = \int_V \Big(\underbrace{\frac{1}{2}\rho u^2}_{=0} + \frac{3}{2}P \Big) d^3r = \frac{3}{2} P \frac{4\pi}{3} R^3$$

$$V = \int_V (\rho \vec{x}\vec{g})\, d^3x = \int_0^R \rho g r\, 4\pi r^2 dr = -\int \rho^2 G M$$

$$= \int \rho \Big(-\frac{G\rho \frac{4\pi}{3} r^3}{r} \Big) 4\pi r^3 dr = -G\rho^2 \frac{16\pi^2}{3} \int r^4 dr = -\frac{16\pi^2}{15} G\rho^2 R^5$$

$S = 0$

With $2K + V = 0$

$$\Rightarrow 4\pi R^3 P = \frac{16}{15}\pi^2 G\rho^2 R^5 \quad \Rightarrow \quad P = \frac{4\pi}{15} G\rho^2 R^2$$

Chose EoS: isothermal: $P = \rho \frac{k_B T}{m}$

insert, blabla

Hydrostatic Equilibrium

Radial part of spherical Euler equations:

$$\frac{Du_r}{Dt} = \frac{u_\phi^2}{r} - \frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial \Phi}{\partial r}$$

For a system in global equilibrium: $\vec{u} = 0$

$$\Rightarrow \boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \Phi}{\partial r}}$$

Poisson equation: $\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \rho}$

Lane - Emden equation

Combination of poisson + hydrost. equil, polytropic EoS
Can be nondimensionalised.

$$\rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G \rho$$

Isothermal case: $\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{1}{\rho} a^2 \frac{\partial \rho}{\partial r} = -\frac{\partial \Phi}{\partial r}$

$$\Rightarrow \rho = \rho_0 e^{-\Phi/a^2}$$

Singular solutions: $\rho = A r^{-\alpha}$ for isothermal:

$$\frac{\partial \Phi}{\partial r} = -a^2 \frac{1}{\rho}\frac{\partial \rho}{\partial r} = -a^2 \frac{1}{A r^{-\alpha}} \frac{\partial (A r^{-\alpha})}{\partial r} = a^2 \frac{\alpha}{r}$$

$$\rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a^2 \frac{\alpha}{r}\right) = 4\pi G A r^{-\alpha}$$

Accretion Disks

= Structures formed by diffused material in orbital motion around a central massive object. Disk spirals inward because of center object's gravitational forces. ⇒ Dominant force contribution comes from central object.

Assume axissymmetry: $\vec{\nabla} \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(A_r r) + \frac{\partial A_z}{\partial z}$

⇒ Mass conservation: $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r \rho) + \frac{\partial}{\partial z}(\rho v_z) = 0$

Get out z dependance: define surface mass density

$$\Sigma(r) = \int_{-\infty}^{\infty} \rho(r,z)\, dz$$

Take Momentum Euler equations in $\partial r, \partial \theta$:

i) $\rho\left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial}{\partial r} v_r - \frac{v_\theta^2}{r}\right] = -\frac{\partial P}{\partial r} - \rho \frac{GM_*}{r^2}$

$\rightarrow \boxed{\Sigma\left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial}{\partial r} v_r - \frac{v_\theta^2}{r}\right] = -\Sigma \frac{GM_*}{r^2} - \int dz \frac{\partial P}{\partial r}}$

ii) $\rho\left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial}{\partial r} v_\theta + \frac{v_r v_\theta}{r}\right] = 0$: No forces in θ-direction!

$\rightarrow \boxed{\Sigma\left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r}\right] = 0}$

Accretion rate:

$\int \frac{\partial}{\partial z}(\rho v_z) = 0$ because $= \int \frac{\partial}{\partial z}(\rho v_z)\, dz = \rho v_z \Big|$

$\Rightarrow \dot{M} = \int \frac{\partial \rho}{\partial t} dV = \int d\theta \int dz \int r\, dr \frac{\partial \rho}{\partial t} = -\int dz \int d\theta \int r\, dr \frac{1}{r}\frac{\partial}{\partial r}(r v_r \rho)$

$= 2\pi r v_r \int \rho\, dz = 2\pi \Sigma r v_r$

$$\boxed{\dot{M} = 2\pi \Sigma r v_r}$$

Solutions:

1) Static equilibrium in radial direction

Set $v_r = 0$, $\frac{\partial}{\partial t}() = 0$, $\Rightarrow \dot{M} = 0$ $\quad\quad \Pi = \int_{-\infty}^{\infty} P\,dz$

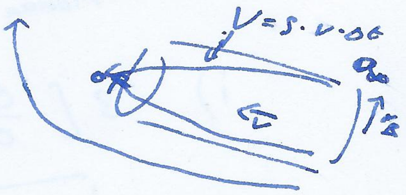
Only equation left: $\boxed{\frac{v_\theta^2}{r} - \frac{GM}{r^2} = \frac{1}{\Sigma}\frac{\partial \Pi}{\partial r}}$

Centrifugal equilibrium

Bondi Accretion

Very simplified model: Assume (wrongly) spherical accretion.
Ignore radiation and magnetic fields;
Assume infinite medium with known ρ_∞, a_∞, with particles moving towards center.
Now determine which particles will fall in:

$$\frac{1}{2} m a_\infty^2 = \frac{GMm}{r_B} \to r_B \approx \frac{2GM}{a_\infty^2}$$



Then the accretion rate will be:

$$\dot{M}_B = \frac{\Delta m}{\Delta t} = \frac{[(\pi r_B^2)\cdot \overset{\rho\cdot V}{v}\cdot\Delta t]\cdot\rho_\infty}{\Delta t} = \frac{4\pi G^2 M^2 a_\infty \rho_\infty}{a_\infty^4} = \frac{4\pi G^2 M^2 \rho_\infty}{a_\infty^3}$$