The Riemann Problem

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1. Quasi-linear Equations

$$\underline{U}_t + \underline{\underline{A}}\, \underline{U}_x + \underline{B} = 0 \qquad [1]$$

with $\underline{U} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$, $\underline{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, $\underline{\underline{A}} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}$

- linear with constant coefficients: $a_{ij} = \text{const}$, $b_i = \text{const}$
- linear with variable coefficients: $a_{ij} = a_{ij}(x,t)$, $b_i = b_i(x,t)$
- if $\underline{B} = \alpha + \beta \cdot \underline{U}$, still linear system
- quasi-linear: $\underline{\underline{A}} = \underline{\underline{A}}(\underline{U})$
- homogeneous: $\underline{B} = 0$

Def: Conservation laws

Conservation Laws are systems of PDEs that can be written in the form

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

$\underline{U}$: vector of conserved variables

$\underline{F}$: vector of fluxes

Def: Jacobian Matrix

The Jacobian of the flux function $\underline{F}(\underline{U})$ is the matrix

$$a_{ij}(\underline{U}) = \frac{\partial F_i}{\partial u_j}$$

Conservation laws can be written in quasi-linear form:

$$\frac{\partial \underline{U}}{\partial t} + \frac{\partial \underline{E}}{\partial x} = \underline{0} = \frac{\partial \underline{U}}{\partial t} + \frac{\partial \underline{E}}{\partial \underline{U}}\frac{\partial \underline{U}}{\partial x} = \frac{\partial \underline{U}}{\partial t} + A(\underline{U})\frac{\partial \underline{U}}{\partial x} = \underline{0}$$

Def: Eigenvalues

The Eigenvalues λ_i of a matrix $\underline{\underline{A}}$ are the solutions of the characteristic polynomial

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

The Eigenvalues of the coefficient matrix $\underline{\underline{A}}$ of a system of the form [1] are called the eigenvalues of the system.

Def Eigenvectors

A right Eigenvector of a matrix $\underline{\underline{A}}$ corresponding to an eigenvalue λ_i of $\underline{\underline{A}}$ satisfies $\underline{\underline{A}}\,\underline{K}^{(i)} = \lambda_i \underline{K}^{(i)}$

A left eigenvector of $\underline{\underline{A}}$ corresponding to λ_i satisfies $\underline{L}^{(i)}\underline{\underline{A}} = \lambda_i \underline{L}^{(i)}$

Def: Hyperbolic System

A system [1] is said to be hyperbolic at a point (x,t) if $\underline{\underline{A}}$ has m real eigenvalues $\lambda_1, \dots, \lambda_m$ and a corresponding set of m linearly independent right eigenvectors $\underline{K}^{(1)}, \dots, \underline{K}^{(m)}$. If the λ_i are all distinct, the system is called strictly hyperbolic.

2. The Linear Advection Equation

General, time-dependent linear advection eqn in 3D:

$$u_t + a(\underline{x},t)\,u_x + b(\underline{x},t)\,u_y + c(\underline{x},t)\,u_z = 0$$

If the coefficients are sufficiently smooth, it can be expressed as a conservation law with source terms:

$$u_t + (au)_x + (bu)_y + (cu)_z = u\,(a_x + b_x + c_z)$$

Simplest form: $u_t + a u_x = 0, \quad -\infty < x < \infty,\ t > 0$

IC: $u(x,0) = u_0(x), \quad a = \text{const}$

Characteristics and the General Solution

Characteristics may be defined as curves $x = x(t)$ in the $t-x$ plane along which the PDE becomes an ODE.

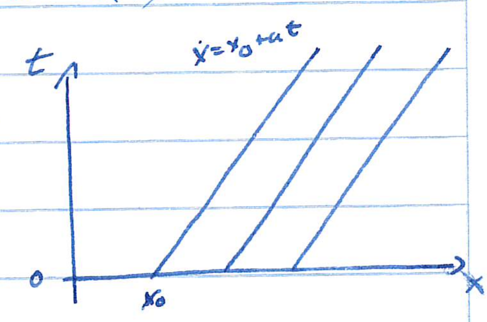
Consider $x = x(t) \rightarrow u = u(x(t), t)$

Then $$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}$$

If now $\dfrac{dx}{dt} = a \Rightarrow \dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$ advection equation

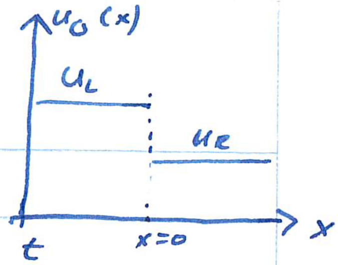
$\Rightarrow u$ is constant along the curve $x = x(t)$

For constant coefficients, the characteristics are parallel.

