Radiative Transfer

1) Radiative Transfer Equation

[Mihalas §33, Rybicky & Lightman 2]

Define the specific intensity for radiation:

$$dE = I_\nu(\vec{x}, n)\, dA\, d\Omega\, d\nu\, dt$$

with $[I_\nu] = \dfrac{erg}{cm^2\, rad^2\, s\, Hz}$ (Energy on surface per solid angle per second for a frequency)

Deriving the radiative transfer equation:

a) Using the energy - intensity relation for photons, show that $I_\nu = I_\nu(f)$

b) Then use the collisionless Boltzmann equations to obtain the radiative transfer equation

a) We know:

$$dE = I_\nu\, dA\, d\Omega\, d\nu\, dt$$

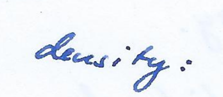
but also:

$$dE = h\nu\, dN \quad \cdot 2 \leftarrow 2 \text{ spins}$$

with dN = number density: $dN = f(\vec{x}, \vec{p})\, d^3x\, d^3p$

For photons:

$$d^3x = dA \cdot c\, dt$$



≙ photons travelling through

$$d^3p = p^2\, dp\, d\Omega \quad \text{in radial coordinates}$$

Using $p = \dfrac{h\nu}{c} \Rightarrow dp = \dfrac{h}{c}\, d\nu$

$$\Rightarrow d^3p = \frac{h^3}{c^3} \nu^2\, d\nu\, d\Omega$$

put it back together:

$$dN = f \cdot d^3x\, d^3p = f \cdot dA \cdot c\,dt \cdot \frac{h^3}{c^3} \nu^2 d\nu\, d\Omega$$

$$\Rightarrow dE = I_\nu\, d\Omega\, dA\, d\nu\, dt$$

$$= 2 \frac{h\nu}{c} dN f = 2 h\nu \cdot f \frac{h^3}{c^2} \nu^2 dA\, dt\, d\nu\, d\Omega = 2 \frac{h^4 \nu^3}{c^2} \cdot f \cdot d\Omega\, dA\, d\nu\, dt$$

$$\Rightarrow \boxed{I_\nu = 2 \cdot \frac{h^4}{c^2} \nu^3 \cdot f}$$ radiation specific intensity

Usually for bosons: $f = \mathcal{N} h^{-3} = \frac{h^{-3}}{1 - e^{-\varepsilon/k_B T}}$

(Blackbody radiation: $\varepsilon = h\nu$)

b) Boltzmann equation: (collisionless)

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{x}} + \vec{a} \frac{\partial f}{\partial \vec{v}} = 0$$

No accelerations on photons (ignore GR): $\vec{a} = 0$

Consider photons only: $\vec{v} = \vec{n} \cdot c$

Then using the result from before:

(Some factors cancel out, $\vec{v} = \vec{n} c$)

$$\boxed{\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \vec{n} \frac{\partial I_\nu}{\partial \vec{x}} = 0 = \frac{d I_\nu}{ds}}$$ radiative transfer equation in vacuum

where ds is a differential element along the way of the ray.

(This is valid for any curvilinear coordinate system: The lhs of the equation above however is derived in cartesian coordinates:

$$\frac{d}{ds} = \left(\frac{\partial t}{\partial s} \frac{\partial}{\partial t} + \frac{\partial x}{\partial s} \frac{\partial}{\partial x} + \frac{\partial y}{\partial s} \frac{\partial}{\partial y} + \frac{\partial z}{\partial s} \frac{\partial}{\partial y} \right) = \frac{1}{c} \frac{\partial}{\partial t} + \vec{n} \cdot \vec{\nabla}$$

)

Interaction between radiation and matter

When not in vacuum: Consider absorption and emission:

i) Emission: (neglect induced emission)

Assuming that emission happens spontaneously and isotropically, then the "additional intensity picked up along the path" can be described as:

$$dI_\nu = j_\nu \, ds$$

With j_ν: emission coefficient

ii) Absorption:

The energy taken out from the beam is proportional to the beam itself:

$$dI_\nu = -\alpha_\nu I_\nu \, ds$$

With α_ν: absorption coefficient

Put all together, we get the radiative transfer equation outside of a vacuum:

$$\boxed{\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu}$$

We can now define the optical depth τ_ν to be:

$\tau_\nu \equiv \alpha_\nu \, ds$. With $S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$ = source function, we get: ! isotropic !

