One-Dimensional Euler Equations

1. Conservative Formulation

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \qquad \underline{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

$$E = \rho\left(\tfrac{1}{2}u^2 + e\right)$$

$$e = e(\rho, p) = \frac{p}{(\gamma-1)\rho} \quad \text{for ideal gases}$$

$$a = \sqrt{\frac{\gamma p}{\rho}} \quad \text{sound speed}$$

We can express the conservation law in quasi-linear form:

$$\underline{U}_t + \underline{\underline{A}}(u)\underline{U}_x = 0$$

where $A_{ij} = \dfrac{\partial f_i}{\partial u_j}$

To compute A_{ij}: First express all quantities of $\underline{F}$ via quantities of $\underline{U} = (\rho, \rho u, E) = (u_1, u_2, u_3)$

Using $E = \frac{1}{2}\rho u^2 + \rho e = \frac{1}{2}\rho u^2 + \rho\frac{p}{(\gamma-1)\rho} = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}$

$$\Rightarrow p = (\gamma-1)\left[E - \tfrac{1}{2}\rho u^2\right] = (\gamma-1)\left[u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right]$$

Which gives us:

$$\underline{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} = \begin{bmatrix} u_2 \\ u_2^2/u_1 + (\gamma-1)[u_3 - \frac{1}{2}u_2^2/u_1] \\ \frac{u_2}{u_1}\left(u_3 + (\gamma-1)[u_3 - \frac{1}{2}u_2^2/u_1]\right) \end{bmatrix}$$

$$= \begin{bmatrix} u_2 \\ \frac{1}{2}(3-\gamma)\, u_2^2/u_1 + (\gamma-1)u_3 \\ \gamma \frac{u_2}{u_1} u_3 - \frac{1}{2}(\gamma-1)\frac{u_2^3}{u_1^2} \end{bmatrix}$$

Computing $A_{ij} = \dfrac{\partial f_i}{\partial u_j}$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1^2} & 2\frac{u_2}{u_1} + (\gamma-1)\frac{-u_2}{u_1} & (\gamma-1) \\ -\gamma \frac{u_2}{u_1^2} u_3 + (\gamma-1)\frac{u_2^3}{u_1^3} & \gamma \frac{u_3}{u_1} - \frac{3}{2}(\gamma-1)\frac{u_2^2}{u_1^2} & \gamma \frac{u_2}{u_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\frac{u_2^2}{u_1^2} & (3-\gamma)\frac{u_2}{u_1} & \gamma-1 \\ -\gamma\frac{u_2 u_3}{u_1^2} + (\gamma-1)\frac{u_2^3}{u_1^3} & \gamma\frac{u_3}{u_1} + \frac{3}{2}(1-\gamma)\frac{u_2^2}{u_1^2} & \gamma\frac{u_2}{u_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -\gamma\frac{uE}{\rho} + (\gamma-1)u^3 & \gamma\frac{E}{\rho} + \frac{3}{2}(1-\gamma)u^2 & \gamma u \end{bmatrix}$$

The Eigenvalues of the Jacobian matrix are

$$\lambda_1 = u-a, \quad \lambda_2 = u, \quad \lambda_3 = u+a$$

The corresponding eigenvectors are

$$\underline{K}_1 = \begin{bmatrix} 1 \\ u-a \\ \frac{E+p}{\rho} - ua \end{bmatrix}, \quad \underline{K}_2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \quad \underline{K}_3 = \begin{bmatrix} 1 \\ u+a \\ \frac{E+p}{\rho} + ua \end{bmatrix}$$

(Proof is easy, but long calculation by hand)

2. Primitive Variable Formulation

Use $\underline{W} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}$

Getting equations from conservative to primitive variables:

$$= \frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} = 0$$

The first equation, $\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} u + \rho \frac{\partial u}{\partial x} = 0}$ is fine in primitive variables too.

For the second:

$$\rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x + p_x = 0$$

$$= u\underbrace{\left[\rho_t + \rho_x u + \rho u_x\right]}_{=0:\ \text{first equation}} + \rho\left[u_t + u u_x + \frac{1}{\rho} p_x\right] = 0$$

$$\Rightarrow \boxed{u_t + u u_x + \frac{1}{\rho} p_x = 0}$$

Third equation:

$$E_t + u_x(E+p) + u(E_x + p_x) = 0$$

$$E = \frac{1}{2}\rho u^2 + \rho e = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}$$

$$E_x = \frac{1}{2}u^2\rho_x + \rho u u_x + \frac{p_x}{\gamma-1}$$

$$E_t = \frac{1}{2}u^2\rho_t + \rho u u_t + \frac{p_t}{\gamma-1}$$

$$\Rightarrow \frac{1}{2}u^2\rho_t + \rho u u_t + \frac{p_t}{\gamma-1} + u_x\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1} + p\right) +$$