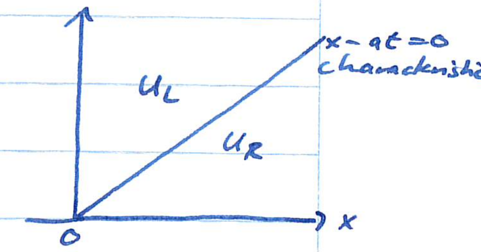


If u is given the initial value $u(t=0) = u_0$, then along $x(t) = x_0 + at$, the solution is $u(x,t) = u_0(x - at) = u_0(x_0)$

Similarly: $u(x,t) = u_0(x - at, 0)$ everywhere



The Riemann Problem



Special Initial Value Problem (IVP):

$$u_t + a u_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_L & \text{if } x<0 \\ u_R & \text{if } x>0 \end{cases}$$

Using the general solution:

$$u(x,t) = u_0(x-at) = \begin{cases} u_L & \text{if } x-at<0 \\ u_R & \text{if } x-at>0 \end{cases}$$

3. Linear Hyperbolic Systems

Extend analysis to sets of m hyperbolic PDEs of the form

$$\underline{U}_t + \underline{\underline{A}}\,\underline{U}_x = \underline{0} \qquad [2]$$

with $\underline{\underline{A}} = \text{const.}$

Def: Diagonalisable System

A matrix $\underline{\underline{A}}$ is said to be diagonalisable if $\underline{\underline{A}}$ can be expressed as

$$\underline{\underline{A}} = \underline{\underline{K}}\,\underline{\underline{\Lambda}}\,\underline{\underline{K}}^{-1} \quad \text{or} \quad \underline{\underline{\Lambda}} = \underline{\underline{K}}^{-1}\underline{\underline{A}}\,\underline{\underline{K}}$$

in terms of a diagonal matrix $\underline{\underline{\Lambda}}$, whose diagonal elements are the eigenvalues λ_i of $\underline{\underline{A}}$, and the matrix $\underline{\underline{K}}$, whose columns $\underline{K}^{(i)}$ are the right eigenvectors of $\underline{\underline{A}}$ corresponding to the eigenvalues λ_i.

A system [2] is called diagonalisable if $\underline{\underline{A}}$ is.

From the assumption of hyperbolicity, $\underline{\underline{A}}$ has m real eigenvalues λ_i and m linearly independent eigenvectors, and thus is diagonalisable.

Now make use of $\underline{\underline{K}}$ and $\underline{\underline{K}}^{-1}$: Because $\underline{\underline{A}} = \text{const}$, $\underline{\underline{K}}$ is also const $\Rightarrow \underline{\underline{K}}_x = 0, \underline{\underline{K}}_t = 0$

Now define $\underline{W} = \underline{\underline{K}}^{-1} \underline{U}$ or $\underline{U} = \underline{\underline{K}} \underline{W}$; Then

$$\underline{\underline{K}} \underline{W}_t + \underline{\underline{A}} \underline{\underline{K}} \underline{W}_x = 0$$

$$\Rightarrow \underline{\underline{K}}^{-1} \underline{\underline{K}} \underline{W}_t + \underline{\underline{K}}^{-1} \underline{\underline{A}} \underline{\underline{K}} \underline{W}_x = \underline{W}_t + \underline{\underline{\Lambda}} \underline{W}_x = 0$$

$$\Rightarrow \underline{W}_t + \underline{\Lambda} \underline{W}_x = 0$$

(canonical form or characteristic form of the system)

In this form, the system is decoupled: The i-th PDE of the system is

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, \quad i = 1, \dots, m$$

which is identical to the linear advection eqn. The characteristic speed is now λ_i, the characteristic curves satisfy $\frac{dx}{dt} = \lambda_i$.

To obtain the solution in original variables $\underline{U}$, we must calculate explicitly $\underline{U} = \underline{\underline{K}} \underline{W}$ or $U_j = \sum_i w_i K_{ij}$

w_i is the coefficient of $\underline{K}^{(i)}$ in an eigenvector expansion of the vector $\underline{U}$.

