Equations of Fluid Dynamics

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1. Euler Equations

A system of non-linear hyperbolic conservation laws that govern the dynamics of a compressible material, for which the effects of body forces, viscous stresses and heat flux are neglected.
Set of variables to describe the flow can be chosen:

- primitive variables: density ρ, pressure p and velocity $\underline{v}$
- conserved variables: density ρ, momentum $\rho\underline{v}$, total energy per unit mass E

Conservation laws:

$$\frac{\partial \underline{U}}{\partial t} + \frac{\partial \underline{F}(\underline{U})}{\partial x} + \frac{\partial \underline{G}(\underline{U})}{\partial y} + \frac{\partial \underline{H}(\underline{U})}{\partial z} = 0 \qquad [1]$$

with

$$\underline{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} \quad \underline{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{bmatrix} \quad \underline{G} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ v(E+p) \end{bmatrix} \quad \underline{H} = \begin{bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ w(E+p) \end{bmatrix}$$

or in short:

$$\underline{U} = \begin{pmatrix} \rho \\ \rho\underline{v} \\ E \end{pmatrix} ; \quad \underline{F} = \begin{pmatrix} \rho\underline{v} \\ \rho\underline{v}\otimes\underline{v} + p\delta_{ij} \\ \underline{v}(E+p) \end{pmatrix} \quad \text{with } \frac{\partial \underline{U}}{\partial t} + \underline{\nabla}\cdot\underline{F} = 0$$

$E = \rho\left(\frac{1}{2}\underline{V}^2 + e\right)$; $\underline{V} = (u, v, w)$

total energy per unit mass

$\frac{1}{2}\underline{V}^2$ = specific kinetic energy; e = specific internal energy

Any set of PDEs written in the form [1] is called a system of conservation laws in differential form.

However, these PDEs are insufficient to completely describe the physicall processes involved. There are more unknowns than equations, closure relations are required.

Equations of state:

thermally ideal gases: $T = \frac{p v}{R}$

calorically ideal gases: $e = \frac{p v}{\gamma - 1} = \frac{p}{\rho(\gamma - 1)}$

2. Viscous Stresses

To introduce viscous stresses, the momentum equation of the Euler equations. The stresses in a fluid, given by tensor $\underline{\underline{S}}$, are due to the effects of pressure and viscosity:

$$\underline{\underline{S}} = -p\,\underline{\underline{I}} + \underline{\underline{\Pi}}$$

$\underline{\underline{\Pi}}$: viscous stress tensor

Newtonian approximation: $\underline{\underline{\Pi}}$ is related to the derivatives of the velocity field $\underline{v}$ via the deformation tensor

$$\underline{\underline{D}} = \begin{bmatrix} u_x & \frac{1}{2}(v_x + u_y) & \frac{1}{2}(w_x + u_z) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(w_y + v_z) \\ \frac{1}{2}(u_z + w_x) & \frac{1}{2}(v_z + w_y) & w_z \end{bmatrix}$$

with $u_x = \frac{\partial u}{\partial x}$

or: $D_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$

Such that

$$\underline{\underline{\Pi}} = 2\eta \underline{\underline{D}} + (\eta_b - \tfrac{2}{3}\eta)(\operatorname{div} \underline{v}) \underline{\underline{I}}$$

with η: coefficient of shear viscosity
η_b: coefficient of bulk viscosity, $=0$ for monoatomic gases

Sutherland formula: $\eta = C_1 \left[1 + \frac{C_2}{T}\right]^{-1} \sqrt{T}$

The Navier-Stokes equations, modified Euler equations, can be written in differential conservational form:

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v} + p \underline{\underline{I}} - \underline{\underline{\Pi}}) = \underline{0}$$

3. Heat Conduction

Influx of energy contributes to the rate of change of total energy E. Let $\underline{Q} = (q_1, q_2, q_3)$ be the energy flux vector, which results from

i) heat flow due to temperature gradients
ii) diffusion processes in gas mixtures
iii) radiation

Consider only heat flow due to temperature gradients:

$$\underline{Q} = -\kappa \underline{\nabla} T, \quad \kappa: \text{thermal conductivity}$$

Similarity to viscosity: Prandtl number

$$P_r \equiv \frac{c_p \eta}{k} = \frac{4\gamma}{9\gamma - 5} = \text{constant}^*$$

The energy equation is then:

$$\frac{\partial E}{\partial t} + \underline{\nabla} \cdot (\underline{v}(E+p)) = -\underline{Q}$$

* assuming that specific heat at constant pressure c_p is constant

4. Integral Form of Equations

Problem: Differential forms assume smoothness, thus don't allow discontinuous solutions like shocks.