$$+ u\left(\frac{1}{2}u^2\rho_x + \rho u u_x + \frac{p_x}{\gamma-1} + p_x\right) = 0$$

$$= \underline{\frac{1}{2}}u^2\rho_t + \underline{\rho u u_t} + \underline{\frac{p_t}{\gamma-1}} + \underline{\frac{1}{2}\rho u^2 u_x} + \underline{\frac{\gamma}{\gamma-1}u_x p} + \underline{\frac{1}{2}u^3\rho_x} + \rho u^2 u_x + \underbrace{u\frac{\gamma}{\gamma-1}p_x}_{=\rho u\left[\frac{\gamma}{\gamma-1}\frac{p_x}{\rho}\right] = \rho u\frac{p_x}{\rho}\left[1+\ldots\right.}$$

$$= \frac{1}{2}u^2\Big[\underbrace{\rho_t + \rho u_x + u\rho_x}_{=0:\ 1\text{st equ}}\Big] + \rho u\Big[\underbrace{u_t + u u_x + \frac{1}{\rho}p_x}_{=0:\ 2\text{nd equ}}\Big] +$$

$$+ \frac{1}{\gamma-1}\left[p_t + \gamma p u_x + u p_x\right] = 0$$

$$\text{with } a^2 = \frac{\gamma p}{\rho} \Rightarrow \boxed{p_t + \rho a^2 u_x + u p_x = 0}$$

Which leaves us with the equations:

$$\boxed{\begin{aligned} &\rho_t + u\rho_x + \rho u_x = 0 \\ &u_t + u u_x + \frac{1}{\rho}p_x = 0 \\ &p_t + \rho a^2 u_x + u p_x = 0 \end{aligned}}$$

or $\underline{W}_t + \underline{\underline{A}}(\underline{W})\,\underline{W}_x = 0$ with

$$\underline{W} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \quad \text{and} \quad \underline{\underline{A}} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{bmatrix}$$

Computing eigenvalues:

$$\det|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0 = \begin{vmatrix} u-\lambda & \varrho & 0 \\ 0 & u-\lambda & 1/\varrho \\ 0 & \varrho a^2 & u-\lambda \end{vmatrix}$$

$$= (u-\lambda)^3 + 0 + 0 - 0 - \varrho a^2 \cdot \frac{1}{\varrho}(u-\lambda) =$$

$$= (u-\lambda)\left[(u-\lambda)^2 - a^2\right]$$

$$\Rightarrow \lambda_1 = u$$

$$(u-\lambda_{2,3})^2 - a^2 = 0 \quad \Rightarrow \quad u - \lambda_{2,3} = \pm a$$

$$\Rightarrow \lambda_{2,3} = u \pm a$$

Computing eigenvectors

$\lambda = u$

$$\begin{pmatrix} u & \varrho & 0 \\ 0 & u & 1/\varrho \\ 0 & \varrho a^2 & u \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = u \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$ux + \varrho y = ux \quad \Rightarrow y = 0$$

$$uy + \frac{1}{\varrho} z = uy \quad \Rightarrow z = 0$$

$\Rightarrow$ valid for all x $\Rightarrow \underline{K}(\lambda = u) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\lambda = u \pm a: \quad \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (u \pm a) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$ux + \rho y = (u \pm a)x \quad \Rightarrow \quad \rho y = \pm a x$$

$$uy + \frac{1}{\rho} z = (u \pm a) y \quad \Rightarrow \quad \frac{1}{\rho} z = \pm a y$$

$$\rho a^2 y + uz = (u \pm a) z \quad \Rightarrow \quad \rho a^2 y = \pm a z$$

Let $x = 1$

$\hookrightarrow$ $y = \mp a/\rho$, $z = \pm \rho a y = \pm \rho a (\mp a/\rho) = \mp a^2$

$$\Rightarrow K(\lambda = u + a) = \begin{pmatrix} 1 \\ -a/\rho \\ a^2 \end{pmatrix}$$

$$\Rightarrow K(\lambda = u - a) = \begin{pmatrix} 1 \\ a/\rho \\ a^2 \end{pmatrix}$$

Similarly, the left eigenvectors are

$$\underline{L}^{(1)} = \beta_1 (0, 1, -1/\rho a)$$

$$\underline{L}^{(2)} = \beta_2 (1, 0, -1/a^2)$$

$$\underline{L}^{(3)} = \beta_3 (0, 1, 1/\rho a)$$

By choosing appropriate normalisation parameters, the left and right eigenvectors are bi-orthonormal:

$$\underline{L}^{(j)} \cdot \underline{K}^{(i)} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Characteristic equations:

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ define characteristic directions $\frac{dx}{dt} = \lambda_i$. For a set of partial differential equations, a characteristic equation says that in a direction, $\underline{L}^{(i)} \cdot d\underline{W} = 0$

$$\Rightarrow \quad \begin{aligned} dp - \rho a\, du &= 0 \quad \text{along} \quad dx/dt = \lambda_1 = u - a \\ dp - a^2 d\rho &= 0 \quad \text{along} \quad dx/dt = \lambda_2 = u \\ dp + \rho a\, du &= 0 \quad \text{along} \quad dx/dt = \lambda_3 = u + a \end{aligned}$$

These relations hold true along characteristic directions

Entropy Formulation

Use entropy instead of pressure as primitive variable: $\underline{W} = (\rho, u, s)^T$

with $s = c_v \ln\left(\frac{p}{\rho^\gamma}\right) + C_0 \iff p = C_1 \rho^\gamma e^{s/c_v}$

The entropy satisfies the following PDE:

$$s_t + u s_x = 0$$

Proof: $s = c_v \ln(p/\rho^\gamma) + C_0 = c_v(\ln p - \gamma \ln \rho) + C_0$

Then $s_t = c_v\left(\frac{1}{p} p_t - \frac{\gamma}{\rho} \rho_t\right) = \frac{c_v}{p}\left(p_t - \underbrace{\frac{\gamma p}{\rho}}_{=a^2} \rho_t\right)$

$= \frac{c_v}{p}\left(p_t - a^2 \rho_t\right)$

analogously: $s_x = \frac{c_v}{p}\left(p_x - a^2 \rho_t\right)$

Using $p_t + \rho a^2 u_x + u p_x = 0$:

$$s_t + u s_x = \frac{c_v}{p}\left[p_t + u p_x - a^2 \rho_t - a^2 \rho_x u\right]$$

$$= \frac{c_v}{p}\left[-\rho a^2 u_x - u p_x + u p_x - a^2 \rho_t - a^2 \rho_x u\right]$$

$$= \frac{c_v}{p}\left[-\rho a^2 (\underbrace{\rho_t + \rho_x u + \rho u_x}_{=0})\right] = 0$$

$$\Rightarrow \boxed{s_t + u s_x = \frac{ds}{dt} = 0}$$

$$\Rightarrow p = C_1 \rho^\gamma e^{s/c_v} = C \rho^\gamma$$ in regions of smooth flow.

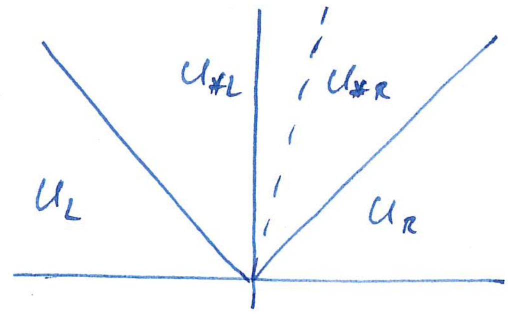
The eigenvalues for the entropy formulation are again $\lambda_1 = u - a$, $\lambda_2 = u$, $\lambda_3 = u + a$

with eigenvectors

$$\underline{K}^{(1)} = \begin{bmatrix} 1 \\ -a/\rho \\ 0 \end{bmatrix}, \quad \underline{K}^{(2)} = \begin{bmatrix} -\frac{\partial p}{\partial s} \\ 0 \\ a^2 \end{bmatrix}, \quad \underline{K}^{(3)} = \begin{bmatrix} 1 \\ a/\rho \\ 0 \end{bmatrix}$$

3. Elementary Wave Solutions of the Riemann Problem

The solution of the Riemann problem is described as a set of elementary waves (rarefactions, contacts, shocks).



There are three waves associated with the three characteristic fields corresponding to the three Eigenvectors.

Reminder: $\underline{K}^{(1)} = \begin{pmatrix} 1 \\ u-a \\ \frac{E+p}{\rho} - ua \end{pmatrix}$, $\underline{K}^{(2)} = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}$, $\underline{K}^{(3)} = \begin{pmatrix} 1 \\ u+a \\ \frac{E+p}{\rho} + ua \end{pmatrix}$

$$\lambda_1 = u - a \quad , \quad \lambda_2 = u , \quad \lambda_3 = u + a$$

The $\underline{K}^{(2)}$ characteristic field is linearly degenerate:

$$\nabla \lambda_2 \cdot \underline{K}^{(2)} = 0$$

$$\lambda_2 = u = \frac{u_2}{u_1}$$

$$\nabla \lambda_2 = \begin{pmatrix} \partial \lambda_2 / \partial u_1 \\ \partial \lambda_2 / \partial u_2 \\ \partial \lambda_2 / \partial u_3 \end{pmatrix} = \begin{pmatrix} -u_2/u_1^2 \\ 1/u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho u/\rho^2 \\ 1/\rho \\ 0 \end{pmatrix} = \begin{pmatrix} -u/\rho \\ 1/\rho \\ 0 \end{pmatrix}$$

$$\nabla \lambda_2 \cdot \underline{K}^{(2)} = -u/\rho + u/\rho + 0 = 0 \qquad \forall\, u, \rho, E$$