But also use the characteristic solution:

$$U_j = \sum_i w_i K_{ij} = \sum_i w_i^{(0)}(x - \lambda_i t) K_{ij}$$

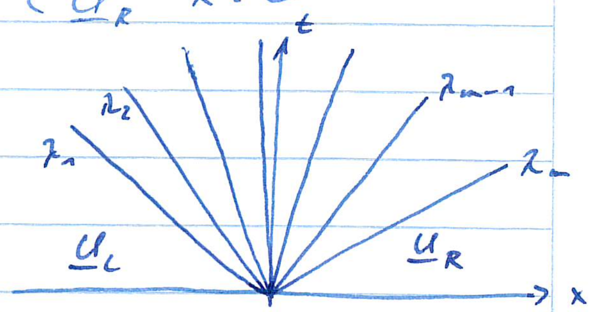
$\Rightarrow$ given a point (x, t) in the x-t plane, the solution $\underline{U}(x,t)$ at this point depends only on the initial data at the m points $x_0^{(i)} = x - \lambda_i t$. These are the intersections of the characteristics of speed λ_i with the x-axis. The solution for $\underline{U}$ can be seen as the superposition of m waves, each of which is advected independently without change in shape. The i-th wave has shape $w_i^{(0)}(x)\,\underline{K}^{(i)}$ and speed λ_i.

The Riemann Problem

For a hyperbolic, constant coefficient system:

$$\underline{U}_t + \underline{A}\,\underline{U}_x = 0$$

$$\underline{U}(x,0) = \underline{U}^{(0)}(x) = \begin{cases} \underline{U}_L & x < 0 \\ \underline{U}_R & x > 0 \end{cases}$$



Assume the system is strictly hyperbolic and that the m eigenvalues are indexed such that $\lambda_1 < \lambda_2 < \ldots < \lambda_m$.

The structure of the solution consists of m waves emanating from the origin, one for each eigenvalue λ_i. Each wave i carries a jump discontinuity in $\underline{U}$ propagating with speed λ_i

Because the eigenvectors $\underline{K}^{(i)}$ are linearly independent, we can express any state $\underline{U} = \sum_i^m \gamma_i \underline{K}^{(i)}$ as linear combinations. That allows us to define $\underline{U}_L = \sum_i \alpha_i \underline{K}^{(i)}$ and $\underline{U}_R = \sum_i \beta_i \underline{K}^{(i)}$

From the general solution, we know that $\underline{U}(x,t) = \sum_i w_i \underline{K}^{(i)}$ with $w_i(x,t) = w_i^{(0)}(x - \lambda_i t) = \begin{cases} \alpha_i & \text{if } x - \lambda_i t < 0 \\ \beta_i & \text{if } x - \lambda_i t > 0 \end{cases}$

We may rewrite/separate it as follows: For every given point (x,t) there is an eigenvalue λ_I such that $\lambda_I < \frac{x}{t} < \lambda_{I+1}$, i.e. $x - \lambda_i t > 0 \ \forall i \leq I$.

$$\Rightarrow \underline{U}(x,t) = \sum_{i=I+1}^{m} \alpha_i \underline{K}^{(i)} + \sum_{i=1}^{I} \beta_i \underline{K}^{(i)}$$

4. Conservation Laws

System of conservation laws: $\underline{U}_t + \underline{F}(\underline{U})_x = 0$

$$= \underline{U}_t + \frac{\partial F}{\partial \underline{U}} \frac{\partial \underline{U}}{\partial x}$$

$$= \underline{U}_t + \underline{A}\, \underline{U}_x = 0$$

The system is hyperbolic if $\underline{A}(\underline{U})$ has real eigenvalues and a complete set of linearly independent eigenvectors $\underline{K}^{(i)}(\underline{U})$. We assume them to be ordered: $\lambda_1(\underline{U}) < \lambda_2(\underline{U}) < \ldots < \lambda_m(\underline{U})$

4.1 Non-Linearities and Shock Formation

Under certain circumstances, assuming that the fluxes $f_i = f_i(u)$ is an inadequate representation of the physical problem being modelled, i.e. shock waves, which contain viscous dissipation and heat conduction. A more appropriate flux function for a model conservation law would also