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu \equiv S_\nu - I_\nu}$$

This form allows for a formal solution of the radiative transfer equation $\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu$

Using the method of variation of constants:

homogenise: $\frac{dI_\nu}{d\tau_\nu} = -I_\nu \quad \Rightarrow \quad I_\nu = K e^{-\tau_\nu} \overset{!}{=} K(\tau_\nu)\, e^{-\tau_\nu}$

then $\frac{dI_\nu}{d\tau_\nu} = -K(\tau_\nu)\, e^{-\tau_\nu} + \frac{dK}{d\tau_\nu} e^{-\tau_\nu}$

$= -K(\tau_\nu) e^{-\tau_\nu} + S_\nu$ (insert into original equation)

$\Rightarrow \frac{dK}{d\tau_\nu} = S_\nu e^{\tau_\nu} \quad \Rightarrow \quad K(\tau_\nu) = \int_0^{\tau_\nu} S_\nu e^{\tau} d\tau + \underbrace{I_\nu(0)}_{\text{integration constant}}$

$$\Rightarrow \boxed{I_\nu = I_\nu(0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{\tau - \tau_\nu} d\tau}$$

Rough estimates:

Approximate $S_\nu = (S_\nu)_0$ = constant and uniform everywhere.

Then: $I_\nu = I_\nu(0) e^{-\tau_\nu} + (S_\nu)_0 \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau)} d\tau$

$= I_\nu(0) e^{-\tau_\nu} + (S_\nu)_0 \left(1 - e^{-\tau_\nu}\right)$

Obviously the outcome depends heavily on the optical thickness τ_ν:

- $\tau_\nu \ll 1$: optically thin medium (thin slab)

  $I_\nu \simeq I_\nu(0)\,(1 - \tau_\nu) + (S_\nu)_0\, \tau_\nu$

  $\approx \underbrace{I_\nu(0)}_{\text{initial radiation}} + \underbrace{(S_\nu)_0\, \tau_\nu}_{\text{the whole emission}}$

- $\tau_\nu \gg 1$: optically thick medium

  $I_\nu \simeq (S_\nu)_0$

  The radiation is then isotropic because the source function is isotropic.

Moments of the specific intensity

$$J_\nu = \frac{1}{4\pi}\int_\Omega I_\nu \, d\Omega$$ mean intensity

$$F_\nu = \int_\Omega \vec{n}\, I_\nu \, d\Omega$$ radiation flux

$$\bar{\bar{P}}_\nu = \frac{1}{c}\int_\Omega \vec{n}\otimes\vec{n}\; I_\nu \, d\Omega$$ Radiation pressure tensor

with $P_{ij} = \int_{4\pi} n_i n_j \frac{I_\nu}{c}\, d\Omega$

Other definitions:

$$u_\nu = \frac{I_\nu}{c}$$ radiation energy density
$(I_\nu dA\, dt = u_\nu dV)$

$$n_\nu = \frac{u_\nu}{h\nu} = \frac{I_\nu}{h\nu c}$$ photon number density

$$E_\nu = \int_\Omega u_\nu \, d\Omega$$ total radiation energy density.

Moments of the radiative transfer equation

[Mihalas 337]

i) Zeroth order

Multiply rad. tr. eq. by $d\Omega$, integrate:

$$\underset{①}{\int_{4\pi} \frac{1}{c}\frac{\partial I_\nu}{\partial t}\, d\Omega} + \underset{②}{\int_{4\pi} \vec{n}\cdot\vec{\nabla} I_\nu\, d\Omega} = \underset{③}{\int_{4\pi} j_\nu\, d\Omega} - \underset{④}{\int_{4\pi} \alpha_\nu I_\nu\, d\Omega}$$

- If g isotropic, then $\int g\, f(\Omega)\, d\Omega = g \int f(\Omega)\, d\Omega$
- $\vec{x}$ and $\vec{v}$ are completely independant variables!

