The Riemann Problem for the Euler Equations: Exact Solution

There is no exact closed-form solution to the Riemann problem for the Euler equations, not even for ideal gases. However, it is possible to devise iterative schemes whereby the solution can be computed numerically.

1) Solution Strategy

Riemann problem for 1D Euler equations:

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

$$\underline{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \qquad \underline{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix}$$

$$\underline{U}(x, 0) = \underline{U}^{(0)}(x) = \begin{cases} \underline{U}_L & \text{if } x < 0 \\ \underline{U}_R & \text{if } x > 0 \end{cases}$$

or with primitive variables:

$$\underline{W} = [\rho, u, p]^T$$

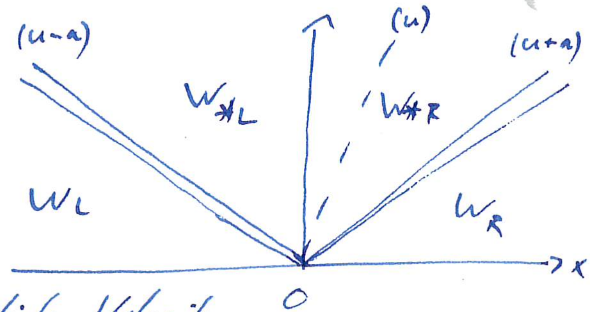
For ideal gases: $e = \frac{p}{(\gamma - 1)\rho}$

When no vacuum is present, the exact solution of the Riemann problem has three waves, which are associated with the Eigenvalues

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a$$

In general, the speeds of these waves are not the characteristics' speeds given by the eigenvalues.

The middle wave is always a contact wave, while the left and right waves are either shock or rarefaction waves.



2) Equations for Pressure and Particle Velocity

The solution for pressure p_* of the Riemann problem is given by the root of the algebraic equation

$$f(p, \underline{W}_L, \underline{W}_R) \equiv f_L(p, \underline{W}_L) + f_R(p, \underline{W}_R) + \Delta u = 0$$

$$\Delta u \equiv u_R - u_L$$

$$f_{L,R}(\rho, \underline{W}_{L,R}) = \begin{cases} (p - p_{L,R}) \left[\dfrac{A_{L,R}}{p + B_{L,R}} \right]^{1/2} & \text{if } p > p_{L,R} \text{ (shock)} \\[2ex] \dfrac{2 a_{L,R}}{(\gamma - 1)} \left[\left(\dfrac{p}{p_{L,R}} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right] & \text{if } p \le p_{L,R} \text{ (rarefaction)} \end{cases}$$

with $A_{L,R} = \dfrac{2}{(\gamma+1)\rho_{L,R}}$; $B_{L,R} = \dfrac{(\gamma-1)}{(\gamma+1)} p_{L,R}$

The solution for the particle velocity u_* in the star region is

$$u_* = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[f_R(p_*) - f_L(p_*)\right]$$

$$\text{with } f_{R,L} = \frac{2a_{R,L}}{\gamma - 1}\left[\left(\frac{p_*}{p_{L,R}}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] \qquad [*]$$

Once the equations for p_* and u_* are solved, the remaining unknowns are found by using standard gas dynamics relations. Here for $\rho_{*L,R}$:

If $p_* > p_{L,R}$ (shock):

$$\rho_{*L,R} = \rho_{L,R}\left[\frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_{L,R}}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_{L,R}} + 1}\right]$$

If $p_* \leq p_{L,R}$ (rarefaction):

$$\rho_{*L,R} = \rho_{L,R}\left(\frac{p_*}{p_{R,L}}\right)^{1/\gamma}$$

Remember that across the contact wave, p^* and u^* are constant, so you only need to solve 1 equation which contains both $\underline{W}_L$ and $\underline{W}_R$!