The conservation law may be rewritten as

$$u_t + \lambda(u) u_x = 0 \qquad \lambda(u) = \frac{df}{du} = f'(u)$$

Assume inviscid system (no u_x dependance of flux). Now consider characteristic curves $x = x(t)$ stasisfying

$$\frac{dx}{dt} = \lambda(u), \quad x(0) = x_0$$

$$\Rightarrow \frac{du}{dt} = u_t + \lambda(u) u_x = 0$$

$\Rightarrow$ u is constant along these characteristic curves

$\Rightarrow$ the slope $\lambda(u)$ is constant along these curves (because u is constant, $\lambda(u)$ is same everywhere)

$\Rightarrow$ characteristic curves are straight lines

Along each curve: $u(x,t) = u_0(x_0)$, $\lambda(u) = \lambda(u_0(x_0))$

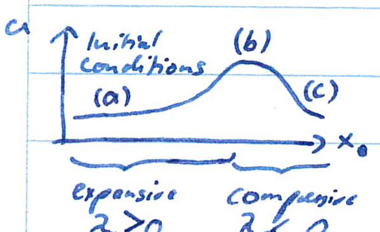
$\Rightarrow$ $x = x_0 + \lambda(u_0(x_0))t$, $u(x,t) = u_0(x - \lambda(u_0(x_0))t)$

Wave Steepening

In the non-linear case the characteristic speed $\lambda(u)$ is a function of the solution itself and in general not constant any more. The initial function is not translated without a distortion any longer. This is a distinguishing feature of non-linear problems.

Assume $\lambda(u)$ is an increasing function with u.

Expansive region: characteristics at (a) will expand slower than (b).



Compressive region: (b) will travel faster than (c). Will get narrower and steeper as time evolves, and eventually produce folding over of the solution profile, with corresponding crossing of characteristics, and triple-valued solution.

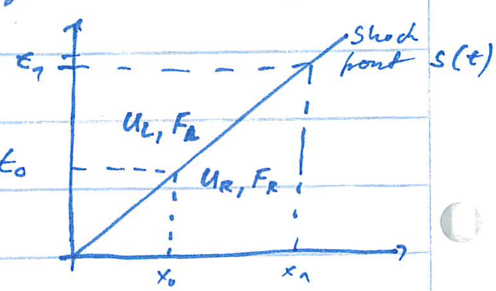
This situation can be rescued either by introducing a more accurate physical model, i.e. include viscosity which includes a term $\propto u_{xx}$, which on our example will have the exact opposite effect on wave steepening or wave easing regions, respectively, since u_{xx} has the exact opposite sign than u_x there.

Or you can stick to the inviscid model by allowing discontinuities, i.e. shocks, to be formed as a process of increasing compression.

Shock Waves

Consider a solution $u(x,t)$ such that $u(x,t)$, $f(u)$ are continuous everywhere except on a line $s=s(t)$ on the $x-t$ plane across which $u(x,t)$ has a jump discontinuity.

Select $x_L < s(t) < x_R(t)$, and let the shock propagate at a speed S such that its position can be inferred by
$x_1 = x_0 + S \cdot (t_1 - t_0)$



For any conservation law, we can integrate:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\underline{F}(\underline{u}) = 0 \quad \Rightarrow \quad \int_{x_0}^{x_1}\int_{t_0}^{t_1} \left[\underline{u}_t + \underline{F}(\underline{u})_x\right] dx\, dt = 0$$

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1} \left[\underline{u}_t + \underline{F}(\underline{u})_x\right] dx\, dt = \int_{x_0}^{x_1} dx \left[\underline{u}(t_1,x) - \underline{u}(t_0,x)\right] + \int_{t_0}^{t_1} dt \left[\underline{F}(t,x_1) - \underline{F}(t,x_0)\right]$$

Assume that the shock is infinitely thin and that before (u_R) and after (u_L) the shock, the state $\underline{u}$ is homogeneous over space and $\underline{F}$ homogeneous over time.