① $\int_{4\pi} 1/c\, \frac{\partial I_\nu}{\partial t}\, d\Omega = \frac{\partial}{\partial t}\int_{4\pi} \frac{I_\nu}{c}\, d\Omega = \frac{\partial}{\partial t}\int u_\nu\, d\Omega = \underline{\underline{\frac{\partial}{\partial t} E_\nu}}$

② Use product rule: $\vec{\nabla}(\vec{n}\cdot I_\nu) = \vec{n}\cdot\vec{\nabla} I_\nu + I_\nu \cdot \underbrace{\vec{\nabla}\cdot\vec{n}}_{=0}$

$=0$: $\vec{n}\cdot c = \vec{v}$, $\vec{\nabla}\cdot\vec{v} = 0$ by definition

$$\int_{4\pi} \vec{n}\cdot\vec{\nabla} I_\nu\, d\Omega = \int_{4\pi} \vec{\nabla}(\vec{n}\cdot I_\nu)\, d\Omega = \vec{\nabla}\int(\vec{n}\cdot I_\nu)\, d\Omega = \underline{\underline{\vec{\nabla}\vec{F}_\nu}}$$

③ $\int_{4\pi} j_\nu\, d\Omega = j_\nu \int d\Omega = 4\pi j_\nu$

j_ν is isotropic

④ $\int_{4\pi} \alpha_\nu I_\nu\, d\Omega = \alpha_\nu \int I_\nu\, d\Omega = \alpha_\nu \cdot c \cdot \int u_\nu\, d\Omega = \underline{\underline{\alpha_\nu c E_\nu}}$

$$\Rightarrow \boxed{\frac{\partial E_\nu}{\partial t} + \vec{\nabla}\vec{F}_\nu = 4\pi j_\nu - \alpha_\nu c E_\nu}$$

radiation energy conservation equation

ii) first order

Multiply rad. transf. eq. by $\vec{n}\, d\Omega$ and integrate:

$$\int_{4\pi} \vec{n} \frac{1}{c} \frac{\partial I_\nu}{\partial t} d\Omega + \int_{4\pi} \vec{n} \cdot (\vec{n} \cdot \vec{\nabla} I_\nu)\, d\Omega = \int_{4\pi} \vec{n}\, j_\nu\, d\Omega - \int \alpha_\nu \vec{n} I_\nu\, d\Omega$$

① ② ③ ④

① $\int_{4\pi} 1/c\, \vec{n} \frac{\partial I_\nu}{\partial t} d\Omega = \frac{1}{c} \frac{\partial}{\partial t} \int I_\nu \vec{n}\, d\Omega = \underline{\frac{1}{c} \frac{\partial}{\partial t} \vec{F}_\nu}$

② Show that $\vec{n} \cdot (\vec{n} \cdot \vec{\nabla} I_\nu) = \vec{\nabla} (\vec{n} \otimes \vec{n}) I_\nu$:

$$[\vec{\nabla} (\vec{n} \otimes \vec{n}) I_\nu]_j = \partial_i (n_j n_i I_\nu) = \underbrace{(\partial_j n_j)}_{=0} n_i I_\nu + n_j (\partial_j n_i I_\nu) =$$

$$= n_j [\underbrace{(\partial_j n_i)}_{=0} I_\nu + n_i \partial_j I_\nu] = n_j n_i \partial_j I_\nu \overset{i \leftrightarrow j}{=} n_j n_i \partial_i I_\nu =$$

$$= [\vec{n} \cdot (\vec{n} \cdot \vec{\nabla}) I_\nu]_j$$

$$\Rightarrow \int_{4\pi} \vec{n} (\vec{n} \cdot \vec{\nabla} I_\nu)\, d\Omega = \int \vec{\nabla} (\vec{n} \otimes \vec{n}) I_\nu\, d\Omega = \underline{c \vec{\nabla} \overline{\overline{P}}_\nu}$$

③ $\int_{4\pi} \vec{n}\, j_\nu\, d\Omega = j_\nu \int \vec{n}\, d\Omega = \underline{0}$

④ $\int_{4\pi} \alpha_\nu \vec{n} I_\nu\, d\Omega = \alpha_\nu \int_{4\pi} \vec{n} I_\nu\, d\Omega = \underline{\alpha_\nu \vec{F}_\nu}$

Put all together:

$$\boxed{\frac{1}{c} \frac{\partial \vec{F}_\nu}{\partial t} + c \vec{\nabla} \overline{\overline{P}}_\nu = - \alpha_\nu \vec{F}_\nu}$$

radiation momentum conservation equation

Closure relation

Unfortunately, for each moment of the radiative transfer equation a new variable will be introduced, such that the system of equations can never be fully solved, we will always have a variable too many.

