Collisionless Fluids

Collisionless fluids: Fluids for which the collisions are negligible because they are rare to take place in a time of interest.

When is a system collisionless?
→ Mean free path much larger than system size
= collision time much larger than system lifetime

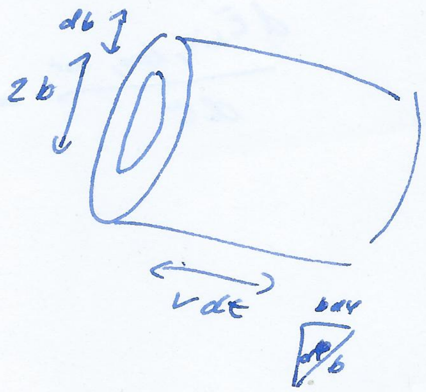
Relaxation time

= time it takes for a particle to feel the effects of its surroundings (≡ binary collisions) and lose energy.

If fluid collisionless, t_R > orbital time. Then two-body interactions can be ignored.

Consider a star moving with a velocity v. Calculate the loss of kinetic energy due to binary collisions.

The number of field stars in the cylindrical shell is:



$$dN = n \, 2\pi \, b \cdot db \cdot v \, dt$$

$$\Rightarrow \frac{dN}{dt} = 2\pi n b v \, db$$

Where n is the particle (star) density)

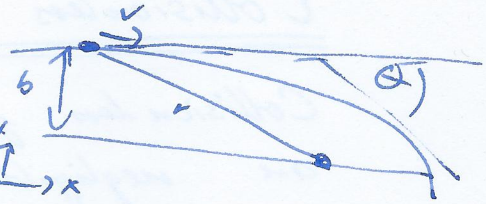
$\Rightarrow$ Mass flux $\dot{m} = m \frac{dN}{dt} = 2\pi m n b v \, db$

Describe loss of kinetic energy per collision as $E_{kin, loss} = \frac{1}{2} m (\Delta v)^2$

$$\Rightarrow \frac{dE_{kin, loss, tot}}{dt} = \frac{1}{2} \int (\Delta v)^2 \dot{m} = \int_0^\infty (\Delta v)^2 \pi b v m n \, db$$

Now to find an expression for ΔV:

Estimate ΔV per collision by applying the Born approximation: Assume the field star's trajectory (we're in the frame where the subject star is at rest) is almost not perturbed $\Rightarrow$ Assume velocity direction is unchanged.



Then the vertical force on the field star is:

$$F_\perp = -\frac{Gm^2}{r^2}\cdot\frac{r_\perp}{r} = -\frac{Gm^2}{\sqrt{v^2t^2+b^2}^3}\cdot b$$

With $F_\perp = \frac{dp}{dt} = m\frac{dv}{dt} \approx m\frac{\Delta v}{\Delta t}$, we get the total energy loss: $\Delta V = \int_{-\infty}^{\infty} \frac{F}{m}\,dt = -Gm\int \frac{1}{\frac{1}{b}\sqrt{v^2t^2+b^2}^3}\,dt =$

$$= -\frac{Gm}{b^2}\int\frac{1}{(v^2t^2/b^2+1)^{3/2}}\,dt = -\frac{Gm}{b^2}\left[\frac{t}{\sqrt{v^2t^2/b^2+1}}\cdot\frac{v^2}{b^2}\right]_{-\infty}^{\infty}$$

$$= -\frac{Gm}{b^2}\left[\frac{1\cdot\mathrm{sign}(t)}{\sqrt{v^2/b^2+1/t^2}}\right]_{-\infty}^{\infty} = \frac{-2Gm}{b^2}\,\frac{b}{v} = -\frac{2Gm}{bv}$$

So we have an expression for $\frac{dE_{kin,loss,tot}}{dt}$:

$$\frac{dE_{kin,loss,tot}}{dt} = \int_0^\infty \frac{4\pi G^2m^2}{b^2v^2}\cdot n\cdot m\cdot b\cdot v\,db = 4\pi\int\frac{G^2m^3}{bv}\,n\,db$$

Now integrating from 0 to ∞ is physical nonsense.
We perform the integration from b_{min} to b_{max}:

$$\frac{dE_{kin}}{dt} = \frac{4\pi G^2 m^3 n}{v} \ln(b_{max}/b_{min})$$

Let's find expressions for b_{max} and b_{min}:
b_{max} can be the cut-off where the influence is negligible, or even simpler, the system size R.
for b_{min}, we take the b_{90} parameter: The distance needed for 90° scattering. After long enough time, we can assume that all stars with $b < b_{90}$ have been scattered away, and thus don't contribute.