The $\underline{K}^{(1)}$, $\underline{K}^{(3)}$ characteristic fields are genuinely non-linear:

$$\nabla \lambda_{1,3} \cdot \underline{K}^{(1,3)} \neq 0$$

$$\lambda_{1,3} = u \pm a = \frac{u_2}{u_1} \pm a$$

$$\nabla \lambda_{1,3} = \begin{pmatrix} -u_2/u_1^2 \\ 1/u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -u/\rho \\ 1/\rho \\ 0 \end{pmatrix}$$

$$\nabla \lambda_{1,3} \cdot \underline{K}^{(1,3)} = -\frac{u}{\rho} + \frac{u \pm a}{\rho} + 0 \neq 0 \qquad \forall\, u, E$$

=0 for ρ=0 when

$\Rightarrow$ The wave associated with the $\underline{K}^{(2)}$ characteristic will be a contact wave

$\Rightarrow$ The waves associated with the $\underline{K}^{(1)}$, $\underline{K}^{(3)}$ characteristic will either be shock or rarefaction wave.

Let's look at the waves individually:

i) Contact wave

Using generalised Riemann invariants $\frac{dw_1}{k_1} = \frac{dw_2}{k_2} = \dots = \frac{dw_m}{k_m}$

$\underline{W} = (\rho, \rho u, E)^T$, $\underline{K} = (1, u, \frac{1}{2}u^2)$

$$\Rightarrow \quad \frac{d\rho}{1} = \frac{d(\rho u)}{u} = \frac{dE}{1/2\, u^2}$$

From this:

$$\frac{d\rho}{1} = d\rho = \frac{1}{u}(d\rho\, u + \rho\, du) = d\rho + \rho\, du$$

$$\Rightarrow \rho\, du = 0 \quad \Rightarrow du = 0 \quad \Rightarrow u = const$$

$$\frac{d\rho}{1} = d\rho = \frac{dE}{1/2\, u^2} = \frac{1}{1/2\, u^2}\, d\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) =$$

$$= \frac{1}{1/2\, u^2}\left[\frac{1}{2}u^2 d\rho + \rho u\, du + \frac{dp}{\gamma-1}\right]$$

$$= d\rho + \frac{2\rho}{u}\underbrace{du}_{=0} + \frac{2dp}{(\gamma-1)u^2}$$

$$\Rightarrow dp = 0 \quad \Rightarrow \quad p = const$$

$\Rightarrow$ p and u are constant across the contact wave

ii) Rarefaction Wave

Is associated with the $\underline{K}^{(1)}, \underline{K}^{(3)}$ eigenvectors.
Consider the entropy formulation:

$$\underline{W} = (\rho, u, s)^T, \qquad s = c_v \ln(P/\rho^\gamma) + C_0$$

and $\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a$

$$\underline{K}^{(1)} = \begin{pmatrix} 1 \\ -a/\rho \\ 0 \end{pmatrix}, \quad \underline{K}^{(2)} = \begin{pmatrix} -\partial p/\partial s \\ 0 \\ a^2 \end{pmatrix}, \quad \underline{K}^{(3)} = \begin{pmatrix} 1 \\ a/\rho \\ 0 \end{pmatrix}$$

For the left rarefaction wave: $\lambda_1, \underline{K}^{(1)}$

Generalised Riemann Invariants:

$$\frac{d\rho}{1} = \frac{du}{-a/\rho} = \frac{ds}{0}$$

$\rightarrow ds$ must $\rightarrow 0 \quad \Rightarrow \quad s = \text{const}$

$du = -a/\rho \, d\rho \quad \Rightarrow \quad u + \int \frac{a}{\rho} d\rho = \text{const}$

Since $s = \text{const} = C_v \ln(P/\rho^\gamma) + C_0$:

$\ln(P) - \ln(\rho^\gamma) = \text{const}$

$P = C \rho^\gamma$

$$\Rightarrow a = \sqrt{\frac{dP}{d\rho}} = \sqrt{C\gamma\rho^{\gamma-1}}$$

$$\Rightarrow \int \frac{a}{\rho} d\rho = \int \sqrt{C\gamma}\, \rho^{\frac{\gamma-1}{2}-1} d\rho = \int \sqrt{C\gamma}\, \rho^{\frac{\gamma-3}{2}} d\rho$$

$$= \frac{2}{\gamma-1}\sqrt{C\gamma}\, \rho^{\frac{\gamma-1}{2}} = \frac{2}{\gamma-1}\sqrt{C\gamma\rho^{\gamma-1}} = \underline{\underline{\frac{2a}{\gamma-1}}}$$

$$\Rightarrow \boxed{\begin{aligned} u + \frac{2a}{\gamma - 1} &= \text{const} \\ S &= \text{const} \end{aligned}}$$