To solve the equations, we need a closure relation, namely an (approximative) model for the pressure tensor $\bar{\bar{P}}_\nu$.

Usually used models are:

i) isotropic model:

$$\bar{\bar{P}}_\nu = P_\nu \bar{\bar{\mathbb{1}}} \simeq \frac{1}{3} E_\nu \bar{\bar{\mathbb{1}}}$$

ii) M1 closure

$$\bar{\bar{P}}_\nu = [\chi_\nu \bar{\bar{\mathbb{1}}} + (1-\chi_\nu) n \otimes n] E_\nu$$

Coupling to Hydrodynamics.

The exchange of momentum and energy between radiation and matter manifests in sink-source terms for the Euler equations. We describe the source (heating) terms as Γ_{rad} and the sink (cooling) terms as Λ_{rad}.

Furthermore, the radiative transfer equations have been set for a specific frequency ν. To obtain the full contribution, we must integrate the rad. transf. equations over all frequencies $d\nu$.

i) Momentum conservation:

Euler: $$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}(\rho(\vec{v}\otimes\vec{v}) + P\bar{\bar{1}}) = \rho\vec{g} + \Gamma_{rad} - \Lambda_{rad}$$

RT: $$\frac{1}{c^2}\frac{\partial F_\nu}{\partial t} + \vec{\nabla}\bar{\bar{P}}_\nu = \underbrace{-\frac{\alpha_\nu}{c}\vec{F}_\nu}$$

Momentum loss: The loss of momentum (sink) of the radiation will be the source for the gas!

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}(\rho(\vec{v}\otimes\vec{v}) + \bar{\bar{P}}) = \rho\vec{g} + \int_0^\infty \frac{1}{c}\alpha_\nu F_\nu \, d\nu}$$

ii) Energy conservation:

Euler: $$\frac{\partial E}{\partial t} + \vec{\nabla}(E + \bar{\bar{P}})\vec{v} = \rho\vec{g}\vec{v} + \Gamma_{rad} - \Lambda_{rad}$$

RT: $$\frac{\partial E_\nu}{\partial t} + \vec{\nabla}\cdot\vec{F}_\nu = 4\pi j_\nu - \alpha_\nu c E_\nu$$

$$\Rightarrow \boxed{\frac{\partial E}{\partial t} + \vec{\nabla}(E + \bar{\bar{P}})\vec{v} = \rho\vec{g}\vec{v} - \int_0^\infty 4\pi j_\nu \, d\nu + \int_0^\infty \alpha_\nu c E_\nu \, d\nu}$$

Notes:

- There is no mass conservation for photons.
- The gas never creates radiation flux, it always emits isotropically.
  $\Rightarrow$ Radiation always loses momentum (in a specific direction) over time.

Equillibria

[Rybicki; Lightman 1.5]

Thermal radiation is radiation emitted by matter in thermal equilibrium. Consider a box at temperature T where we do not let radiation in or out until a radiation equilibrium has been established inside the box.

Then I_ν must be:

i) only dependant on T

Proof: Stick a second box at same temperature next to the first box and put a filter between them that only lets photons of frequency ν pass: If $I_\nu \neq I_\nu'$, then energy will flow spontanously from one box to another, violating the second law of thermodynamics;

$$\Rightarrow \quad I_\nu \equiv B_\nu(T)$$

ii) isotropic

it does not depend on the shape of the box (same argument as before)

Kirchhoff's Law:

Material in LTE must radiate isotropically and only depending on the ambient temperature $\Rightarrow$ The source function must be black body radiation:

$$S_\nu = B_\nu(T) = \frac{j_\nu}{\alpha_\nu} \quad \Rightarrow \quad j_\nu = B_\nu(T)\,\alpha_\nu$$

Note: $I_\nu = B_\nu$ implies $\tau_\nu \to \infty$. In this case, radiation <u>and</u> matter are in LTE.