[*] You can also separate the equations:
$u_* = u_L - f_L(p_*)$, $u_* = u_R + f_R(p_*)$

2.1) f_L for a left shock

Define comoving frame of reference:

$$\hat{u}_L = u_L - S_L, \quad \hat{u}_* = u_* - S_L$$

Using Rankine - Hugoniot:

$$\rho_L \hat{u}_L = \rho_{*L} \hat{u}_*$$

$$\rho_L \hat{u}_L^2 + p_L = \rho_{*L} \hat{u}_*^2 + p_*$$

$$\hat{u}_L (\hat{E}_L + p_L) = \hat{u}_* (\hat{E}_{*L} + p_*)$$

Define mass flux $Q_L \equiv \rho_L \hat{u}_L = \rho_{*L} \hat{u}_*$

Use $\rho_L \hat{u}_L^2 + p_L = \rho_{*L} \hat{u}_*^2 + p_*$

$$= (\rho_L \hat{u}_L) \hat{u}_L + p_L = (\rho_{*L} \hat{u}_* + p_*)$$

$$= Q_L \hat{u}_L + p_L = Q_L \hat{u}_* + p_*$$

$$\Rightarrow \boxed{\begin{aligned} Q_L &= \frac{p_* - p_L}{\hat{u}_L - \hat{u}_*} = - \frac{p_* - p_L}{\hat{u}_* - \hat{u}_L} \\ u_* &= u_L - \frac{p_* - p_L}{Q_L} \end{aligned}}$$

Now substitute $\hat{u}_L = Q_L/\varrho_L$, $\hat{u}_* = \frac{Q_L}{\varrho_{*L}}$

$$\rightarrow Q_L = - \frac{p_* - p_L}{\hat{u}_L - \hat{u}_*} = - \frac{p_* - p_L}{Q_L/\varrho_L - \frac{Q_L}{\varrho_{*L}}}$$

$$\Rightarrow Q_L^2 = - \frac{p_* - p_L}{1/\varrho_L - 1/\varrho_{*L}} = - \frac{p_* - p_L}{\frac{\varrho_{*L} - \varrho_L}{\varrho_* \varrho_L}} = \frac{- \varrho_* \varrho_L (p_* - p_L)}{\varrho_{*L} - \varrho_L}$$

Now using $\varrho_{*L} = \varrho_L \left[\frac{\frac{\gamma - 1}{\gamma + 1} + \frac{p_*}{p_L}}{\frac{\gamma-1}{\gamma+1} \frac{p_*}{p_L} + 1} \right]$:

$$Q_L^2 = \frac{- \varrho_L^2 \left(\frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_L}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_L} + 1} \right) (p_* - p_L)}{\varrho_L \left[\frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_L}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_L} + 1} - 1 \right]}$$

$$= \frac{- \varrho_L^2 \left(\frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_L}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_L} + 1} \right) (p_* - p_L)}{\varrho_L \left[\frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_L} - \frac{\gamma-1}{\gamma+1}\frac{p_*}{p_L} - 1}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_L} + 1} \right]}$$

$$= \frac{- \varrho_L \left(\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_L} \right) (p_* - p_L)}{\left(\frac{\gamma-1}{\gamma+1} - 1 \right) \left(1 - \frac{p_*}{p_L} \right)}$$

$$= - \frac{\varrho_L \left(\frac{\gamma-1}{\gamma+1} p_L + p_* \right) (p_* - p_L)}{\frac{-2}{\gamma+1} (p_L - p_*)}$$

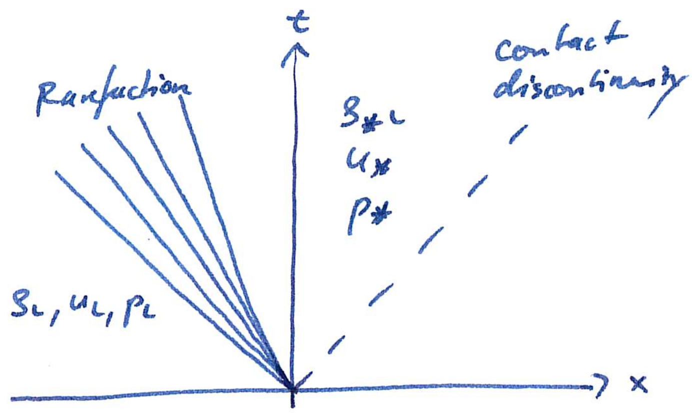
$$Q_L^2 = \frac{\frac{\gamma-1}{\gamma+1} p_L + p_*}{\frac{2}{(\gamma+1)} \frac{1}{\rho_L}} = \frac{p_* + B_L}{A_L}$$

with $A_L \equiv \frac{2}{(\gamma+1)\rho_L}$, $B_L = \frac{\gamma-1}{\gamma+1} p_L$

$$\Rightarrow \boxed{u_* = u_L - \frac{p_* - p_L}{\sqrt{\frac{p_* + B_L}{A_L}}} = u_L - f_L(p_*, \underline{W}_L)}$$

2.2) function f_L for left rarefaction

For rarefactions, $\underline{W}_L$ is connected to $\underline{W}_{*L}$ via the isentropic law and the generalized Riemann invariants.