Similarly for the right wave:

$$\boxed{\begin{aligned} u - \frac{2a}{\gamma - 1} &= \text{const} \\ S &= \text{const} \end{aligned}}$$

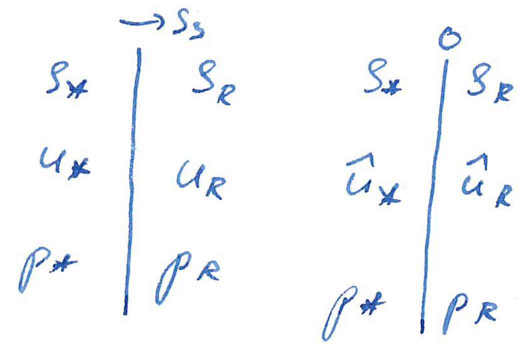
iii) Shock Waves

In the context of 1D Euler equations, shock waves are discontinuous waves associated with the genuinely non-linear fields 1 and 3.

All three quantities ρ, u and p change across a shock wave.

Consider the $\underline{K}^{(3)}$ characteristic field and assume the corresponding wave is a right-facing shock wave travelling at a constant speed S_3.

We denote the state ahead of the shock by $\underline{W}_R = (\rho_R, u_R, p_R)^T$ and the state behind the shock by $\underline{W}_* = (\rho_*, u_*, p_*)^T$.



It is convenient to transform the problem to a new frame of reference, comoving with the shock:

$\hat{u}_* = u_* - S_3$, $\hat{u}_R = u_R - S_3$

Now use Rankine-Hugoniot:

$$\underline{F}_R - \underline{F}_* = S(\underline{U}_R - \underline{U}_*)$$

with $\underline{U} = (\rho, \rho u, E)^T$, $\underline{F} = (\rho u, \rho u^2 + p, u(E+p))$

1st eqn:

$$\rho_R u_R - \rho_* u_* = S\rho_R - S\rho_*$$

$$\Rightarrow \rho_R (u_R - S) = \rho_* (u_* - S)$$

$$\Rightarrow \boxed{\rho_R \hat{u}_R = \rho_* \hat{u}_*}$$

2nd eqn:

$$\rho_R u_R^2 + p_R - \rho_* u_*^2 - p_* = S\rho_R u_R - S\rho_* u_*$$

$$\rho_R (u_R^2 - u_R S) + p_R - \rho_* (u_*^2 - u_* S) - p_* = 0$$

Use 1st eqn:

$$\rho_R (u_R - S) - \rho_* (u_* - S) = 0$$

$$\Rightarrow S\rho_R (u_R - S) - \rho_* (u_* - S) = 0$$

$$= \rho_R (u_R S - S^2) - \rho_* (u_* S - S^2) = 0$$

Subtract this from the second equation:

$$\rho_R (u_R^2 - 2u_R S + S^2) + p_R - \rho_* (u_*^2 - 2u_*^2 S + S^2) - p_* = 0$$

$$\Rightarrow \boxed{\rho_R \hat{u}_R^2 + p_R = \rho_* \hat{u}_*^2 + p_*}$$

Third equation:

For both R and $*$ indices, we have:

$$u(E+p) - SE = E(u-S) + up =$$
$$= \hat{u}\, E + up = \hat{u}\left(\tfrac{1}{2}\rho u^2 + \rho e\right) + up =$$
$$= \hat{u}\left(\tfrac{1}{2}\rho u^2 + \rho e - \rho u S + \tfrac{1}{2}\rho S^2 + \rho u S - \tfrac{1}{2}\rho S^2\right) + up - Sp + Sp$$
$$= \hat{u}\left(\tfrac{1}{2}\rho(u^2 - 2uS + S^2) + \rho e\right) + (u-S)p + \hat{u}(\rho u S - \tfrac{1}{2}\rho S^2) + Sp$$
$$= \hat{u}\left(\tfrac{1}{2}\rho \hat{u}^2 + \rho e\right) + \hat{u}p + S(\rho u \hat{u} - \tfrac{1}{2}\rho S \hat{u} + p)$$
$$= \hat{u}(\hat{E} + p) + S(\rho \hat{u}(u-S) + p + \tfrac{1}{2}\rho \hat{u} S)$$
$$= \hat{u}(\hat{E} + p) + S(\rho \hat{u}^2 + p) + \tfrac{1}{2}S^2 \rho \hat{u}$$

From the first two equations we know:

$$\rho_R \hat{u}_R = \rho_* \hat{u}_*$$
$$\rho_R \hat{u}_R^2 + p_R = \rho_* \hat{u}_*^2 + p_*$$

