Methods for Multidimensional PDEs

We now have a look at numerical methods for multidimensional hyperbolic conservation laws. For Cartesian geometries, we have

$$\underline{U}_t + \underline{F}(\underline{U})_x + \underline{G}(\underline{U})_y + \underline{H}(\underline{U})_z = 0$$

$\underline{U}$ is the vector of conserved variables and $\underline{F}$, $\underline{G}$, and $\underline{H}$ are fluxes in the x, y, z direction, respectively.

We will have a look at two ways of solving this type of PDEs:

1) dimensional splitting or method of fractional steps
   Here one applies one-dimensional methods in each coordinate direction

2) finite volume method, where all the intercell fluxes are involved in updating the solution in a single step.

1. Dimensional Splitting

1.1. Splitting Schemes for 2D Systems

For non-linear systems, dimensional splitting is not exact, but one may construct approximate splitting schemes.

Consider the 2D IVP

$$\begin{cases} \underline{U}_t + \underline{F}(\underline{U})_x + \underline{G}(\underline{U})_y = 0 \\ \underline{U}(x, y, t^n) = \underline{U}^n \end{cases}$$

The initial data at a time t^n is given by the set $\underline{U}^n$ of discrete cell average values $\underline{U}^n_{i,j}$.

The dimensional splitting approach replaces the IVP by a pair of one-dimensional IVPs. The simplest version would be

$$\left.\begin{array}{ll} \text{PDE:} & \underline{U}_t + \underline{F}(\underline{U})_x = \underline{0} \\ \text{IC:} & \underline{U}^n \end{array}\right\} \xrightarrow{\Delta t} \underline{U}^{n+1/2}$$

and

$$\left.\begin{array}{ll} \text{PDE:} & \underline{U}_t + \underline{G}(\underline{U})_y = 0 \\ \text{IC:} & \underline{U}^{n+1/2} \end{array}\right\} \xrightarrow{\Delta t} \underline{U}^{n+1}$$

In the first step one solves a one-dimensional problem in the x-direction for a time step Δt ("x-sweep") for all cells. The solution is denoted $\underline{U}^{n+1/2}$.

In the next step one solves the one-dimensional problem in the y-direction ("y-sweep"), where one takes the solution $\underline{U}^{n+1/2}$ of the x-sweep as initial conditions.

If $\mathcal{X}^{(t)}$ and $\mathcal{Y}^{(t)}$ are approximate solution operators for IVPs of the x- and y-sweeps, then the splitting can be written as

$$\underline{U}^{n+1} = \mathcal{Y}^{(\Delta t)} \mathcal{X}^{(\Delta t)} (\underline{U}^{n})$$

$$= \mathcal{X}^{(\Delta t)} \mathcal{Y}^{(\Delta t)} (\underline{U}^{n})$$

These splitting schemes are first-order in time provided the operators $\mathcal{X}$ and $\mathcal{Y}$ are at least first order in time.

Let us demonstrate that on the simple case of linear advection. More complicated cases require more math, but follow the same principles.

We write the advection equation:

$$\underline{u}_t + u\underline{u}_x + v\underline{u}_y = 0$$

as

$$\underline{u}_t = (\mathcal{X} + \mathcal{Y})\underline{u}$$

i.e. the operators are $\mathcal{X} = -u\frac{\partial}{\partial x}$, $\mathcal{Y} = -v\frac{\partial}{\partial y}$

Assuming the operators don't depend on time, we see that

$$\underline{u}_{tt} = (\underline{u}_t)_t = [(\mathcal{X} + \mathcal{Y})\underline{u}]_t$$

$$= (\mathcal{X} + \mathcal{Y})\underline{u}_t = (\mathcal{X} + \mathcal{Y})^2\underline{u}$$

and in general

$$\partial_t^k \underline{u} = (\mathcal{X} + \mathcal{Y})^k \underline{u}$$

Suppose now we are at $t=0$ and want to compute the error introduced by the splitting over one time step Δt.

Let $\underline{u}(x, \Delta t)$ be the exact solution and $\underline{u}^{n+1}$ the solution computed through dimensional splitting. The One-Step-Error OSE is then

$$OSE \equiv \underline{u}(x, \Delta t) - \underline{u}^{n+1}$$

To obtain the accuracy of the method, we Taylor-expand.

First the exact solution:

$$\underline{u}(x, \Delta t) = \sum_{j=0}^{\infty} \frac{\partial_t^j \underline{u}(x,0)}{j!} \cdot \Delta t^j$$

$$= e^{\Delta t \, \partial_t} \underline{u}(x,0) = e^{\Delta t (\mathcal{X} + \mathcal{Y})} \underline{u}(x,0)$$

Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, and $\partial_t \underline{u} = (\mathcal{X} + \mathcal{Y}) \underline{u}$.