Then $u(t_1, x) = u_R, \quad u(t_0, x) = u_L, \quad x \in [x_0, x_1]$

$F(t, x_1) = F_R, \quad F(t, x_0) = F_L, \quad t \in [t_0, t_1]$

$$\Rightarrow \int_{x_0}^{x_1} dx \left[\underline{u}(t_1, x) - \underline{u}(t_0, x) \right] + \int_{t_0}^{t_1} dt \left[\underline{F}(t, x_1) - \underline{F}(t, x_0) \right] =$$

$$= [\underline{u}_R - \underline{u}_L](x_1 - x_0) + [\underline{F}_R - \underline{F}_L](t_1 - t_0) = 0$$

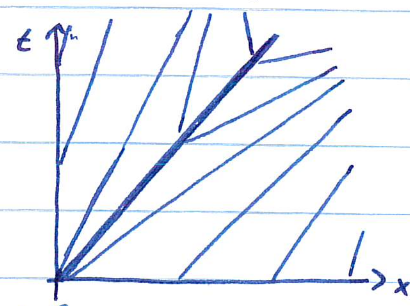
$$\Rightarrow \Delta \underline{F} = \underline{F}_R - \underline{F}_L = \frac{x_1 - x_0}{t_1 - t_0} [\underline{u}_R - \underline{u}_L] = S \Delta u$$ Rankine Hugoniot

For physical shocks, we also need the so-called entropy condition: Assume $u_L > u_R$. For a convex flux $\lambda'(u) = f''(u) > 0$ (e.g. Burger's equ: $f(u) = \frac{1}{2}u^2$), the characteristic speeds on the left are greater than those on the right, i.e. $\lambda_L \equiv \lambda(u_L) > \lambda_R \equiv \lambda(u_R)$

The entropy condition is: $\lambda_L > S > \lambda_R$

Otherwise, you get unphysical solutions, albeit mathematically valid. Assume $u_R > u_L$ for a convex flux; Then $\lambda_L < \lambda_R$, the characteristics diverge from the discontinuity. You get rarefaction shocks.

To explain the problem: Characteristics carry information about the solution. Diverging characteristics like they appear for rarefaction shocks generate new information to be carried, which is unphysical.



4.2. Elementary-Wave Solutions of the Riemann Problem

The Riemann problem for a general $m \times m$ nonlinear hyperbolic system with data $\underline{U}_L$, $\underline{U}_R$ is the IVP

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

$$\underline{U}(x, 0) = \underline{U}^{(0)} = \begin{cases} \underline{U}_L & \text{if } x < 0 \\ \underline{U}_R & \text{if } x > 0 \end{cases}$$

The similarity solution consists of $m+1$ constant states separated by m waves. For each eigenvalue λ_i there is a wave family.

- For linear systems with constant coefficients: Each wave is a discontinuity of speed $S_i = \lambda_i$
- For non-linear systems: The waves may be discontinuities such as shock waves, contact waves, or smooth transition waves such as rarefactions.

Shock wave:

$\underline{U}_L$, $\underline{U}_R$ are connected through a single jump discontinuity*

The following conditions apply:

- Rankine-Hugoniot $\quad \underline{F}_R - \underline{F}_L = S_i (\underline{U}_R - \underline{U}_L)$
- Entropy condition $\quad \lambda_i(\underline{U}_L) > S_i > \lambda_i(\underline{U}_R)$

Contact Wave

$\underline{U}_L$, $\underline{U}_R$ are connected through a jump discontinuity of speed S_i in a linearly degenerate field i:

The following conditions apply:

- Rankine-Hugoniot $\quad \underline{F}_R - \underline{F}_L = S_i (\underline{U}_R - \underline{U}_L)$
- constancy of Generalised Riemann Invariants across the wave:

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \dots = \frac{dw_m}{k_m^{(i)}}$$

* in a genuinely non-linear field i ; [genuinely non-lin.: see later]

• parallel characteristic condition $\lambda_i(\underline{u}_L) = S_i = \lambda_i(\underline{u}_R)$