Consequences of Blackbody Radiation ($I_\nu \doteq B_\nu$)

$$\vec{F}_{\nu\,\mathrm{tot}} = \int_{4\pi} \vec{n}\, I_\nu \, d\Omega = \int_{4\pi} \vec{n}\, B_\nu(T)\, d\Omega = B_\nu \int \vec{n}\, d\Omega = \underline{\underline{0}}$$

$\rightarrow$ blackbody radiation produces no net flux

$$E_\nu = \int_{4\pi} \frac{I_\nu}{c} d\Omega = B_\nu(T) \int_{4\pi} \frac{1}{c} d\Omega = \underline{\underline{\frac{4\pi}{c} B_\nu(T)}}$$

In LTE, the pressure tensor is diagonal: $\overline{\overline{P}}_\nu = P_\nu \overline{\overline{\mathbb{1}}}$

So it suffices to compute only one component (it is also isotropic)

$$\overline{\overline{P}}_\nu = \int_{4\pi} \frac{I_\nu}{c} \vec{n} \otimes \vec{n}\, d\Omega = \frac{B_\nu(T)}{c} \int_{4\pi} \vec{n} \otimes \vec{n}\, d\Omega$$

Using diagonality and isotropy: $\vec{n} \otimes \vec{n} = n_i n_i$; compute only for n_z^2

Use $n_z = \cos\theta$ $\rightarrow$

$$\int_{4\pi} n_z^2\, d\Omega = \int \cos^2\theta \sin\theta\, d\theta\, d\varphi = 2\pi \int \cos^2\theta \sin\theta\, d\theta = \frac{4\pi}{3}$$

$$\Rightarrow \overline{\overline{P}}_\nu = \underline{\underline{\frac{4\pi}{3} \frac{B_\nu(T)}{c} \mathbb{1}}} \; ; \qquad P_\nu = \underline{\underline{\frac{1}{3} E_\nu}}$$

Stephan's Law

By the first law of thermodynamics:

$$dU = -p\, dV + \delta Q = -p\, dV + T\, dS$$

Now consider the closed box in equilibrium again. We can set $U = uV$ (u = specific energy), then $p = \frac{1}{3} u$. Since the energy density is a product of blackbody radiation, we can set $u = u(T)$.

Rearange the equation for dS and insert previous definitions:

$$dS = \frac{1}{T} dU + \frac{p}{T} dV = \frac{1}{T} d(uV) + \frac{u}{3T} dV =$$

$$= \frac{V}{T} du + \frac{u}{T} dV + \frac{u}{3T} dV \qquad \Big| \; du = \frac{du}{dT} dT$$

$$= \frac{V}{T}\frac{du}{dT} dT + \frac{4}{3}\frac{u}{T} dV$$

Now we will use that dS is a perfect differential, therefore

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}:$$

$$\frac{\partial S}{\partial T}\Big|_{\underset{dV=0}{V}} = \frac{V}{T}\frac{du}{dT} \qquad \rightarrow \qquad \frac{\partial^2 S}{\partial V \partial T} = \frac{1}{T}\frac{du}{dT}$$

$$\frac{\partial S}{\partial V}\Big|_T = \frac{4}{3}\frac{u}{T} \qquad \rightarrow \qquad \frac{\partial^2 S}{\partial T \partial V} = -\frac{4}{3}\frac{u}{T^2} + \frac{4}{3T}\frac{du}{dT}$$

$$\Rightarrow \frac{1}{T}\frac{du}{dT} = -\frac{4}{3}\frac{u}{T^2} + \frac{4}{3T}\frac{du}{dT}$$

$$\Rightarrow \frac{du}{dT} = 4\frac{u}{T} \qquad \Rightarrow \qquad \frac{du}{u} = 4\frac{dT}{T} \qquad \Rightarrow \boxed{u = aT^4}$$

u corresponds to the energy density $E_{rad} = \int_0^\infty E_\nu \, d\nu$

with $E_\nu = \frac{4\pi}{c} B_\nu$

$$\Rightarrow B(T) = \frac{ca}{4\pi} T^4 \qquad \text{with} \qquad B(T) = \int_0^\infty B_\nu \, d\nu$$

We can derive an expression for B_ν using the same steps as on page 1 with $f = \frac{-h^{-3}}{1 - e^{h\nu/k_B T}}$

$$I_\nu = 2\frac{h^4}{c^2} \nu^3 \cdot f = B_\nu = -\frac{2h\nu^3}{c^2} \frac{1}{1 - e^{h\nu/k_B T}}$$

Diffusion Limit

The diffusion limit describes the situation where the optical depth is large, photon mean free paths are small and photons diffuse through the material in a random walk. (e.g. interior of a star).

