Exercise 3: Virial Theorem

a) Derive the Reynolds transport theorem

[Mihalas & Mihalas p. 60]

Let I be the integral form of an equation, i.e.
$I = \int_{V(t)} \alpha \, d^3x$, where α is any single-valued scalar, vector, or tensor field. Also let $V(t)$ be some finite (material) volume with $dV(t=0) = dV_0$.

Furthermore, let us introduce the "color function" ψ:

$$\psi(\vec{x}) = \begin{cases} 1 & \vec{x} \in V(t) \\ 0 & \text{else} \end{cases}$$

such that
$$I = \int_{V(t)} \alpha \, d^3x = \int_{\mathbb{R}^3} \alpha \psi \, d^3x$$

This moves the t-dependance of the volume into the integral.

By definition, $\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + \vec{v} \cdot \vec{\nabla} \psi = 0$ because ψ is moving along with the particle flow.

Then: $\frac{dI}{dt} = \frac{d}{dt}\left[\int_{V(t)} \alpha \, d^3x\right] = \frac{d}{dt}\left[\int_{\mathbb{R}^3} \alpha \Psi \, d^3x\right]$

$$= \int_{\mathbb{R}^3} \frac{d}{dt}[\alpha \Psi] \, d^3x =$$

$$= \int_{\mathbb{R}^3}\left[\frac{\partial \alpha}{\partial t}\Psi + \alpha \frac{\partial \Psi}{\partial t}\right] d^3x$$

Now use that $\frac{D\Psi}{Dt} = 0 = \frac{\partial \Psi}{\partial t} + \vec{v}\cdot\vec{\nabla}\Psi \Rightarrow \frac{\partial \Psi}{\partial t} = -\vec{v}\cdot\vec{\nabla}\Psi$

$$\hookrightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3}\left[\frac{\partial \alpha}{\partial t}\Psi + \alpha\frac{\partial \Psi}{\partial t}\right] d^3x = \int_{\mathbb{R}^3}\left[\frac{\partial \alpha}{\partial t}\Psi - \alpha\vec{v}\cdot\vec{\nabla}\Psi\right] d^3x$$

Furthermore use that:

$$\vec{\nabla}\cdot(\Psi\alpha\vec{v}) = \Psi\vec{\nabla}(\alpha\vec{v}) + \alpha\vec{v}\cdot(\vec{\nabla}\Psi)$$

$$\Rightarrow -\alpha\vec{v}(\vec{\nabla}\Psi) = \Psi\vec{\nabla}(\alpha\vec{v}) - \vec{\nabla}\cdot(\Psi\alpha\vec{v})$$

$$\hookrightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3}\left[\frac{\partial \alpha}{\partial t}\Psi - \alpha\vec{v}(\vec{\nabla}\Psi)\right] d^3x =$$

$$= \int_{\mathbb{R}^3}\left[\frac{\partial \alpha}{\partial t}\Psi + \Psi\vec{\nabla}(\alpha\vec{v}) - \vec{\nabla}\cdot(\Psi\alpha\vec{v})\right] d^3x$$

$$= \int_{\mathbb{R}^3}\left[\Psi\left(\frac{\partial \alpha}{\partial t} + \vec{\nabla}(\alpha\vec{v})\right) - \vec{\nabla}\cdot(\Psi\alpha\vec{v})\right] d^3x$$

In the next step, show that $\int_{\mathbb{R}^3}\vec{\nabla}\cdot(\Psi\alpha\vec{v})\,d^3x = 0$ using the divergence theorem:

$$\int_V \vec{\nabla}\cdot\vec{g}\,d^3x = \int_{S(V)} \vec{g}\cdot\vec{n}\,dS$$

$$\int_{\mathbb{R}^3} \vec{\nabla} \cdot (\alpha \psi \vec{v}) d^3x = \int_{S(\mathbb{R}^3)} \alpha \psi \vec{v} \vec{n} \, d^3x$$

Remember that $\psi(\vec{x}) = 0$ for $x \notin V(t)$:

$$\Rightarrow \int_{S(\mathbb{R}^3)} \alpha \psi \vec{v} \vec{n} \, d^3x = 0$$

The surface that is being integrated over is clearly outside of $V(t)$.

$$\Rightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3} \psi \left[\frac{\partial \alpha}{\partial t} + \vec{\nabla} \cdot (\alpha \cdot \vec{v}) \right] d^3x$$

$$= \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \vec{\nabla} \cdot (\alpha \cdot \vec{v}) \right] d^3x$$

This is the Reynolds transport theorem.

Now consider a specific variable, i.e. β for which is: $\alpha = \rho \cdot \beta$ where ρ is the density.

