Waves, Instabilities and Shocks

Acoustic Waves

[Mihalas 169]

Acoustic waves are small-amplitude disturbances that propagate in a compressible medium (through the interplay between fluid inertia and the restoring force of pressure.)

Basic assumptions: Medium is homogeneous, isotropic and extends infinitely ($\rho(\vec{x}) = \rho_0 \; \forall \vec{x}$). No external forces act: $\vec{a}=0$.

Linearised ~~fluid~~ equations

Consider the Euler equations in 1D for the isothermal case, when $P = \rho a^2$: ($\Rightarrow$ no energy equation)

3D

[1] $\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$

[2] $\partial_t \vec{v} + \vec{v}\cdot(\vec{\nabla}\vec{v}) + \frac{1}{\rho}\vec{\nabla}P = 0$

1D

$\longrightarrow \quad \partial_t \rho + \partial_x(\rho v) = 0$

$\longrightarrow \quad \partial_t v + v\partial_x v + \frac{1}{\rho}\partial_x P = 0$

Now starting from an equilibrium situation ($\rho = \rho_0, \vec{v}=0$) we add small perturbances: $\rho = \rho_0 + \delta\rho$, $v = v_0 + \delta v = \delta v$

Where we define "small" as: $|\delta\rho| \ll \rho_0$, $|\delta v| \ll a$

(a = speed of sound)

Now insert the chosen values into the mass conservation equation [1] and the momentum conservation equation [2] and drop all second order terms ($\hat{=}$ linearisation)

Note: We assume that $\delta\rho$ and δv are smooth, such that $\partial_{t,x}\,\delta\rho, v \ll \rho, v$

i) Mass conservation equation:

$$\frac{\partial}{\partial t}(\delta\rho) + (\rho_0 + \delta\rho)\,\partial_x(\delta v) + \delta v\,\partial_x(\delta\rho) = 0$$

$$= \frac{\partial}{\partial t}(\delta\rho) + \rho_0\,\partial_x(\delta v) + \underbrace{\delta\rho\,\partial_x(\delta v) + \delta v\,\partial_x(\delta\rho)}_{\text{second order terms}} = 0$$

$$\rightarrow \boxed{\frac{\partial}{\partial t}(\delta\rho) + \rho_0\,\partial_x(\delta v) = 0}$$

ii) Momentum equation

$$\frac{\partial}{\partial t}(\delta v) + \underbrace{\delta v\frac{\partial}{\partial x}(\delta v)}_{\substack{\approx 0 \\ \text{second order}}} + \frac{1}{\rho_0+\delta\rho}\frac{\partial}{\partial x}\underbrace{(\rho_0+\delta\rho)a^2}_{P=\rho a^2} = 0$$

$$\rightarrow \frac{\partial}{\partial t}(\delta v) + \frac{1}{\rho_0+\delta\rho}\frac{\partial}{\partial x}(\delta\rho\, a^2) = 0$$

Approximate $\frac{1}{\rho_0+\delta\rho}$:

$$\frac{1}{\rho_0+\delta\rho} = \frac{1}{\rho_0}\frac{1}{1+\frac{\delta\rho}{\rho_0}} \quad \text{using Taylor/geometric series}$$

$$= \frac{1}{\rho_0}\sum_{n=0}^{\infty}\left(-\frac{\delta\rho}{\rho_0}\right)^n \approx \frac{1}{\rho_0}\left(1-\frac{\delta\rho}{\rho_0}\right) \quad \text{keeping only linear term}$$

$$\Rightarrow \partial_t(\delta v) + \underbrace{\frac{1}{\rho_0}\left(1-\frac{\delta\rho}{\rho_0}\right)\partial_x\,\delta\rho\, a^2}_{\text{another second order term}} \approx 0$$

$$\Rightarrow \boxed{\partial_t(\delta v) + \frac{a^2}{\rho_0}\partial_x(\delta\rho) = 0}$$

To obtain the wave equation, we derivate the previously obtained equations:

$$\partial_t(\delta\rho) + \rho_0 \partial_x(\delta v) = 0 \xrightarrow{\partial_t} \partial_t^2(\delta\rho) + \rho_0 \partial_t \partial_x(\delta v) = 0$$

$$\partial_t(\delta v) + \frac{a^2}{\rho_0}\partial_x(\delta\rho) = 0 \xrightarrow{\rho_0 \partial_x} \rho_0 \partial_t \partial_x(\delta v) + a^2 \partial_x(\delta\rho) = 0$$

Now subtract the lower line from the upper line and obtain:

$$\boxed{\partial_t^2(\delta\rho) - a^2\partial_x^2(\delta\rho) = 0}$$ wave equation

Now let's assume the perturbations propagate via planar waves. This ansatz means:

$$\delta\rho = \Delta\rho \exp[i(kx-\omega t)] \;; \quad \delta v = \Delta v \exp[i(kx-\omega t)]$$

Where $\Delta\rho$ and Δv describe amplitudes.