Rarefaction Wave

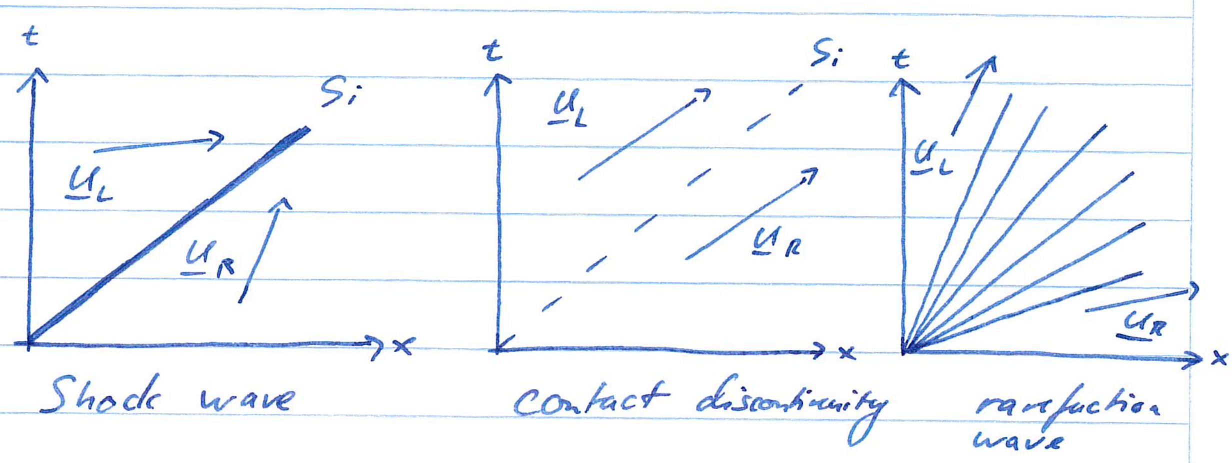
$\underline{u}_L$, $\underline{u}_R$ are connected through a smooth transition in a genuinely non-linear field i. The following conditions apply:

- constancy of the generalised Riemann Invariants across the wave:

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \dots = \frac{dw_m}{k_m^{(i)}}$$

- divergence of characteristics: $\lambda_i(\underline{u}_L) < \lambda_i(\underline{u}_R)$

The solution of the general Riemann problem contains m waves of any type of the above.



Difference contact/shock:

Contact: lin. deg. field: $\nabla\lambda_i \cdot \underline{K} = 0$

shock: genuinely non-linear: $\nabla\lambda_i \cdot K \neq 0$

Which one you get obviously depends on the equation you're solving and the eigenvalue/vector you're working with

Characteristic Fields

Consider a hyperbolic system of conservation laws: $\underline{U}_t + \underline{F}(\underline{U})_x = 0$ with real eigenvalues $\lambda_i(\underline{U})$ and right eigenvectors $\underline{K}^{(i)}(\underline{U})$. The characteristic speed $\lambda_i(\underline{U})$ defines a characteristic field.

Def: Linearly degenerate fields

A λ_i-characteristic field is said to be linearly degenerate if $\nabla \lambda_i(\underline{U}) \cdot \underline{K}^{(i)}(\underline{U}) = 0, \ \forall \underline{U} \in \mathbb{R}^m$

Def: Genuinely non-linear fields

A λ_i-characteristic field is said to be genuinely nonlinear if $\nabla \lambda_i(\underline{U}) \cdot \underline{K}^{(i)}(\underline{U}) \neq 0, \ \forall \underline{U} \in \mathbb{R}^m$

$\nabla \lambda_i(\underline{U})$ denotes the gradient of the eigenvalue:

$$\nabla \lambda_i(\underline{U}) = \left(\frac{\partial \lambda_i}{\partial u_1}, \frac{\partial \lambda_i}{\partial u_2}, \dots, \frac{\partial \lambda_i}{\partial u_m} \right)^T$$

Note that for a linear system $\underline{U}_t + \underline{\underline{A}}\,\underline{U}_x = \underline{0}$ the eigenvalues are all constant $\Rightarrow \nabla \lambda_i = 0 \Rightarrow$ All characteristic fields of a linear hyperbolic system with constant coefficients are linearly degenerate.

Generalised Riemann Invariants

For a general quasi-linear hyperbolic system

$$\underline{W}_t + \underline{\underline{A}}(\underline{W})\underline{W}_x = \underline{0}$$

with $\underline{W} = [w_1, w_2, \dots, w_m]^T$, λ_i, $\underline{K}^{(i)} = [k_1^{(i)}, k_2^{(i)}, \dots, k_m^{(i)}]^T$

Recall that any system of conservation laws may be expressed in quasi-linear form via the Jacobian matrix.

The Generalised Riemann Invariants are relations that hold true for certain waves across the wave structure*

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \frac{dw_3}{k_3^{(i)}} = \dots = \frac{dw_m}{k_m^{(i)}}$$

They relate ratios of changes dw_j of a quantity w_j to the respective component $k_j^{(i)}$ of the right eigenvector $\underline{K}^{(i)}$ corresponding to a λ_i-wave family.

* across the wave: Valid for both left/right state of particular wave
certain waves are contact and rarefaction waves.