Now use that on Rankine Hugeniot:

$$u_R(E_R + p_R) - u_*(E_* + p_*) = SE_R - SE_*$$

$$\Rightarrow u_R(E_R + p_R) - SE_R = u_*(E_* + p_*) - SE_*$$

$$\Rightarrow \hat{u}_R(\hat{E}_R + p_R) + S(\rho_R \hat{u}_R^2 + p_R) + \tfrac{1}{2}S^2 \rho_R u_R =$$
$$= \hat{u}_*(\hat{E}_* + p_*) + S\underbrace{(\rho_* \hat{u}_*^2 + p_*)}_{\text{equal on both sides}} + \tfrac{1}{2}S^2 \underbrace{\rho_* u_*}_{\text{equal on both sides}}$$

$$\Rightarrow \boxed{\hat{u}_R(\hat{E}_R + p_R) = \hat{u}_*(E_* + p_*)}$$

The third equation can be re-written:

$$\hat{u}(\hat{E} + p) = \hat{u}\left(\left(\tfrac{1}{2}\rho\hat{u}^2 + e\right) + p\right) =$$

$$= \hat{u}\rho\left(\tfrac{1}{2}\hat{u}^2 + (e + p/\rho)\right)$$

$$= \hat{u}\rho\left(\tfrac{1}{2}\hat{u}^2 + h\right)$$

with $h = e + p/\rho$ the specific enthalpy. The equation is then

$$\hat{u}_R \rho_R \left(\tfrac{1}{2}\hat{u}_R^2 + h_R\right) = \hat{u}_* \rho_* \left(\tfrac{1}{2}\hat{u}_*^2 + h_*\right)$$

with $\hat{u}_R \rho_R = \hat{u}_* \rho_*$:

$$\Rightarrow \tfrac{1}{2}\hat{u}_R^2 + h_R = \tfrac{1}{2}\hat{u}_*^2 + h_*$$

To find an expression for $\hat{u}_*^2, \hat{u}_R^2$, use the first two equations:

$$\rho_* \hat{u}_*^2 = \rho_R \hat{u}_R^2 + p_R - p_*$$

$$= \frac{(\rho_R \hat{u}_R)^2}{\rho_R} + p_R - p_* = \frac{\rho_*^2 \hat{u}_*^2}{\rho_R} + p_R - p_*$$

$$\rho_* \hat{u}_*^2 \left(1 - \frac{\rho_*}{\rho_R}\right) = \rho_* u_*^2 \left(\frac{\rho_R - \rho_*}{\rho_R}\right) = p_R - p_*$$

$$\Rightarrow \hat{u}_*^2 = \frac{\rho_R}{\rho_*} \frac{p_R - p_*}{\rho_R - \rho_*}$$

analogously: $\hat{u}_R^2 = \dfrac{\rho_*}{\rho_R} \dfrac{p_R - p_*}{\rho_R - \rho_*}$

This gives us: $\tfrac{1}{2}\hat{u}_R^2 + h_R = \tfrac{1}{2}\hat{u}_*^2 + h_*$

$$\rightarrow h_R - h_* = \tfrac{1}{2}\left(\hat{u}_*^2 - \hat{u}_R^2\right) = \frac{1}{2}\frac{p_R - p_*}{\rho_R - \rho_*}\left(\frac{\rho_R}{\rho_*} - \frac{\rho_*}{\rho_R}\right)$$

$$= \frac{1}{2}\frac{\Delta p}{\Delta \rho}\left(\frac{\rho_R^2 - \rho_*^2}{\rho_* \rho_R}\right) = \frac{1}{2}(p_* - p_R)\left(\frac{\rho_R + \rho_*}{\rho_* \rho_R}\right)$$

We can use that result to get an expression for the specific internal energy:

$$h = e + p/\rho$$

$$h_* - h_R = \frac{1}{2}(p_* - p_R)\frac{\rho_* + \rho_R}{\rho_* \rho_R} = e_* + p_*/\rho_* - e_R - p_R/\rho_R$$

$$\Rightarrow e_* - e_R = \frac{1}{2}(p_* - p_R)\frac{\rho_* + \rho_R}{\rho_* \rho_R} + \frac{p_R}{\rho_R} - \frac{p_*}{\rho_*}$$

$$= \frac{1}{2}\frac{1}{\rho_* \rho_R}\left(p_* \rho_* + p_* \rho_R - p_R \rho_* - p_R \rho_R + 2 p_R \rho_* - 2 p_* \rho_R\right)$$

$$= \frac{1}{2}\frac{1}{\rho_* \rho_R}\left(p_* \rho_* - p_* \rho_R + p_R \rho_* - p_R \rho_R\right)$$