Inserting the ansatz into the wave equation, we get:

$$\partial_t(\delta\rho) = -i\omega\,\delta\rho \;, \quad \partial_t^2(\delta\rho) = -\omega^2\delta\rho$$

$$\partial_x(\delta\rho) = ik\,\delta\rho \;, \quad \partial_x^2(\delta\rho) = -k^2\delta\rho$$

$$\Rightarrow -\omega^2\delta\rho + a^2k^2\delta\rho = 0$$

If we want to have a wave, we must demand $\delta\rho \neq 0$, leaving us with the dispersion relation:

$$\boxed{\omega^2 = a^2k^2}$$

To establish a connection between the amplitudes Δv and $\Delta\rho$ of the plane wave ansatz, we insert the expressions for δv and $\delta\rho$ into the linearised momentum equation:

$$\partial_t(\delta v) + \frac{a^2}{\rho_0}\partial_x(\delta\rho) = 0$$

$$\delta v = \Delta v \exp[i(kx-\omega t)] \quad \rightarrow \quad \partial_t(\delta v) = -i\omega\,\delta v$$

$$\delta\rho = \Delta\rho \exp[i(kx-\omega t)] \quad \rightarrow \quad \partial_x(\delta\rho) = ik\,\delta\rho$$

$$\Rightarrow \quad -i\omega\,\delta v + \frac{a^2}{\rho_0} ik\,\delta\rho = 0$$

$$\Rightarrow \quad \cancel{\omega}\,\Delta v = \frac{a^2}{\rho_0}\frac{k}{\omega}\Delta\rho$$

Using the dispersion relation: $\omega^2 = a^2k^2 \quad \Rightarrow \frac{k}{\omega} = \pm\frac{1}{a}$

$$\Rightarrow \quad \Delta v = \pm\frac{a}{\rho_0}\Delta\rho$$

This shows the perturbation condition for the velocity:

With $\left|\frac{\Delta\rho}{\rho_0}\right| = \left|\frac{\delta\rho}{\rho_0}\right| = \left|\frac{\Delta v}{a}\right| = \left|\frac{\delta v}{a}\right|$ $\quad |\Delta v| = |\Delta v|\cdot|1| = |\Delta v|\cdot|e^{i(kx-\omega t)}| = |\delta v|$

And the demand $\quad \left|\frac{\delta\rho}{\rho_0}\right| \ll 1 \quad \Rightarrow \quad \left|\frac{\delta v}{a}\right| \ll 1$

$$\Rightarrow \quad |\delta\rho| \ll \rho_0 \quad \Rightarrow \quad |\delta v| \ll a$$

Thermal Instability

[Shu 106]

To discuss thermal instabilities, we first need to have a look at heating and cooling processes: A fluid in thermal equilibrium is a fluid where heating and cooling balance each other.

Cooling and heating enter into effect through the energy conservation equation (Euler):

$$\frac{\partial E}{\partial t} + \vec{\nabla}(\vec{v}(E+P)) = \rho \vec{g}\vec{r} + \left(\frac{\delta Q}{dt}\right)$$

more precisely with the source term $\frac{\delta Q}{dt}$ = heating - cooling

$\frac{\delta Q}{dt} = 0$ is the case when heating and cooling balance each other <u>or</u> there are only elastic collisions in the gas.

i) cooling function

Cooling of the gas takes place due to inelastic collisions: atoms/molecules are excited by a collision and the photon emitted during de-excitation gets "lost". (This requires an optically thin medium so the photon can escape.)

Let's simplify the situation by assuming only a gas made of hydrogen and electrons, where the number of hydrogen atoms and electrons is the same.