Then:

$$I(t) = \int_{V(t)} \rho \beta \, d^3x$$

$$\dot{I}(t) = \int_{V(t)} \left[\frac{\partial}{\partial t}(\rho \beta) + \vec{\nabla}(\rho \beta \vec{v}) \right] d^3x$$

$$= \int_{V(t)} \left[\beta \frac{\partial \rho}{\partial t} + \rho \frac{\partial \beta}{\partial t} + \beta \vec{\nabla}(\rho \vec{v}) + (\rho \vec{v})(\vec{\nabla} \beta) \right] d^3x$$

$$= \int_{V(t)} \left[\rho \left(\frac{\partial \beta}{\partial t} + \vec{v}(\vec{\nabla} \cdot \beta) \right) + \beta \underbrace{\left(\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) \right)}_{[*]} \right] d^3x$$

If the Euler equations are used, then the term [*] is the mass conservation equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

$$\Rightarrow \dot{I}(t) = \int_V \rho \left(\frac{\partial \beta}{\partial t} + \vec{v}(\vec{\nabla} \cdot \beta) \right) d^3x = \int_V \rho \frac{D\beta}{Dt} d^3x$$

b) Derive the virial theorem

Start with the "scalar moment of inertia": $I = \int_{V(t)} \rho |x|^2 \, d^3x$

Then: $\dot{I}(t) = \int_{V(t)} \rho \frac{D}{Dt}(|x|^2)\, d^3x = \int \rho \frac{D}{Dt}(\vec{x}\cdot\vec{x})\, d^3x$

$$= \int_{V(t)} \rho \, 2 \cdot \vec{x} \cdot \frac{D\vec{x}}{Dt} \, d^3x = 2\int \rho \, \vec{x}\cdot\vec{v} \, d^3x$$

where it was used that $\frac{D\vec{x}}{Dt} = \vec{v}$
(by definition of the comoving derivation; see Mihalas 55-56)

$$\Rightarrow \tfrac{1}{2}\dot{I} = \int_{V(t)} \rho \, \vec{x}\cdot\vec{v} \, d^3x$$

Then:

$$\tfrac{1}{2}\ddot{I} = \tfrac{1}{2}\frac{d}{dt}\dot{I} = \int_{V(t)} \rho \frac{D(\vec{x}\cdot\vec{v})}{Dt} d^3x =$$

$$= \int_{V(t)} \rho \left[\vec{v}\frac{D\vec{x}}{Dt} + \vec{x}\frac{D\vec{v}}{Dt} \right] = \int_{V(t)} \rho \left[v^2 + \vec{x}\frac{D\vec{v}}{Dt} \right] d^3x$$

For the last part of the integral, use the Euler equations again: $\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \rho \vec{a}$

$$\Rightarrow \int_{V(t)} \rho \, \vec{x} \frac{D\vec{v}}{Dt} d^3x = \int_{V(t)} \vec{x}\left[-\vec{\nabla} P + \rho\vec{a}\right] d^3x = \int_{V(t)} \rho \cdot \vec{x}\cdot\vec{a} \, d^3x$$

$$= \int_{V(t)} \rho \cdot \vec{x}\cdot\vec{a}\, d^3x - \int_{V(t)} \vec{x}\cdot\vec{\nabla} P \, d^3x$$

For the last term, once again use that

$$\vec{\nabla}(P\vec{x}) = \vec{x}(\vec{\nabla}P) + P(\underbrace{\vec{\nabla}\cdot\vec{x}}_{=3}) = \vec{x}(\vec{\nabla}P) + 3P$$

$$\rightarrow -\vec{x}\cdot(\vec{\nabla}P) = 3P - \vec{\nabla}(P\vec{x})$$

Then:

$$-\int_{V(t)} \vec{x}\cdot\vec{\nabla}P \, d^3x = \int_V 3P\, d^3x - \int_V \nabla(P\vec{x})\, d^3x \quad \text{[divergence theorem again]}$$

$$= \int_V 3P\, d^3x - \int_{S(V)} P\vec{x}\cdot\vec{n}\, dS$$

Putting all together:

$$\frac{1}{2}\ddot{I} = \int_{V(t)} \rho \cdot v^2 \, d^3x + \int_{V(t)} \rho \cdot \vec{x} \cdot \vec{a} \, d^3x + \int_{V(t)} 3P \, d^3x - \int_{S(V)} P \, \vec{x} \cdot \vec{n} \, dS$$

$$= \underbrace{\underbrace{\int_V \rho \cdot v^2 d^3x}_{\substack{2 \cdot \text{Macroscopic} \\ \text{kinetic energy} \\ 2 \cdot \frac{1}{2} \rho v^2 dV}} + \underbrace{\int_V 3P d^3x}_{\substack{2 \cdot \text{Microscopic} \\ \text{kinetic Energy} \\ 2 \cdot \frac{3}{2} P dV \\ \text{"} \equiv 2 \cdot \frac{3}{2} N k_B dT \text{"}}}}_{\equiv 2K} + \underbrace{\underbrace{\int_V \rho \cdot \vec{x} \cdot \vec{a} \, d^3x}_{\substack{\text{Force term:} \\ \vec{a} \text{ comes from} \\ \text{any force(s)}}}}_{\equiv V} - \underbrace{\underbrace{\int_S P \vec{x} \cdot \vec{n} \, dS}_{\text{Surface term}}}_{\equiv S}$$