Define substantial/material/Lagrangian/comoving derivative:

scalar $\varphi(\underline{x}, t)$: $\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \underline{V}\, \underline{\nabla} \phi$

vector $\underline{V}(\underline{x}, t)$: $\frac{D\underline{V}_i}{Dt} = \frac{\partial \underline{V}_i}{\partial t} + \underline{V} \cdot (\underline{\nabla}\, \underline{V}_i)$ componentwise

$$\frac{D\underline{V}_i}{Dt} = \frac{\partial \underline{V}_i}{\partial t} + \underline{V}_j \frac{\partial}{\partial x_j} \underline{V}_i$$

To derive the integral form of the equations, we will need the Reynold's transport theorem:

Let $I = \int_{V(t)} \alpha \, d^3x$ be the integral form of an equation, where α is any scalar, vector or tensor field. Then:

$$\frac{dI}{dt} = \frac{d}{dt}\left[\int_{V(t)} \alpha \, d^3x\right] = \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla}(\alpha \cdot \underline{V})\right] d^3x$$

Proof: Let $V(t)$ be some finite volume with $dV(t=0) = dV_0$. Furthermore, let $\Psi(\underline{x})$:

$$\Psi(\underline{x}) = \begin{cases} 1 & \underline{x} \in V(t) \\ 0 & \text{else} \end{cases}$$

such that $I = \int_{V(t)} \alpha \, d^3x = \int_{\mathbb{R}^3} \alpha \Psi \, d^3x$ to move the time dependance of the volume into the integral.

By definition: $\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\partial t} + \underline{v}\,\underline{\nabla}\Psi = 0$ because Ψ is comoving.

Then: $\frac{dI}{dt} = \frac{d}{dt}\left[\int_{V(t)} \alpha\, d^3x\right] = \frac{d}{dt}\left[\int_{\mathbb{R}^3} \alpha \Psi\, d^3x\right] = \int_{\mathbb{R}^3} \frac{d}{dt}[\alpha \Psi]\, d^3x$

$$= \int_{\mathbb{R}^3} \left[\frac{d\alpha}{dt}\Psi + \alpha \frac{d\Psi}{dt}\right] d^3x$$

Using $\frac{D\Psi}{Dt} = 0 \Rightarrow \frac{\partial \Psi}{\partial t} = -\underline{v}\cdot\underline{\nabla}\Psi$

$$\Rightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3} \left[\frac{\partial\alpha}{\partial t}\Psi - \alpha\underline{v}\cdot\underline{\nabla}\Psi\right] d^3x$$

Using $\underline{\nabla}\cdot(\Psi\alpha\underline{v}) = \Psi\underline{\nabla}\cdot(\alpha\underline{v}) + \alpha\underline{v}\cdot\underline{\nabla}\Psi$

$\Rightarrow -\alpha\underline{v}\cdot\underline{\nabla}\Psi = \Psi\underline{\nabla}(\alpha\underline{v}) - \underline{\nabla}(\Psi\alpha\underline{v})$

$$\Rightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3} \left[\frac{\partial\alpha}{\partial t}\Psi + \Psi\underline{\nabla}(\alpha\underline{v}) - \underbrace{\underline{\nabla}(\Psi\alpha\underline{v})}\right] d^3x$$

$\int_{\mathbb{R}^3} \underline{\nabla}(\Psi\alpha\underline{v})\, d^3x = \int_{S(\mathbb{R}^3)} \alpha\Psi\underline{v}\,\underline{n}\, dA = 0$

Surface of $\mathbb{R}^3$ is clearly outside $V(t)$, so $\Psi = 0$ always

$$= \int_{\mathbb{R}^3} \Psi\left[\frac{\partial\alpha}{\partial t} + \underline{\nabla}\cdot(\alpha\underline{v})\right] d^3x$$

$$= \int_{V(t)} \left[\frac{\partial\alpha}{\partial t} + \underline{\nabla}\cdot(\alpha\cdot\underline{v})\right] d^3x$$

$$= \int_{V(t)} \frac{\partial\alpha}{\partial t}\, d^3x + \underbrace{\int_{S(V(t))} \alpha\cdot\underline{v}\cdot\underline{n}\, d^2x}_{\text{using Gauss' theorem}}$$