We can describe the cooling function as:

$\left.\frac{\delta Q}{dt}\right|_{cool}$ = # hydrogen · # e^- · probability they'll collide

The probability they'll collide must include the following two things:

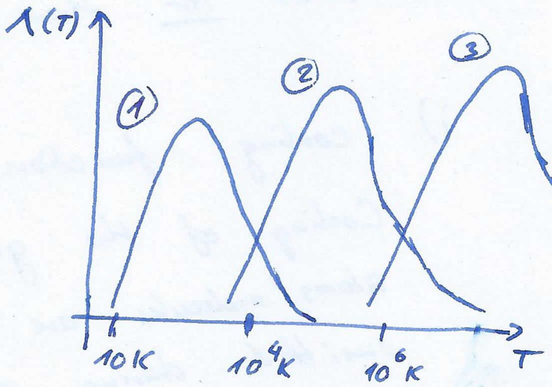
- It will depend on the distribution function, because it must depend on the velocities of the participants.
  The distribution function on the other hand is determined by the ambient temperature: $f = f(T)$
- The collisions must be able to excite a participant, meaning they must have enough energy to transfer. This will be an exponential cut-off for participants with energies lower than the excitation energy E_b.

We call this "probability function" the cooling function Λ.
As argumented above, we demand: $\Lambda = \Lambda(T)$.

This gives us:

$$\left.\frac{dQ}{dt}\right|_{cool} = n_H \cdot n_e \, \Lambda(T) = n_H^2 \Lambda(T)$$

There are 3 typical "peaks" of the cooling function at different temperatures, corresponding to different excitations:



① At $\sim 10-100$K, rotational levels of molecules and fine-structure levels of atoms begin to be excited by collisional impacts with excitation energies ~ 0.01eV

② At $\sim 10^4$ K, enough particles in the distribution have sufficient energies to collisionally excite the lower-lying electronic states of the common elements like hydrogen and Helium; The excitation energies are ~ 10eV

③ At $\sim 10^6$ K, the inner shells of elements like oxygen and iron can be excited with energies ~ 100 eV.

ii) heating function

The main heating process is due to photoionisation, where a photon is absorbed and an electron "freed".
Following the same argumentation, but correcting that in this case, the "collisions" are between atoms/molecules and photons, we can thus describe the heating sources as:

$$\frac{\delta Q}{dt}\Big|_{heat} = n_H \, n_\gamma \, \Gamma(T)_{n_\gamma} \quad \text{where } \Gamma(T) \text{ is the heating function.}$$

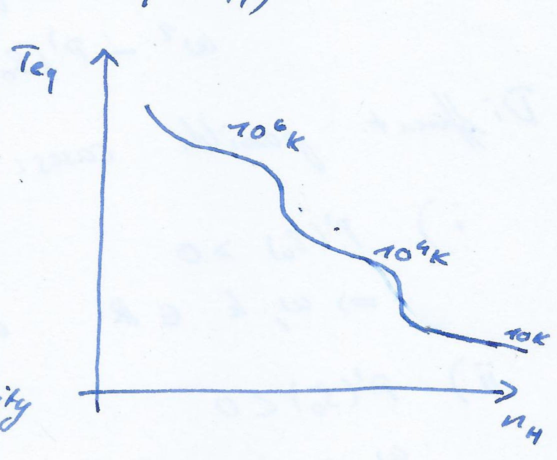
In the interstellar medium, we can use the fixed heating function $\Gamma_{ISRF.} = \Gamma(T, n_\gamma)$ (the photon density is already accounted for.)
[ISRF: InterStellar Radiation Field]

Returning to our source term, for an equilibrium state we demand:

$$\frac{\delta Q}{dt} = 0 = \frac{\delta Q}{dt}\Big|_{heat} - \frac{\delta Q}{dt}\Big|_{cool} \overset{ISM}{=} n_H \Gamma_{ISRF} - n_H^2 \Lambda(T)$$

$$\Rightarrow \quad \Lambda(T) = \frac{\Gamma_{ISRF}}{n_H} \qquad \Rightarrow \quad T_{eq} = T_{eq}(n_H)$$

To sum up: Using the nature of the heating and cooling processes in interstellar matter, we have shown that the equilibrium temperature T_{eq} only depends on the (number) density of the gas.



Corresponding to the "peaks" of the cooling function $\Lambda(T)$, 3 "plateaus" of T_{eq} will exist over a range of densities, because at those points, cooling effects are available to balance out any heating processes.