← my "interpretation"

$$\frac{1}{2}\ddot{I} = 2K + V - S$$

For an equillibrium state, we must demand: $\ddot{I} = 0$

[Note: $\dot{I} = 2\int_V \vec{x} \cdot \rho \cdot \vec{v} \, d^3x$ represents the sum of the dot products of all particle's positions and their velocities, so it sort of represents the system's total action. Demanding $\ddot{I} = 0$ would mean that the total action of the system is constant over time.

Furthermore: A system in equilibrium must be bound. If it is, then $\langle \dot{I} \rangle = \frac{1}{T}\int_0^T \dot{I} \, dt = \frac{1}{T}(I(T) - I(0)) \xrightarrow{T \to \infty} 0$. The average only diverges if $I(t)$ is not bound ($I(T) \to \infty$), which would mean $\vec{V} \to \infty$ since $I = \int_V \rho |\vec{x}|^2 d^3x$]

c) Stellar mass - radius relation

Assuming equillibrium state and a uniform, spherical system

1) Calculate gravitational acceleration

Poisson equation: $\Delta\varphi = 4\pi G\rho = \vec{\nabla}(\vec{\nabla}\varphi) = \vec{\nabla}(-\vec{g}) = -\vec{\nabla}\cdot\vec{g}$

Assume spherical symmetry: $\vec{g} = g\vec{e}_r$

Integrate poisson equation:

$$\int_V \Delta\varphi \, dV = \int_V -\vec{\nabla}\vec{g}\, dV \overset{\text{div. th.}}{=} -\int_S \vec{g}\cdot\vec{n}\, dS = -\int g\cdot\vec{e}_r\cdot\vec{e}_r\, dS = -\int g\, dS = -4\pi r^2 g$$

$$= \int_V 4\pi G\rho\, dV = \frac{16\pi^2}{3} G\rho r^3$$

$$\Rightarrow \quad g = \frac{-4\pi G\rho r^3}{3r^2} = -\frac{4\pi}{3} G\rho r$$

2) Virial theorem: $\frac{1}{2}\ddot{I} = 2K + V - S$

Assuming equilibrium: $\ddot{I} = 0$

Assuming static system: $\vec{v} = 0$

$$K = \int_V \tfrac{3}{2} P\, d^3x + \underbrace{\int_V \rho v^2\, d^3x}_{=0} = \frac{3}{2}\int_V P\, d^3x = \frac{3}{2}\,\frac{4\pi}{3} R^3 P = 2\pi R^3 P$$

$$V = \int_V \rho(\vec{g}\cdot\vec{x})\, d^3x = \int_V \rho\cdot g\cdot\vec{e}_r\cdot r\cdot\vec{e}_r\cdot r^2 \sin\theta\, dr\, d\varphi\, d\theta =$$

$$= -\frac{4\pi}{3} G\rho^2 \cdot 4\pi\int_0^R r^4\, dr = -\frac{16\pi^2}{15} G\rho^2 R^5$$

$$S = \int_S P\vec{x}\cdot\vec{n}\, d^3x = 0$$ no pressure on surface

$$\Rightarrow \quad 2K + V = 0 = 4\pi R^3 P - \frac{16}{15}\pi^2 G\rho^2 R^5$$

$$\Rightarrow \quad P = \frac{4\pi}{15} G\rho^2 R^2$$

3) Chose an equation of state

E.g. isothermal EoS: $T = T_0$

$$PV = N k_B T_0$$

With $\frac{N}{V} = \frac{\rho}{m}$ $\left[\text{or } N = \frac{M}{m} \, ; \, N/V = \frac{M/V}{m} = \frac{\rho}{m} \right]$

$$P = \frac{\rho}{m} k_B T_0$$

$$= \frac{4\pi}{15} G \rho^2 R^2 \qquad \text{(from before)}$$

$$\Rightarrow \frac{k_B T_0}{m} = \frac{4\pi}{15} G \rho R^2 =$$

$$= \frac{4\pi}{15} G R^2 \cdot \frac{M}{\frac{4\pi R^3}{3}} = \underline{\underline{\frac{1}{5} G \frac{M}{R}}}$$

mass - radius relation

Note: Other EoS can be chosen. The isothermal one is the simplest. The point is that it needs a relation between the pressure P and the density, so the total mass can be calculated.