That means for a system of size L and the time scale of interest T:

$$cT \gg L$$

$$\lambda_\nu = \frac{1}{\alpha_\nu} \ll L \qquad \lambda_\nu: \text{ mean free path}$$

$$\tau_\nu \gg 1 \qquad \Rightarrow S_\nu \rightarrow B_\nu$$

Now consider the energy conservation equation:

$$\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + c\vec{\nabla}\bar{\bar{P}}_\nu = -\alpha_\nu \vec{F}_\nu$$

For a first order of magnitude estimate, we approximate: $\|P_\nu\| \sim E_\nu$, $\|F\| \simeq \varepsilon_\nu c E_\nu$; then replace differential quotients by difference quotients for the estimate: (ε_ν is some factor):

$$\frac{c\varepsilon_\nu E_\nu}{cT} + c\frac{E_\nu}{L} = -\frac{1}{\lambda_\nu}\cdot c\varepsilon_\nu E_\nu$$

With $\lambda_\nu \ll L \ll cT \Rightarrow \frac{1}{\lambda_\nu} \gg \frac{1}{L} \gg \frac{1}{cT}$ we can see two things:

i) The first term of the equation, $\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t}$, is negligible

ii) In order for the other two terms to balance, we need $\varepsilon_\nu \ll 1$

Our equation then is:

$$c\vec{\nabla}\bar{\bar{P}}_\nu \simeq -\alpha_\nu \vec{F}_\nu$$

By demanding $\tau \gg 1$, we are in the blackbody radiation case; Thus we may then use that $I_\nu \approx B_\nu$, $E_\nu \approx \frac{4\pi}{c} B(T)$ and $P_\nu \approx \frac{1}{3} E_\nu$

Then: $c \vec{\nabla} \bar{\bar{P}}_\nu \approx \frac{1}{3} c \vec{\nabla} E_\nu = -\alpha_\nu \vec{F}_\nu \Rightarrow \boxed{\vec{F}_\nu \approx -\frac{c}{3\alpha_\nu} \vec{\nabla} E_\nu}$

With an approximation for the flux F_ν, we can also approximate the energy conservation equation:

$$\frac{\partial E_\nu}{\partial t} + \vec{\nabla} \vec{F}_\nu = 4\pi j_\nu - \alpha_\nu c E_\nu \approx \frac{\partial E_\nu}{\partial t} + \vec{\nabla}\left(-\frac{c}{3\alpha_\nu} \vec{\nabla} E_\nu\right)$$

To find the temperature dependance of the radiation flux $\vec{F}_{rad}$, we calculate:

$$\vec{F}_{rad} = \int_0^\infty \vec{F}_\nu \, d\nu = \int_0^\infty -\frac{c}{3\alpha_\nu} \vec{\nabla} E_\nu \, d\nu$$

With $E_\nu = \frac{4\pi}{c} B_\nu(T) \longrightarrow \vec{\nabla} E_\nu = \frac{4\pi}{c} \frac{\partial B_\nu}{\partial T} \vec{\nabla} T$

$$\Rightarrow \vec{F}_{rad} = -\int_0^\infty \frac{c}{3\alpha_\nu} \cdot \frac{4\pi}{c} \frac{\partial B_\nu}{\partial T} \vec{\nabla} T \, d\nu = -\frac{4\pi}{3} \vec{\nabla} T \int \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T} d\nu$$

We define the Rosseland mean absorption coefficient α_R:

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty 1/\alpha_\nu \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

Furthermore, using that $\int \frac{4\pi}{c} \int B_\nu \, d\nu = \frac{ca}{4\pi} T^4$, we express

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu \, d\nu = \frac{\partial}{\partial T}\left(\frac{ca}{4\pi} T^4\right) = 4 \frac{aT^3}{\pi}$$

$$\Rightarrow \frac{1}{\alpha_R} = \left(\frac{caT^3}{\pi}\right)^{-1} \int_0^\infty 1/\alpha_\nu \frac{\partial B_\nu}{\partial T} d\nu$$