Now let's return to thermal instabilities.

Once again we linearise the Euler equations by adding small perturbances to the density and velocity; But this time, we can't assume an isothermal situation for obvious reasons. Instead, we will use a general equation of state: $P = P(\rho, T)$.

As demonstrated earlier, for an equilibrium state (which will be our starting point), $T = T_{eq} = T_{eq}(\rho) \Rightarrow P = P(\rho)$

$$\Rightarrow \frac{\partial P}{\partial x} = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial x} = P'(\rho)\frac{\partial \rho}{\partial x}$$

Repeating the same steps as for sound waves, we obtain:

$$\frac{\partial^2}{\partial t^2}(\delta \rho) = P'(\rho_0)\frac{\partial^2}{\partial x^2}(\delta \rho)$$

[With sound waves, we used the isothermal equation of state, $P = \rho a^2$
$\Rightarrow P' = \frac{\partial P}{\partial \rho} = a^2$]

Now use the planar wave ansatz: $\delta \rho = \Delta \rho \exp[i(kx - \omega t)]$, but this time, do not restrict k, ω to be real; instead: $k, \omega \in \mathbb{C}$.

Demanding that the amplitude $\Delta \rho \neq 0$ yields:

$$\omega^2 - P'(\rho_0)\, k^2 = 0$$

Different possible cases:

i) $P'(\rho_0) > 0$

$\Rightarrow \omega, k \in \mathbb{R}$ and $\omega = \pm\sqrt{P'}\, k$ with $\sqrt{P'} = c_s$ speed of sound

ii) $P'(\rho_0) < 0$

ω or k will be complex; choose $k \in \mathbb{R}$, $\omega = i\gamma$ with $\gamma \in \mathbb{R}$

then $\gamma = \sqrt{-P'}\, k \quad \Rightarrow \delta \rho = \Delta \rho\, e^{ikx} e^{\gamma t}$

In the case of $\gamma > 0$ [$k > 0$], this solution gives that $\delta \rho \propto e^{\gamma t}$: The perturbance $\delta \rho$ will grow exponentially, therefore the system will not be stable!

Jeans Instability

The Jeans instability considers the self-gravitation of a system. Unfortunately this poses a problem for our initial equilibrium state: There can't be a well-defined gravitational potential ϕ for a uniform, infinitely extended medium.

For a solution, we use the "Jeans swindle": Rewrite the poisson equations to $\Delta\phi = 4\pi G\rho \longrightarrow 4\pi G(\rho-\rho_0) = 4\pi G\,\delta\rho$

The resulting effect of this swindle is to ignore the zeroth-order equations and only consider only the first-order perturbation equations.

Not neglecting accelerations, the Euler equations in 1D are:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{\partial\phi}{\partial x} = g$$

Using the small perturbation ansatz: $\rho = \rho_0 + \delta\rho$, $v = \delta v$, $\phi = \delta\phi$ and the isothermal EoS $P = \rho a^2$, we arrive to the following equations:

$$\frac{\partial^2}{\partial x^2}(\delta\phi) = 4\pi G(\delta\rho) \qquad \text{Poisson equation}$$

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0\frac{\partial}{\partial x}(\delta v) = 0 \qquad \text{Momentum equation}$$

$$\frac{\partial}{\partial t}(\delta v) + \frac{a^2}{\rho_0}\frac{\partial}{\partial x}(\delta\rho) = -\frac{\partial}{\partial x}(\delta\phi) \qquad \text{Energy equation}$$

Now do $\frac{\partial}{\partial t}[\text{momentum eq}] - \rho_0\cdot\frac{\partial}{\partial x}[\text{energy equation}]$ to obtain:

$$\frac{\partial^2}{\partial t^2}(\delta\rho) + \rho_0\frac{\partial^2}{\partial x\partial t}(\delta v) - \rho_0\frac{\partial^2}{\partial t\partial x}(\delta v) - a^2\frac{\partial^2}{\partial x^2}(\delta\rho) - \frac{\partial^2}{\partial x^2}(\delta\phi)\cdot\rho_0 = 0$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial t^2}(\delta\rho) - a^2\frac{\partial^2}{\partial x^2}(\delta\rho) - 4\pi G\rho_0(\delta\rho) = 0}$$