$$= \frac{1}{2}\frac{1}{\rho_* \rho_R}\left(p_*(\rho_* - \rho_R) + p_R(\rho_* - \rho_R)\right)$$

$$= \frac{1}{2}\frac{1}{\rho_* \rho_R}(p_* + p_R)(\rho_* - \rho_R)$$

$$= \frac{1}{2}(p_* + p_R)\frac{\rho_* - \rho_R}{\rho_* \rho_R}$$

For ideal gases, we can now use $e = \frac{p}{(\gamma - 1)\rho}$

Then $e_* - e_R = \frac{1}{\gamma - 1}\left(\frac{p_*}{\rho_*}\right) - \frac{1}{\gamma - 1}\left(\frac{p_R}{\rho_R}\right)$

$$= \frac{1}{\gamma - 1}\left(\frac{p_* \rho_R - p_R \rho_*}{\rho_* \rho_R}\right)$$

$$= \frac{1}{2}(p_* + p_R)\frac{\rho_* - \rho_R}{\rho_* \rho_R}$$

$$\Rightarrow \frac{\gamma - 1}{2}(p_* + p_R)(\rho_* - \rho_R) = p_* \rho_R - p_R \rho_*$$

$$p_* S_R - p_R S_* = \frac{\gamma-1}{2}\left(p_* S_* - p_* S_R + p_R S_* - p_R S_R\right)$$

$$\Rightarrow \frac{\gamma-1}{2}\left(p_* S_* - p_R S_R\right) = \frac{\gamma-1}{2}\left(\frac{2}{\gamma-1} p_* S_R - \frac{2}{\gamma-1} p_R S_* + p_* S_R - p_R S_*\right)$$

$$= \frac{\gamma-1}{2}\left(\frac{2+\gamma-1}{\gamma-1} p_* S_R - \frac{2+\gamma-1}{\gamma-1} p_R S_*\right)$$

$$= \frac{\gamma-1}{2}\left(\frac{\gamma+1}{\gamma-1} p_* S_R - \frac{\gamma+1}{\gamma-1} p_R S_*\right)$$

$$\Rightarrow p_* S_* + \frac{\gamma+1}{\gamma-1} p_R S_* = p_R S_R + \frac{\gamma+1}{\gamma-1} p_* S_R$$

$$S_* \left(p_* + \frac{\gamma+1}{\gamma-1} p_R\right) = S_R \left(p_R + \frac{\gamma+1}{\gamma-1}\right) p_*$$

$$\Rightarrow \frac{S_*}{S_R} = \frac{p_R + \frac{\gamma+1}{\gamma-1} p_*}{p_* + \frac{\gamma+1}{\gamma-1} p_R} = \frac{1 + \frac{\gamma+1}{\gamma-1} \frac{p_*}{p_R}}{\frac{p_*}{p_R} + \frac{\gamma+1}{\gamma-1}}$$

$$= \frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_R}}{\frac{\gamma-1}{\gamma+1} \frac{p_*}{p_R} + 1}$$

Or for $\frac{p_*}{p_R}$:

$$p_* S_* + \frac{\gamma+1}{\gamma-1} p_R S_* = p_R S_R + \frac{\gamma+1}{\gamma-1} p_* S_R$$

$$\Rightarrow p_* \left(S_* - \frac{\gamma+1}{\gamma-1} S_R\right) = p_R \left(S_R - \frac{\gamma+1}{\gamma-1} S_*\right)$$

$$\Rightarrow \frac{p_*}{p_R} = \frac{S_R - \frac{\gamma+1}{\gamma-1} S_*}{S_* - \frac{\gamma+1}{\gamma-1} S_R} = \frac{1 - \frac{\gamma+1}{\gamma-1} \frac{S_*}{S_R}}{\frac{S_*}{S_R} - \frac{\gamma+1}{\gamma-1}}$$

This establishes a useful relation between the density and pressure ratios.

The shock speed S_3 can be related to the pressure and density ratios. To demonstrate this, we introduce the Mach numbers:

$$M_R = \frac{u_R}{a_R} \qquad M_S = \frac{S_3}{a_R}$$

with $$u_R^2 = \frac{\rho_*}{\rho_R}\left[\frac{p_R - p_*}{\rho_R - \rho_*}\right] = a_R^2 (M_R - M_S)^2 =$$

$$= \frac{\gamma p_R}{\rho_R}(M_R - M_S)^2$$

$$= \frac{\rho_*}{\rho_R}\frac{p_R}{\rho_R}\left[\frac{1 - p_*/p_R}{1 - \rho_*/\rho_R}\right]$$