Using dimensional splitting, we first compute

$$\underline{u}^{n+1/2} = e^{\Delta t \mathcal{X}} \underline{u}(x,0)$$

as the mid-step, when the y-operator is ignored, and then the full update

$$\underline{u}^{n+1} = e^{\Delta t \mathcal{Y}} \underline{u}^{n+1/2} = e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}} \underline{u}(x,0)$$

Now we can compute the OSE:

$$OSE = \underline{u}(x, \Delta t) - \underline{u}^{n+1}$$
$$= \left[e^{\Delta t(\mathcal{X}+\mathcal{Y})} - e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}}\right] \underline{u}(x,0)$$

We apply the Taylor-expansion again:

$$\underline{u}(x, \Delta t) = \left[1 + \Delta t(\mathcal{X}+\mathcal{Y}) + \frac{\Delta t^2}{2}(\mathcal{X}+\mathcal{Y})^2 + \mathcal{O}(\Delta t^3)\right] \underline{u}(x,$$

$$\underline{u}^{n+1} = \Big[\left(1 + \Delta t \mathcal{Y} + \frac{\Delta t^2}{2}\mathcal{Y}^2 + \mathcal{O}(\Delta t^3)\right)\left(1 + \Delta t \mathcal{X} + \frac{\Delta t^2}{2}\mathcal{X}^2 + \right.$$
$$\left. + \mathcal{O}(\Delta t^3)\Big] \underline{u}(x,0)\right.$$

$$= \left[(1 + \Delta t \mathcal{X} + \frac{\Delta t^2}{2}\mathcal{X}^2 + \Delta t \mathcal{Y} + \Delta t^2 \mathcal{Y}\mathcal{X} + \frac{\Delta t^2}{2}\mathcal{Y}^2 + \mathcal{O}(\Delta t^3)\right] \underline{u}$$

$$= \left[1 + \Delta t(\mathcal{X}+\mathcal{Y}) + \frac{\Delta t^2}{2}(\mathcal{X}^2 + 2\mathcal{Y}\mathcal{X} + \mathcal{Y}^2) + \mathcal{O}(\Delta t^3)\right] \underline{u}(x$$

This gives us

$$OSE = \left[e^{\Delta t(\mathcal{X}+\mathcal{Y})} - e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}} \right] \underline{u}(x,0)$$

$$= \left[\frac{\Delta t^2}{2}(\mathcal{X}+\mathcal{Y})^2 - \frac{\Delta t^2}{2}(\mathcal{X}^2 + 2\mathcal{Y}\mathcal{X} + \mathcal{Y}^2) + \mathcal{O}(\Delta t^3) \right] \underline{u}$$

$$= \frac{\Delta t^2}{2} \left[\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X} - \mathcal{X}^2 - 2\mathcal{Y}\mathcal{X} - \mathcal{Y}^2 \right] \underline{u} + \mathcal{O}(\Delta t^3)$$

$$= \frac{\Delta t^2}{2} \left[\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X} \right] \underline{u}(x,0) + \mathcal{O}(\Delta t^3)$$

$$= \mathcal{O}(\Delta t^2)$$

Note that so far, we assumed the operators $\mathcal{X}$ and $\mathcal{Y}$ to be exact.

So the splitting error depends on the commutator $\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$ of the two operators. If they commute, the splitting is at least $\mathcal{O}(\Delta t^3)$ (actually all the other terms vanish as well, so it is an exact method, but again: Assuming $\mathcal{X}$ and $\mathcal{Y}$ are exact operators and don't depend on t.)

So now what if our discrete scheme is not exact?

Firstly, we are concerned about the accuracy w.r.t. time in this exercise, so we can leave the operators $\mathcal{X}$ and $\mathcal{Y}$ as such.

Secondly, suppose our time discretization scheme is only first order accurate. Then all we need to do is truncate the Taylor expansion at first order, and we obtain

with $\underline{U}(x,\Delta t) = \left[1 + \Delta t(\mathcal{X}+\mathcal{Y}) + \frac{\Delta t^2}{2}(\mathcal{X}+\mathcal{Y})^2 + \mathcal{O}(\Delta t^3)\right]\underline{U}($

and $\underline{U}^{n+1} = \left[(1+\Delta t\,\mathcal{Y} + \mathcal{O}(\Delta t^2))(1+\Delta t\,\mathcal{X} + \mathcal{O}(\Delta t^2))\right]\underline{U}(x,0)$

$$= \left[1 + \Delta t\,\mathcal{X} + \Delta t\,\mathcal{Y} + \mathcal{O}(\Delta t^2)\right]\underline{U}(x,0)$$

Then

$$OSE = \underline{U}(x,\Delta t) - \underline{U}^{n+1}$$

$$= \left[1 + \Delta t(\mathcal{X}+\mathcal{Y}) + \mathcal{O}(\Delta t^2) - 1 - \Delta t(\mathcal{X}+\mathcal{Y})\right]\underline{U}(x,0)$$

$$= \mathcal{O}(\Delta t^2)$$

We again obtain that the One-Step-Error is $\mathcal{O}(\Delta t^2)$.