Using the plane wave ansatz $\delta\rho = \Delta\rho \exp[i(kx - \omega t)]$ such that $\partial_x(\delta\rho) = ik(\delta\rho)$, $\partial_x^2(\delta\rho) = -k^2(\delta\rho)$; $\partial_t(\delta\rho) = -i\omega(\delta\rho)$, $\partial_t^2(\delta\rho) = -\omega^2(\delta\rho)$ and inserting it into the equation, we obtain the dispersion relation:

$$\boxed{\omega^2 = a^2 k^2 - 4\pi G \rho_0}$$

This dispersion relation allows us to define a critical wave length $\lambda_J = \frac{2\pi}{k_J}$ for which the wave doesn't propagate:

$$\omega^2 \equiv 0 = a^2 k_J^2 - 4\pi G \rho_0 \quad \Rightarrow k_J = \sqrt{\frac{4\pi G \rho_0}{a^2}}$$

Now let's differentiate between the cases for $\omega \neq 0$:

i) $k > k_J$

$\rightarrow \omega \in \mathbb{R}$. The waves propagate with $c_s < a$

For $k \gg k_J \rightarrow \frac{\omega}{k} \simeq a$

ii) $k < k_J$

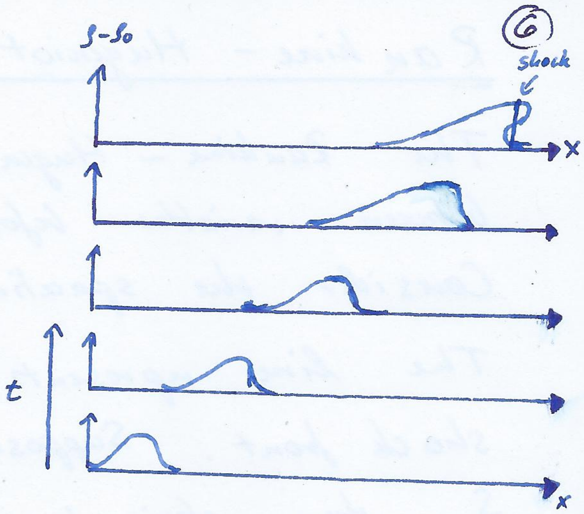
$\omega \notin \mathbb{R}$. Instead define $\omega \equiv i\gamma$ with $\gamma \in \mathbb{R}$ such that $\gamma = \pm\sqrt{4\pi G \rho_0 - a^2 k^2}$ and the perturbation equation reads:

$\delta\rho = \Delta\rho \exp[ikx] \exp[\gamma t]$

$\Rightarrow$ Depending on the sign of γ, the perturbation will be exponentially dampened or grown.

Valid for: Cosmology, Orbiting / Rotating stuff

Shock Formation

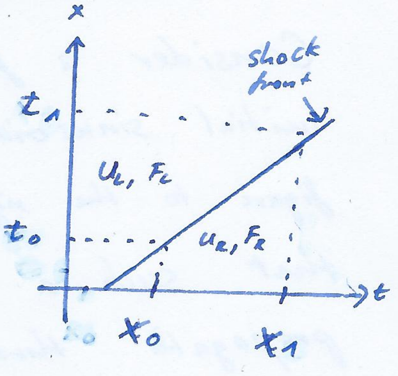


Consider a finite pulse having an initial sinusoidal shape as in the figure to the right below. We have seen that such a small perturbance will propagate through the medium as acoustic waves. But it can be shown that (Mihalas 228-229) the more compressed parts of the pulse will be hotter, have a higher sound speed and therefore moving to the right faster than the less compressed regions. Thus the crest of the pulse continuously gains on the pulse front and the wave front progressively steepens. The wave crest eventually overtakes the pulse front and the solution becomes multiple valued, thus unphysical. At this point, the front steepens into a shock, in which all variables change abruptly within a thin layer.

This indicates that from small ("accoustic") perturbances and therefore waves, a shock front will always form, provided that it has time to form (→ shock formation timescale must be smaller than the structure's lifetime timescale)

[Burger's equation left out]

Rankine - Hugeniot Relations



The Rankine - Hugeniot relations are relations between variables before and after a shock.
Consider the spacetime diagram to the right.
The line represents the propagation of the shock front. Suppose the shock propagates with the velocity S, then obviously its position at a point x_2 can be inferred by
$x_2 = x_1 + S \cdot (t_2 - t_1)$

Now consider a general (set of) conservation laws. We can describe a general conservation law as:

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x} \vec{F}(\vec{u}) = 0$$

where $\vec{u}$ and $\vec{F}$ contain/are conserved variables.
Suppose the shock moves from left to right; then the variables left of the shock (u_L, F_L) have already experienced the shock, and the variables to the right (u_R, F_L) are going to.