Then: $b_{min} = \frac{2Gm}{v^2}$ $\left(\frac{1}{2}mv^2 = \frac{Gm^2}{b_{90}}\right)$

$$\Rightarrow b_{max}/b_{min} = \frac{R}{\frac{2Gm}{v^2}}$$

From the virial theorem we estimate: $2 \cdot \frac{1}{2}Mv^2 \approx \frac{GM^2}{R}$

$$\Rightarrow v^2 \approx \frac{GM}{R}$$

$M = mN$

$$\Rightarrow \frac{b_{max}}{b_{min}} = \frac{R}{\frac{2Gm}{GM/R}} = \frac{M}{2m} = \frac{N}{2}$$

$$\Rightarrow \frac{dE_{kin}}{dt} = \frac{4\pi G^2 m^3}{v} n \ln(N/2)$$

We then define the relaxation time:

$$\frac{1}{t_{relax}} \equiv \frac{1}{E_{kin}} \frac{dE_{kin}}{dt} = \frac{1}{\frac{1}{2}mv^2} \frac{4\pi G^2 m^3}{v} n \ln(N/2) = \frac{8\pi G^2 m^2}{v^3} n \ln(N/2)$$

Furthermore, we define the orbital time $t_{orb} \equiv \frac{R}{v}$

Then $\frac{1}{t_{relax}} = \frac{8\pi G^2 m^2}{v^3} n \ln(N/2) = \frac{1}{t_{orb}} \frac{R}{v} \frac{8\pi G^2 m^2}{v^3} n \ln(N/2)$

$$= \frac{1}{t_{orb}} \frac{8\pi G^2 m^2 R}{v^4} n \ln(N/2) \qquad v^2 \approx \frac{MG}{R}$$

$$= \frac{1}{t_{orb}} \frac{8\pi G^2 m^2 R \cdot R^2}{M^2 G^2} n \ln(N/2)$$

$$= \frac{1}{t_{orb}} \frac{8\pi \, m^2 R^3}{M^2} n \ln(N/2) \qquad R^3 n \approx N, \; M/m = N$$

$$= \frac{1}{t_{orb}} \frac{8\pi N}{N} \ln(N/2)$$

$$\Rightarrow \quad t_{relax} \approx \frac{N \, t_{orb}}{8\pi \ln(N/2)}$$

Collision less Boltzmann Equation

To describe collision less fluids with gravity:

$$\left(\frac{Df}{Dt}\right)_{coll} = 0 \quad ; \quad \vec{a} = -\nabla\phi$$

$$\Rightarrow \frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial x} - \vec{\nabla}\phi\frac{\partial f}{\partial v} = 0$$

Jeans Theorem

1) Any steady state solution of the CBE depends on the phase-space coordinates only through orbital invariants on a static potential

2) Any function of the orbital invariants yields a steady state solution of the CBE.

→ Jeans Theorem provides ansatz for the solution of the CBE.

For any orbital invariant I, it holds:

$$\frac{dI}{dt} = 0 = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial I}{\partial v_i}\frac{\partial v_i}{\partial t} = \frac{\partial I}{\partial t} + \vec{v}\vec{\nabla}I - \nabla\phi\frac{\partial I}{\partial v}$$

For a stationary system: $\frac{\partial I}{\partial t} = 0$. If $f = f(I_i)$, CBE is always satisfied.

$\Rightarrow$ Ansatz $f(\vec{x}, \vec{v}) = f(I_1, I_2, \ldots, I_n)$

Solving for $f(E)$: Eddington Formulae.

Orbital invariants

Spherical system : Total energy, angular momentum are conserved

$\rightarrow \quad I_1 = E, \quad I_{2,3,4} = L_{x,y,z}$