Using the same method:

$$\frac{DI}{Dt} = \frac{D}{Dt}\left[\int_V \alpha\, d^3x\right] = \frac{D}{Dt}\left[\int_{\mathbb{R}^3} \alpha\Psi\, d^3x\right] = \int_{\mathbb{R}^3} \frac{D}{Dt}(\alpha\Psi)\, d^3x$$

$$= \int_{\mathbb{R}^3} \left[\Psi\frac{D\alpha}{Dt} + \alpha\underbrace{\frac{D\Psi}{Dt}}_{=0}\right] d^3x = \int_V \frac{D\alpha}{Dt}\, d^3x$$

$$= \int_V \left(\frac{\partial\alpha}{\partial t} + \underline{\nabla}(\underline{v}\cdot\alpha)\right) d^3x = \int_V \frac{\partial\alpha}{\partial t}\, d^3x + \int_{S(V)} \alpha\underline{v}\cdot\underline{n}\, d^2x$$

4.1 Mass Conservation

Let $\alpha = \rho$, then $I(t)$ is the total mass in the volume V: $I = \int_V \rho \, dV = m$. Assuming no mass is generated or annihilated within V and recalling that the volume/volume's area/surface is moving along with the fluid, we have $\frac{DI}{Dt} = 0$

$$\Rightarrow \frac{DI}{Dt} = 0 = \int_V \frac{\partial \rho}{\partial t} dV + \int_A \underline{n} \cdot (\rho \underline{v}) dA$$

If V is now a fixed ^comoving^ control volume: $\int_V \frac{\partial \rho}{\partial t} dV = \frac{\partial}{\partial t} \int_V \rho \, dV$

$$\Rightarrow \frac{\partial}{\partial t} \int_V \rho \, dV = - \int_A \underline{n} \cdot (\rho \underline{v}) dA$$

4.2 Momentum Conservation

A control volume V with bounding surface A is chosen and the total momentum in V is given by

$$I(t) = \int_V \rho \underline{v} \, dV$$

Newton's law: The rate of change of the momentum in V is equal to the total force acting on V. The total force is divided into surface forces f_s and volume forces f_v:

$$f_s = \int_A \underline{S} \, dA \, , \quad f_v = \int_V \rho g \, dV$$

$\underline{S}$: stress vector, given in terms of stress tensor: $\underline{S} = \underline{\underline{S}} \cdot \underline{n}$

g : specific volume-force vector (inertial, gravitational, EM forces...

This gives: $\frac{DI}{Dt} = f_s + f_v$

with $\alpha = \rho \underline{v}$: $\int_V \frac{\partial}{\partial t}(\rho \underline{v})\, dV = -\int_A \underline{v}\cdot(\underline{n}\cdot \rho \underline{v})\, dA + f_s + f_v$

4.3 Conservation of Energy

The rate of change of total energy $I = \int_V E\, dV$ is equal to the work done, per unit time, by all the forces acting on the volume plus the influx of energy per unit time into the volume.
A force $\underline{f}$ acting on a point moving with velocity $\underline{v}$ produces the work $\underline{f}\cdot\underline{v}$ per unit time.
The surface and volume forces produce the following terms:

$$E_{surf} = -\int_A \underline{v}\cdot\underline{S}\, dA, \qquad E_{vol} = \int_V \rho(\underline{v}\cdot\underline{g})\, dV$$

Influx of energy: $E_{influ} = -\int_A (\underline{n}\cdot\underline{Q})\, dA$
Recall that the energy flow is given by the heat flux.

$$\Rightarrow \quad \frac{DI(t)}{Dt} = E_{surf} + E_{vol} + E_{influ}$$

$$\Rightarrow \quad \frac{d}{dt}\int_V E\, dV = -\int_A [\underline{S}\cdot\underline{v} + \underline{n}\cdot\underline{Q} - (\underline{n}\cdot E\underline{v})\, dA$$

$$+ \int_V \rho(\underline{v}\cdot\underline{g})\, dV$$