Let $\Delta M \equiv (M_R - M_S)$

$$\Rightarrow \frac{\rho_*}{\rho_R} = \gamma (M_R - M_S)^2 \left[\frac{1 - \rho_*/\rho_R}{1 - p_*/p_R}\right] =$$

$$= \gamma \Delta M^2 \frac{1 - \rho_*/\rho_R}{1 - \dfrac{1 - \frac{\gamma+1}{\gamma-1}\frac{\rho_*}{\rho_R}}{\frac{\rho_*}{\rho_R} - \frac{\gamma+1}{\gamma-1}}} =$$

$$= \gamma \Delta M^2 \frac{(1 - \rho_*/\rho_R)\left(\rho_*/\rho_R - \frac{\gamma+1}{\gamma-1}\right)}{\frac{\rho_*}{\rho_R} - \frac{\gamma+1}{\gamma-1} - 1 + \frac{\gamma+1}{\gamma-1}\frac{\rho_*}{\rho_R}}$$

$$= \gamma \Delta M^2 \frac{(1 - \rho_*/\rho_R)\left(\rho_*/\rho_R - \frac{\gamma+1}{\gamma-1}\right)}{\frac{2\gamma}{\gamma-1}\left(\frac{\rho_*}{\rho_R} - 1\right)}$$

$$= \Delta M^2 \frac{\frac{\gamma+1}{\gamma-1} - \frac{\rho_*}{\rho_R}}{2/(\gamma-1)}$$

$$\frac{\rho_*}{\rho_R}\left(1+\frac{\gamma-1}{2}\Delta M^2\right) = \frac{\Delta M^2\,\frac{\gamma+1}{\gamma-1}}{2/\gamma-1} = \frac{\Delta M^2(\gamma+1)}{2}$$

$$\frac{\rho_*}{\rho_R} = \frac{\Delta M^2(\gamma+1)}{2\left(1+\frac{\gamma-1}{2}\Delta M^2\right)} = \underline{\underline{\frac{(\gamma+1)(M_R-M_S)^2}{(\gamma-1)(M_R-M_S)^2+2}}}$$

And for $\frac{p_*}{p_R}$:

$$\frac{p_*}{p_R} = \frac{1-\frac{\gamma+1}{\gamma-1}\frac{\rho_*}{\rho_R}}{\frac{\rho_*}{\rho_R}-\frac{\gamma+1}{\gamma-1}} = \frac{1-\frac{\gamma+1}{\gamma+1}\left(\frac{(\gamma+1)\Delta M^2}{(\gamma-1)\Delta M^2+2}\right)}{\frac{(\gamma-1)\Delta M^2}{(\gamma-1)\Delta M^2+2}-\frac{\gamma+1}{\gamma-1}}$$

$$= \frac{\frac{(\gamma-1)[(\gamma-1)\Delta M^2+2]-(\gamma+1)^2\Delta M^2}{(\gamma-1)[(\gamma-1)\Delta M^2+2]}}{\frac{(\gamma-1)(\gamma+1)\Delta M^2-(\gamma+1)[(\gamma-1)\Delta M^2+2]}{(\gamma-1)[(\gamma-1)\Delta M^2+2]}}$$

$$= \frac{[\gamma^2-2\gamma+1-\gamma^2-2\gamma-1]\Delta M^2+2(\gamma-1)}{-2(\gamma+1)}$$

$$= \frac{-4\gamma\Delta M^2+2(\gamma-1)}{-2(\gamma+1)} = \underline{\underline{\frac{2\gamma(M_R-M_S)^2-(\gamma-1)}{\gamma+1}}}$$

These formulae allow us to relate the shock speed S_3 to the pressure and density ratios:

$$\frac{p_*}{p_R} = \frac{2\gamma\,(M_R - M_S)^2 - (\gamma - 1)}{(\gamma + 1)}$$

$$\Rightarrow (M_R - M_S) = -\sqrt{\frac{\gamma+1}{2\gamma}\,\frac{p_*}{p_R} + \frac{\gamma - 1}{2\gamma}}$$

$M_S > M_R$: condition for shock to form

with $M_R = \frac{u_R}{a_R}$; $M_S = \frac{S_3}{a_R}$:

$$S_3 = u_R + a_R \sqrt{\frac{\gamma+1}{2\gamma}\,\frac{p_*}{p_R} + \frac{\gamma - 1}{2\gamma}}$$

Furthermore, using $\hat{u}_* \varrho_* = \hat{u}_R \varrho_R$:

$$\hat{u}_* = \hat{u}_R \frac{\varrho_R}{\varrho_*} = (u_R - S_3)\frac{\varrho_R}{\varrho_*} = (u_* - S_3)$$

$$\Rightarrow u_* = (u_R - S_3)\frac{\varrho_R}{\varrho_*} + S_3 = \underline{\underline{\left(1 - \frac{\varrho_R}{\varrho_*}\right) S_3 + u_R \frac{\varrho_R}{\varrho_*}}}$$

giving us an expression for the particle velocity u_* after the shock wave.

The derivation for the left-facing S_1 shockwave are completely analogous.