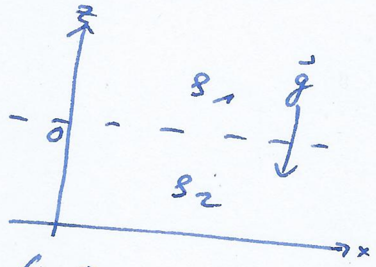
$$\Rightarrow \int_0^\infty 1/\alpha_\nu \frac{\partial B_\nu}{\partial T} d\nu = \frac{caT^3}{\pi \alpha_R}$$

Giving us the expression for the radiation flux:

$$\boxed{\vec{F}_{rad} = -\frac{4c}{3\alpha_R} a T^3 \vec{\nabla} T = -K_{rad} \vec{\nabla} T}$$

Rayleigh - Taylor Instability

Consider two fluids of different densities under the influence of an external gravity field $\vec{g}$.



Suppose the interface is at $z=0$.

Suppose the fluids are at equilibrium at first.

The hydrostatic equilibrium means:

$$\frac{1}{\varsigma}\frac{\partial P}{\partial z} = -g$$

Region 1) : $\frac{\partial P}{\partial z} = -\varsigma_1 g \Rightarrow P_1 = P_0 - \varsigma_1 g z$

2) : $\frac{\partial P}{\partial z} = -\varsigma_2 g \Rightarrow P_2 = P_0 - \varsigma_2 g z$

Interaction between radiation and matter

Considers mainly atomic radiative processes.
3 categories:

- Free-Free radiation: → Thomson scattering
  → Bremsstrahlung
- Bound-Free radiation → Ionisation
- Bound-Bound radiation (line radiation) → Excitation

1) Thomson Scattering

Free electron oscillates in response to an incident EM wave → it radiates because acceleration.
Assumptions: System non-relativistic: $v \ll c$, $h\nu \ll m_e c^2$
Electric field: Planar wave $\vec{E} = E_0\, e^{i(k\vec{x} - \omega t)}$
Ansatz: $\vec{x} = \vec{x}_0 \exp[i(kx - \omega t)]$
Equ of motion: $m\ddot{\vec{x}} = q\vec{E} \Rightarrow \ddot{\vec{x}} = \frac{q\vec{E}_0}{m}$
↳ + ED stuff ...
Thomson cross section: Easy derivation
Determine "electron radius": e^- and e^+ at rest start falling into each other. Closest they can get: Energy conservation:
$2m_e c^2 - \frac{e^2}{r_T} = 0 \Rightarrow r_T = \frac{e^2}{2m_e c^2}$. Factor 2 gone: radiated away. (→ Virial theorem).
Then use rigid sphere cross section: $\sigma_T \approx \pi d^2 = \pi(2r_T)^2 = 4\pi \frac{e^4}{m_e^2 c^4}$

Line Radiation (Bound - bound radiation)

= Transitions in a system remaining in a bound state.
mainly for hydrogen atoms.

Energy levels of excitation: Bohr model

Classical description: $K = V = \frac{1}{2}\frac{p^2}{m_e} = \frac{e^2}{r}$

Using Heisenberg uncertainty: $p\, r \sim h \quad \to \quad \frac{1}{2}\frac{p^2}{m_e} \approx \frac{1}{2}\frac{h^2}{m_e r^2}$

$$\to \quad \frac{1}{2}\frac{h^2}{m_e r^2} = \frac{e^2}{r} \quad \to \quad r_B = \frac{h^2}{2 m_e e^2} \sim \frac{h^2}{m_e e^2}$$

With the Bohr radius, we can compute the Bohr Energy:

$E_B = \frac{e^2}{r_B} \approx 27.2\ \mathrm{eV}$ $= 2 \cdot$ ground state energy of H

Obtain spectrum: Take degeneracy into account (factor 2)

$$E_n = -\frac{13.6\ \mathrm{eV}}{n^2}$$

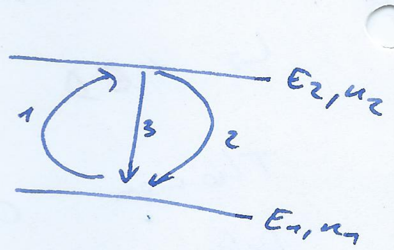
1) Radiationless system without demanding LTE

