Splitting Schemes for PDEs with Source Terms

Let us now have a look at how to solve hyperbolic conservation laws with source terms:

$$\partial_t \underline{U} + \partial_x \underline{F}(\underline{U}) = \underline{S}(\underline{U})$$

For example, the Euler equations in 1D with some velocity-independent acceleration $\underline{a}$ are

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \frac{\partial}{\partial x} \cdot \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} = \begin{pmatrix} 0 \\ \rho a \\ \rho a u \end{pmatrix}$$

The acceleration could be for example gravity, photon/radiation pressure, chemical reactions ...

There are essentially two ways of constructing methods to solve inhomogeneous systems like the one above.

One approach attempts to preserve some coupling between the homogeneous part of the equation and the sources.

Another approach is to split, for a time Δt, into the homogeneous problem and the source problem. This might appear unreasonable, but for a model advection PDE, the splitting is exact. For more general problems, we can construct high-order splitting schemes.

1. Splitting for a model equation

The simplest hyperbolic equation of the form

$$\frac{\partial}{\partial t}\underline{u} + \frac{\partial}{\partial x}\underline{\underline{F}}(\underline{u}) = \underline{S}(\underline{u})$$

is given by

$$\partial_t u + a u_x = \lambda u$$

where a is a constant wave propagation speed and λ is a constant parameter.

This simple model equation will prove very useful in discussing possible strategies for solving the inhomogeneous conservation law.

Consider the initial value problem

$$\begin{cases} \text{PDE}: & \partial_t u + a\partial_x u = \lambda u \\ \text{IC}: & u(x,0) = u_0(x) \end{cases}$$

Using the method of characteristics, we can solve the problem analytically. Let us find a parametrisation

$$x = x(r), \quad t = t(r)$$

such that

$$\frac{du}{dr} = \lambda u$$

Using the chain rule, we have

$$\frac{du}{dr} = \frac{\partial u}{\partial t}\frac{dt}{dr} + \frac{\partial u}{\partial x}\frac{dx}{dr}$$

Because we want

$$\frac{du}{dr} = \lambda u$$

we can compare the expanded $\frac{du}{dr}$ with our equation:

$$\frac{du}{dr} = \frac{dt}{dr}\frac{\partial u}{\partial t} + \frac{dx}{dr}\frac{\partial u}{\partial x} = \lambda u$$

and $$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \lambda u$$

$$\Rightarrow \frac{dt}{dr} = 1 \quad \Rightarrow t = r$$

we omit the integration constant here, because it can be set later with $\frac{dx}{dr}$ being solved.

$$\frac{dx}{dr} = a \qquad \Rightarrow \quad x = ar + C = at + C$$

$$\Rightarrow \quad C = x - at$$

Finally, we can now solve the ODE

$$\frac{du}{dr} = \lambda u \qquad \Rightarrow \quad u = u_0 e^{\lambda r} = u_0 e^{\lambda t}$$

where $u_0 = u_0(C)$, it may vary with $C = x - at$!

So finally:

$$u(x,t) = u_0(x - at)\, e^{\lambda t}$$

Next, we show that this exact solution can be obtained by solving exactly a pair of initial value problems in succession:

The exact solution

$$u(x,t) = u_0(x-at)\,e^{\lambda}$$

of the problem

$$\partial_t u + a\partial_x u = \lambda u, \quad u(x,0) = u_0(x)$$

can be found by solving exactly the following pair of IVPs:

1) $\begin{cases} \partial_t r + a\partial_x r = 0 \\ r(x,0) = u_0(x) \end{cases}$ to get $r(x,t)$

2) $\begin{cases} \dfrac{d}{dt} s = \lambda s \\ s(0) = r(x,t) \end{cases}$ to get $u(x,t)$

<u>Proof</u>

The solution of

$$\partial_t r + a\partial_x r = 0$$

is given by $r(x,t) = u_0(x-at)$.

The exact solution of

$$\frac{d}{dt} s = \lambda s$$

is given by

$$s = S_0 e^{\lambda t}$$

using the initial values

$$s(0) = S_0 = r(x,t) = u_0(x - at)$$

we arrive at

$$s = u_0(x-at) e^{\lambda t}$$

which is the exact solution that we are looking for.

We can express the result of the splitting scheme in succinct form:

$$u(x,t) = S^{(t)} C^{(t)} [u_0(x)]$$

where $C^{(t)}$ is the solution operator for the advection problem applied over a time t and $S^{(t)}$ is the solution operator for the ODE $\frac{ds}{dt} = \lambda s$ applied over a time t.

Similarly, we can demonstrate that we can obtain the exact result as well by solving the IVPs

1) $$\begin{cases} \partial_t u + a\partial_x u = 0 \\ u(x,0) = u_0(x) \end{cases} \Rightarrow \text{get } \bar{u}^{n+1}$$

2) $$\begin{cases} \dfrac{d}{dx}(au) = \lambda u \\ u(x,0) = \bar{u}^{n+1} \end{cases} \Rightarrow \text{get } u^{n+1}$$

in succession.

This result says that the splitting scheme modified by replacing the ODE in time by an ODE in space also gives the exact solution. Again we may express this scheme in succinct form

$$u(x,t) = S^{(x)} C^{(t)} [u_0(x)]$$

2. Numerical Methods Based on Splitting

2.1 Scalar Equations

The splitting scheme

$$u(x,t) = S^{(t)} C^{(t)} [u_0(x)]$$

is exact if the operators S and C are exact.

If the numerical analogues of $S^{(t)}$ and $C^{(t)}$ are still denoted as $S^{(\Delta t)}$ and $C^{(\Delta t)}$, then we can write the numerical splitting as

$$u^{n+1} = S^{(\Delta t)} C^{(\Delta t)} (u^n)$$

Each numerical subproblem is dealt with separately, and for a full time step Δt.
This procedure is exceedingly simple, but is only first-order accurate in time.

A second order accurate scheme is

$$u^{n+1} = S^{(\frac{1}{2}\Delta t)} C^{(\Delta t)} S^{(\frac{1}{2}\Delta t)} (u^n)$$

where S and C are at least second-order accurate operators.

2.2 Schemes for Systems

The generalisation is straightforward:

$$\underline{U}^{n+1} = S^{(\Delta t)} C^{(\Delta t)} (\underline{U}^n)$$

gives a first order accurate method, while

$$\underline{U}^{n+1} = S^{(\frac{1}{2}\Delta t)} C^{(\Delta t)} S^{(\frac{1}{2}\Delta t)} (\underline{U}^n)$$

gives a second order accurate scheme.

Note however that unlike for the model equations, there is no analytical proof that the method is exact even if the operators C, S are exact.