Recall that the accuracy of a multiple-timestep computation is better gauged by the Local Truncation Error: LTE:

$$LTE \equiv \frac{1}{\Delta t} OSE$$

While the OSE decreases $\propto \Delta t^2$ for smaller Δt, to get to a given time T one requires $\frac{\Delta t_{bigger}}{\Delta t_{smaller}}$ more steps to get there, thus introducing an OSE at every time step and accumulating them.

Hence we find that the truncation error for the splitting method is

$$LTE = \frac{1}{\Delta t} OSE = \mathcal{O}(\Delta t)$$

as long as the operators we used are also at least $\mathcal{O}(\Delta t)$.

This form of fractional splitting sometimes called "Godunov splitting", is only first order accurate formally.

It turns out that a slight modification of the splitting idea will yield second order accuracy quite generally, assuming each subproblem is solved with a method of at least this accuracy.

Instead of solving

$$\underline{u}^{n+1/2} = \mathcal{X}\underline{u}^{n} \qquad \text{over step } \Delta t$$

$$\underline{u}^{n} = \mathcal{Y}\underline{u}^{n+1/2} \qquad \text{over step } \Delta t$$

we solve

$$\underline{u}^{n+1/4} = \mathcal{X}\underline{u}^{n} \qquad \text{over step } \Delta t/2$$

$$\underline{u}^{n+3/4} = \mathcal{Y}\underline{u}^{n+1/4} \qquad \text{over step } \Delta t$$

$$\underline{u}^{n+1} = \mathcal{X}\underline{u}^{n+3/4} \qquad \text{over step } \Delta t/2$$

i.e.

$$\underline{u}^{n+1} = e^{1/2\Delta t\mathcal{X}}\, e^{\Delta t\mathcal{Y}}\, e^{\Delta t\mathcal{X}}\, \underline{u}_{n}$$

To compute the One-Step Error for this approach, we again compute the Taylor expansion of this method first:

$$e^{1/2 \Delta t \mathcal{X}} e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}} =$$

$$= \left(1 + \frac{1}{2}\Delta t \mathcal{X} + \frac{1}{8}\Delta t^2 \mathcal{X}^2 + \mathcal{O}(\Delta t^3)\right) \cdot$$
$$\left(1 + \Delta t \mathcal{Y} + \frac{1}{2}\Delta t^2 \mathcal{Y}^2 + \mathcal{O}(\Delta t^3)\right) \cdot$$
$$\left(1 + \frac{1}{2}\Delta t \mathcal{X} + \frac{1}{8}\Delta t^2 \mathcal{X}^2 + \mathcal{O}(\Delta t^3)\right)$$

$$= \left(1 + \Delta t \mathcal{Y} + \frac{1}{2}\Delta t^2 \mathcal{Y}^2 + \frac{1}{2}\Delta t \mathcal{X} + \frac{1}{2}\Delta t^2 \mathcal{X}\mathcal{Y} + \right.$$
$$\left. + \frac{1}{8}\Delta t^2 \mathcal{X}^2 + \mathcal{O}(\Delta t^3)\right) \cdot$$
$$\left(1 + 1/2 \Delta t \mathcal{X} + \frac{1}{8}\Delta t^2 \mathcal{X}^2 + \mathcal{O}(\Delta t^3)\right)$$

$$= \left(1 + \frac{1}{2}\Delta t \mathcal{X} + \frac{1}{8}\Delta t^2 \mathcal{X}^2 + \Delta t \mathcal{Y} + \frac{1}{2}\Delta t^2 \mathcal{Y}\mathcal{X} + \right.$$
$$+ \frac{1}{2}\Delta t^2 \mathcal{Y}^2 + \frac{1}{2}\Delta t \mathcal{X} + \frac{1}{4}\Delta t^2 \mathcal{X}^2 + \frac{1}{2}\Delta t^2 \mathcal{X}\mathcal{Y} +$$
$$\left. + \frac{1}{8}\Delta t^2 \mathcal{X}^2 + \mathcal{O}(\Delta t^3)\right) =$$

$$= \left(1 + \Delta t \mathcal{X} + \Delta t \mathcal{Y} + \frac{1}{2}\Delta t^2 (\mathcal{Y}\mathcal{X} + \mathcal{X}\mathcal{Y} + \mathcal{Y}^2 + \mathcal{X}^2) + \mathcal{O}(\Delta t^3)\right)$$
$$= \left(1 + \Delta t (\mathcal{X} + \mathcal{Y}) + \frac{1}{2}\Delta t^2 (\mathcal{Y}^2 + \mathcal{Y}\mathcal{X} + \mathcal{X}\mathcal{Y} + \mathcal{X}^2) + \mathcal{O}(\Delta t^3)\right)$$
$$= \left(1 + \Delta t (\mathcal{X} + \mathcal{Y}) + \frac{1}{2}\Delta t^2 (\mathcal{X} + \mathcal{Y})^2 + \mathcal{O}(\Delta t^3)\right)$$