There are 3 possible processes:

(1) Collisional excitation (collisions with electrons)

(2) Collisional de-excitation

(3) spontaneous de-excitation



Set up rate equation:

$$\frac{dn_1}{dt} = \underbrace{-n_1 n_e \langle \sigma_{excite}\, v\rangle}_{\text{excitation}} + \underbrace{n_2 n_e \langle \sigma_{de\text{-}ex}\, v\rangle}_{\text{deexcitation}} + \underbrace{n_2 A_{21}}_{\text{spontaneous}}$$

Chemical equilibrium: $\hat{=}$ detailled balance for excitation processes. $\hat{=} \frac{dn_1}{dt} = 0$

$$\Rightarrow n_1 C_{12} = n_2 C_{21} + n_2 A_{21}$$

Which de-excitation process is dominant?
This depends on the electron density:

$$C_{21} = n_e \langle \sigma_{de-ex} v \rangle$$

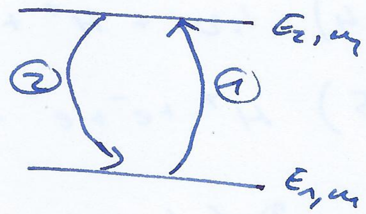
The critical density where $C_{21} = A_{21}$ is:

$$n_{e,crit} \langle \sigma_{de-ex} v \rangle = A_{21} \Rightarrow \underline{n_{e,crit} = \frac{A_{21}}{\langle \sigma_{de-ex} v \rangle}}$$

This is valid also for non-LTE cases; We obtain the LTE solutions for high n_e.

2) Including Radiation

There are two further processes when radiation is included:



(1) photo-excitation

(2) Stimulated emission: The presence of a photon with the right wavelength increases the transition probability for a desexcitation greatly.

This gives us a full rate equation:

$$\frac{dn_1}{dt} = -n_1 n_e \langle \sigma_{ex} v \rangle + n_2 n_e \langle \sigma_{desex} v \rangle + A_{21} n_2 \underbrace{- n_1 \int n_\nu \sigma_\nu^{ex} d\nu}_{\text{photo excitation}}$$

$$+ \underbrace{n_2 \int n_\nu \sigma_\nu^{stim} d\nu}_{\text{Stimulated emission}}$$

We can rewrite that equation using the following:

$$n_\nu = \frac{u_\nu}{h\nu} = \frac{1}{h\nu}\frac{1}{c}\int I_\nu \, d\Omega = \frac{4\pi}{ch\nu} J_\nu$$

$$\rightarrow \quad n_1 \int n_\nu \sigma_\nu^{ex} \, d\nu = n_1 \bar{J} B_{12}$$

$$n_2 \int n_\nu \sigma_\nu^{stim} \, d\nu = n_2 \bar{J} B_{21}$$

$$\Rightarrow \quad \frac{dn_1}{dt} = n_2 C_{21} + n_2 A_{21} - n_1 C_{12} + n_2 \bar{J} B_{21} - n_1 \bar{J} B_{12}$$

Bound-Free Radiation

We have to take into account the following processes:

1) $H^0 + e^- \rightarrow H^+ + e^- + e^-$ collisional ionisation
2) $H^+ + e^- \rightarrow H^0 + h\nu$ collisional recombination
3) $H^0 + h\nu \rightarrow H^+ + e^-$ photoionisation
4) $h\nu + H^+ + e^- \rightarrow H^0 + h\nu$ stimulated recombination
5) $H^+ + e^- + e^- \rightarrow H^0 + e^- + h\nu'$ dielectronic recombination

Dielectronic recombination: Capture of an electron by an ion under emission of a photon

Rate equation:

$$\frac{dn_{H^0}}{dt} = \underset{(1)}{-n_{H^0} n_e \alpha_{ion}} + \underset{(2)}{n_{H^+} n_e \beta_{rec}} - \underset{(3)}{n_{H^0} \int n_\nu \sigma_\nu^{ion} c \, d\nu} +$$

$$+ n_{H^+} n_e \int n_\nu \sigma_\nu^{rec} c \, d\nu + n_{H^+} n_e^2 \delta_{diel}$$