Now lets integrate the conservation law over time from t_0 to t_1 and over space from x_0 to x_1. Because it's a conservation law, we know that the result must be zero:

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x} \vec{F}(\vec{u}) = 0 \quad \Rightarrow \quad \int_{x_0}^{x_1}\int_{t_0}^{t_1} dx\, dt \left(\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x} \vec{F}(\vec{u}) \right) = 0$$

Now evaluate the integral:

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1} dx\, dt \left(\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x} \vec{F}(\vec{u}) \right) = \int_{x_0}^{x_1} dx \left[\vec{u}(t_1, x) - \vec{u}(t_0, x) \right] + \int_{t_0}^{t_1} dt \left[\vec{F}(t, x_1) - \vec{F}(t, x_0) \right]$$

Now furthermore assume that the phase-space volume that contains this part of the shock front is small, so that on each side of the shock (left = after, right = before shock), $u(t,x)$ is homogeneous over space and that F is homogeneous over time. Also assume that the shock is infinitely thin.

We can also identify: $u(t_1,x) = u_R$, $u(t_0,x) = u_L$

$F(t,x_1) = F_R$, $F(t,x_0) = F_L$

This allows to easily integrate the previous expression:

$$\int_{x_0}^{x_1} dx \left[\vec{u}(t_1,x) - \vec{u}(t_0,x) \right] + \int_{t_0}^{t_1} dt \left[\vec{F}(t,x_1) - \vec{F}(t,x_0) \right] = 0$$

$$\Rightarrow [\vec{u}_R - \vec{u}_L](x_1 - x_0) + [\vec{F}_R - \vec{F}_R]\cdot(t_1 - t_0) = 0$$

$$\Rightarrow \boxed{\vec{F}_R - \vec{F}_L = \frac{x_1 - x_0}{t_1 - t_0}(\vec{u}_R - \vec{u}_L) = S\cdot(\vec{u}_R - \vec{u}_L)}$$

Euler equations in $\vec{u}, \vec{F}$ - form:

i) isothermal

$\vec{u} = (\rho, \rho v)$ $\quad \vec{F} = (\rho v, \rho v^2 + P)$

with $P = \rho a^2$

ii) adiabatic

$\vec{u} = (\rho, \rho v, E)$ $\quad \vec{F} = (\rho v, \rho v^2 + P, v\cdot(E+P))$

1) Isothermal Shocks

The Rankine - Hugoniot relations give us:

$$\rho_R v_R - \rho_L v_L = S(\rho_R - \rho_L)$$
$$\rho_R v_R^2 + \rho_R a^2 - \rho_L v_L^2 - \rho_L a^2 = S(\rho_R v_R - \rho_L v_L)$$

Solve in the frame of the shock : $w \equiv v - S$

$$\rightarrow \quad \rho_R(v_R - S) - \rho_L(v_L - S) = \rho_R w_R - \rho_L w_L = 0$$
$$\Rightarrow \quad \boxed{\rho_R w_R = \rho_L w_L}$$

Momentum equation: use $v = w + S$

$$\rightarrow \rho_R (w_R + S)^2 + \rho_R a^2 - \rho_L (w_L + S)^2 - \rho_L a^2 = S(\rho_R (w_R + S) - \rho_L (w_L + S))$$
$$\rho_R [w_R^2 + 2 w_R S + S^2 + a^2] - \rho_L [w_L^2 + 2 w_L S + S^2 + a^2] = \rho_R [w_R S + S^2] - \rho_L [w_L S + S^2]$$
$$\rightarrow \rho_R (w_R^2 + w_R S + a^2) - \rho_L (w_L^2 + w_L S + a^2) = 0$$
$$\rightarrow \rho_R w_R^2 + \rho_R a^2 - \rho_L w_L^2 - \rho_L a^2 = S(\underbrace{\rho_L w_L - \rho_R w_L}_{= 0 \text{ : mass conservation equation}}) = 0$$

$$\Rightarrow \quad \boxed{\rho_R w_R^2 + \rho_R a^2 = \rho_L w_L^2 + \rho_L a^2}$$