Then

$$\underline{u}(x, \Delta t) = \left[1 + \Delta t(\mathcal{X} + \mathcal{Y}) + \frac{1}{2}\Delta t^2(\mathcal{X} + \mathcal{Y})^2 + \mathcal{O}(\Delta t^3)\right]\underline{u}(x,$$

$$\underline{u}^{n+1} = \left[1 + \Delta t(\mathcal{X} + \mathcal{Y}) + \frac{1}{2}\Delta t^2(\mathcal{X} + \mathcal{Y})^2 + \mathcal{O}(\Delta t^3)\right]\underline{u}(x, 0)$$

and

$$OSE = \underline{u}(x, \Delta t) - \underline{u}^{n+1} = \mathcal{O}(\Delta t^3)\,\underline{u}(x, 0) = \mathcal{O}(\Delta t^3)$$

and finally

$$LTE = \frac{1}{\Delta t}\,OSE = \mathcal{O}(\Delta t^2)$$

At this point it is also clear why the operators must be at least of second order as well. If they weren't, we would have to truncate the Taylor expansion of the split method at first order, and the second order terms of the OSE wouldn't cancel out. Hence the OSE would remain $\mathcal{O}(\Delta t^2)$, and the LTE $\mathcal{O}(\Delta t)$.

This method is called "<u>Strang splitting</u>".

A very interesting property is that we can achieve the same effect using the full time step Δt for both operators, but to alternate the order of these steps in different time steps, e.g.

$$\underline{u}^{n+1} = e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}} \underline{u}^{n}$$

$$\underline{u}^{n+2} = e^{\Delta t \mathcal{X}} e^{\Delta t \mathcal{Y}} \underline{u}^{n+1}$$

$$\underline{u}^{n+3} = e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}} \underline{u}^{n+2}$$

$$\underline{u}^{n+4} = e^{\Delta t \mathcal{X}} e^{\Delta t \mathcal{Y}} \underline{u}^{n+3}$$

etc.

If we write out all the operations, we have for an even number of steps:

$$\underline{u}^{n+m} = (e^{\Delta t \mathcal{X}} e^{\Delta t \mathcal{Y}})(e^{\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}})(e^{\Delta t \mathcal{X}} e^{\Delta t \mathcal{Y}}) \dots \quad \underline{u}^{n}$$

$$= (e^{\Delta t \mathcal{X}} e^{2\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}})(e^{\Delta t \mathcal{X}} e^{2\Delta t \mathcal{Y}} e^{\Delta t \mathcal{X}}) \dots \underline{u}^{n}$$

this is essentially the same as Strang splitting with $\frac{1}{2}\Delta t$ replaced with Δt, hence still $O(\Delta t^2)$ in LTE.

Unsplit Finite Volume Methods

In unsplit finite volume methods, the solution is advanced by accounting for all flux contributions in a single step.

Consider a time-dependent two-dimensional system of conservation laws

$$\underline{U}_t + \underline{\underline{F}}(\underline{U})_x + \underline{\underline{G}}(\underline{U})_y = 0$$

An explicit finite volume scheme to solve this problem would read

$$\underline{U}_{i,j}^{n+1} = \underline{U}_{i,j}^{n} + \frac{\Delta t}{\Delta x}\left(\underline{\underline{F}}_{i-1/2,j}^{n} - \underline{\underline{F}}_{i+1/2,j}^{n}\right) + \frac{\Delta t}{\Delta x}\left[\underline{\underline{G}}_{i,j-1/2} - \underline{\underline{G}}_{i,j+1/2}\right]$$

This conservative formula is the natural extension of the one-dimensional conservative formulae.

However, in two dimensions, this doesn't complete the upwinding. For linear advection for example,

with

$$u_t + a_1 u_x + a_2 u_y = 0$$

with $a_1, a_2 > 0$.

The Godunov finite volume scheme then reads

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{a_1 \Delta t}{\Delta x}\left(u_{i-1,j}^{n} - u_{i,j}^{n}\right) + \frac{a_2 \Delta t}{\Delta y}\left(u_{i,j-1}^{n} - u_{i,j}^{n}\right)$$

But this doesn't involve $u_{i-1,j-1}^{n}$, which should be the most obvious upwind value. After all, with $a_1, a_2 > 0$, one expects the fluid to advect diagonally upwards and to the right.