Now define the compression ratio $r \equiv \frac{\rho_L}{\rho_R}$ and the Mach number $M \equiv \frac{w_R}{a}$, we can solve for a relation between the densities and velocities:

From mass cons. eq. : $w_L = \frac{1}{r} w_R$

$$\rightarrow \quad w_L^2 + a^2 = r\left(w_R^2 \frac{1}{r^2} + a^2\right) \quad \Big| : a$$
$$M^2 + 1 = r\left(\frac{M^2}{r^2} + 1\right)$$
$$\Rightarrow \quad r^2 + M^2 - r(M^2 + 1) = 0$$

Solve for r:

$$r = \begin{cases} 1 & \hat{=} \text{ a shock that does nothing} \\ M^2 \end{cases}$$

$r = M^2 \iff \rho_L = M^2 \rho_R$

$\Rightarrow w_L = \frac{1}{M^2} w_R$

For isothermal shocks, the shock's energy goes into compressing the medium.

2) Adiabatic Shocks

The adiabatic Euler equations are described by

$\vec{u} = (\rho, \rho v, E)$ and $\vec{F} = (\rho v, \rho v^2 + P, v(E+P))$

The Rankine–Hugoniot relations give us:

(1) $\rho_L w_L = \rho_R w_R$

(2) $\rho_L w_L^2 + P_L = \rho_R w_R^2 + P_R$

(3) $w_L (E_L + P_L) = w_R (E_R + P_R)$

Using the adiabatic equations $P = (\gamma - 1)\rho\varepsilon$ and $E = \rho\varepsilon + \frac{1}{2}\rho v^2$, we transform eq. (3):

$$w_L\left(\rho_L \varepsilon_L + \frac{1}{2}\rho_L w_L^2 + P_L\right) = w_R\left(\rho_R \varepsilon_R + \frac{1}{2}\rho_R w_R^2 + P_R\right)$$

with $w_L \rho_L = w_R \rho_R$:

$$\varepsilon_L + \frac{1}{2} w_L^2 + \frac{P_L}{\rho_L} = \varepsilon_R + \frac{1}{2} w_R^2 + \frac{P_R}{\rho_R}$$

Now with $P = (\gamma - 1)\rho\varepsilon \rightarrow \frac{P}{\rho} = (\gamma - 1)\varepsilon \rightarrow \varepsilon = \frac{1}{\gamma - 1}\frac{P}{\rho}$

$$\Rightarrow \varepsilon + \frac{P}{\rho} = \frac{1}{\gamma-1}\frac{P}{\rho} + \frac{P}{\rho} = \frac{1 + \gamma - 1}{\gamma - 1}\frac{P}{\rho} = \frac{\gamma}{\gamma-1}\frac{P}{\rho}$$

$$\Rightarrow \boxed{\frac{1}{2} w_L^2 + \frac{\gamma}{\gamma-1}\frac{P_L}{\rho_L} = \frac{1}{2} w_R^2 + \frac{\gamma}{\gamma-1}\frac{P_R}{\rho_R}}$$

We now must introduce simplifying assumptions:

i) Initial state is at rest: $v_R = 0 \Rightarrow w_R = -S$

ii) $\mathcal{M} = \frac{S}{c_s} \gg 1 \Rightarrow \gamma\frac{P}{\rho} = c_s^2 \ll S$

Astrophysical Blast Waves

[Shu 230]

Overview over mathematical methods:

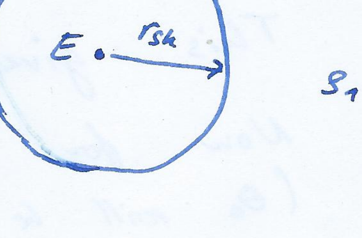
i) Dimensional analysis

If one nondimensionalises any set of equations and boundary conditions properly, one will obtain a minimum number of dimensionless parameters on which the solution of the problem depends. Different physical problems with the same parameters will then have the same solutions (except for a change in scale)

ii) Similarity analysis

A self-similar flow is the case when the same flow at any location and time looks the same as it did at a different location and an earlier time. I.e. "the flow keeps its shape".

Now consider a point release of an enormous amount of energy E into a static medium of homogeneous density ρ_1. ($\approx$ supernova.) In addition, assume that the amount of radiated away energy $\ll E$ and the ram pressure of the shock $\rho_1 v_{sh}^2 \gg P_1$ = outward equilibrium pressure of the medium.



This is called a blast wave.

Now let's try to express the shock front's position r via a dimensionless variable θ. (To demand self-similarity, we'll set the dimensionless variable to constant later.)

To nondimensionalise r, the available parameters would be ρ_1, P_1 and E. But we demanded that $P_1 \to 0$ for our cases, leaving us with ρ_1 and E.

We can not form a quantity of dimension L^{-1} only with ρ_1 $[\rho_1] = M L^{-3}$, and E $[E] = M L^2 T^{-2}$; we'll also need the time.

Now take the ansatz $\theta = r t^l \rho_1^m E^n$ and solve for l, m, n such that $[\theta] = 1$:

M: mass
T: Time
L: Length

$$1 = \underset{\substack{\uparrow \\ r}}{L} \cdot \underset{\substack{\uparrow \\ t}}{T^l} \cdot \underbrace{M^m L^{-3m}}_{\rho_1} \cdot \underbrace{M^n L^{2n} T^{-2n}}_{E}$$

$$= L^{1-3m+2n} M^{m+n} T^{l-2n}$$

This leads to the equations:

$$\underset{\text{for } T}{l = +2n}; \quad \underset{\text{for } M}{m = -n}; \quad \underset{\text{for } L}{2n + 3m + 1 = 0}$$

Solve them:

$$2n - 3m = -1 \overset{m=-n}{=} 2n + 3n = 5n = -1 \quad \Rightarrow n = -1/5$$

$$m = -n \quad \Rightarrow m = 1/5$$

$$l = 2n \quad \Rightarrow l = -2/5$$

This gives us: $\theta = r (\rho_1 E^{-1} t^{-2})^{1/5}$

Now for self similarity, demand $\theta = \theta_0$ (θ_0 will be of order of unity). This gives a solution for $r(t)$:

$$\boxed{r(t) \simeq \left(\frac{E}{\rho_1} t^2\right)^{1/5}}$$ Sodov solution

The velocity of the shock front will then be

$$\frac{dr}{dt} = \frac{2}{5}\left(\frac{E}{\rho_1}\right)^{1/5} t^{-3/5} = u_{sh}$$

Therefore, the shock will become slower over time.

For the derivation of this formula, we demanded that the radiative losses are negligible. But over time, the losses will accumulate: We can define the cooling time t_{cool} with $\int_0^{t_{cool}} \dot{E}_{cool}\, dt = \Delta E_{cool}$. We exit the Sedov regime (= it's not valid anymore) when $\Delta E_{cool} \approx E$. $\dot{E}_{cool}$ can be defined as the rate of energy loss by cooling:

$$\dot{E}_{cool} = \int_0^{R(t)} n_H^2 \Lambda(T)\, 4\pi r^2\, dr$$ (= cooling within the sphere of the blast wave).

After the Sedov regime follows the "snow plow regime": The shock front will become a very thin shell. At this point, energy conservation is not valid anymore; Instead, momentum is conserved.

Let Q_0 be the accumulated momentum at the point when we enter the snow plow regime:

$$Q_0 = \underbrace{\frac{4\pi}{3}\rho_1\, r^3(t_{cool})} \cdot \overbrace{\dot{r}(t_{cool})}^{=\vec{v}} = \text{Constant because conserved}$$

Accumulated mass: The density was ρ_1 in the beginning, so the total mass corresponds to sphere $\cdot\ \rho_1$

Following the same argument for the momentum, we can write:

$$\underbrace{\frac{4}{3}\pi r_{sp}^3 \varrho_1}_{m} \cdot \underbrace{\dot{r}_{sp}}_{v} = Q_0$$

$$\Rightarrow r_{sp}^3 \, dr_{sp} = \frac{Q_0}{\varrho_1} \frac{3}{4\pi} dt \approx \frac{1}{4} \frac{Q_0}{\varrho_1} dt$$

$$\Rightarrow \frac{1}{4} r_{sp}^4 = \frac{1}{4} \frac{Q_0}{\varrho_1} t$$

$$\Rightarrow \boxed{r_{sp}(t) = \left(\frac{Q_0}{\varrho_1}\right)^{1/4} t^{1/4}}$$

We see that the shock front will propagate $\propto t^{1/4}$, thus slower than in the Sedov regime (Sedov: $\propto t^{2/5}$)