Isentropic law: $\quad p = C \rho^\gamma$

C is a constant and evaluated at the initial left data state:

$$p_L = C \rho_L^\gamma \Rightarrow C = p_L / \rho_L^\gamma$$

$$\Rightarrow p_* = C \rho_{*L}^\gamma = \frac{p_L}{\rho_L^\gamma} \rho_{*L}^\gamma$$

$$\Rightarrow \boxed{\rho_{*L} = \rho_L \left(\frac{p_*}{p_L}\right)^{1/\gamma}}$$

Now use the result from generalized Riemann invariants:

$$u + \frac{2a}{\gamma - 1} = \text{const} \quad \text{across wave}$$

$$\Rightarrow \quad u_L + \frac{2a_L}{\gamma - 1} = u_* + \frac{2a_{*L}}{\gamma - 1}$$

Using the definition for the sound speed a:

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \overset{\text{ideal gas}}{=} \sqrt{\frac{\gamma p}{\rho}}$$

$$a_L = \sqrt{\frac{\gamma p_L}{\rho_L}}, \quad a_{*L} = \sqrt{\frac{\gamma p_{*L}}{\rho_{*L}}} \overset{p = c\rho^\gamma}{=} \sqrt{\frac{\gamma p_{*L}}{\rho_L \left(\frac{p_{*L}}{p_L}\right)^{1/\gamma}}}$$

$$= \sqrt{\frac{\gamma p_{*L}}{\rho_L \, p_*^{1/\gamma} p_L^{-1/\gamma}}} = \sqrt{\frac{\gamma}{\rho_L} p_{*L}^{1-1/\gamma} p_L^{1/\gamma}} =$$

$$= \sqrt{\frac{\gamma}{\rho_L} p_{*L}^{1-1/\gamma} p_L^{1/\gamma - 1} p_L} = \sqrt{\frac{\gamma p_L}{\rho_L}\left(\frac{p_{*L}}{p_L}\right)^{1-1/\gamma}}$$

$$= a_L \left(\frac{p_{*L}}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}$$

This gives us:

$$u_L + \frac{2a_L}{\gamma - 1} = u_* + \frac{2a_{*L}}{\gamma - 1}$$

$$\Rightarrow u_* = u_L + \frac{2a_L}{\gamma-1} - \frac{2a_{*L}}{\gamma-1} = u_L + \frac{2}{\gamma-1}(a_L - a_{*L})$$

$$\boxed{\begin{aligned} &= u_L + \frac{2a_L}{\gamma-1}\left[1 - \left(\frac{p_{*L}}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}\right] \\ &= u_L - f_L(p_{*L}, \underline{W}_L) \end{aligned}}$$

2.3) Solution inside the rarefaction wave

Across the rarefaction wave, we have the relations

$$S_{*L} = S_L \left(\frac{\rho_*}{\rho_L}\right)^{1/\gamma}$$
$$a_{*L} = a_L \left(\frac{\rho_*}{\rho_L}\right)^{\frac{\gamma-1}{2\gamma}}$$

The rarefaction wave is enclosed by the head and the tail, which are characteristics of speeds given by

$$S_{HL} = u_L - a_L\,, \quad S_{TL} = u_* - a_{*L}$$

We now find the solution for $\underline{W}_{L\,fan} = (S, u, \rho)^T$ inside the left rarefaction wave.

Consider the characteristic ray through the origin $(0,0)$ and a general point (x,t) inside the fan. The slope of the characteristic is

$$\frac{dx}{dt} = \frac{x}{t} = u - a$$

we are looking for the values of particle speed u and sound speed a.

The generalized Riemann invariants also hold:

$$u_L + \frac{2a_L}{\gamma-1} = u + \frac{2a}{\gamma-1}$$

This gives us:

$$u = \frac{x}{t} + a = u_L + \frac{2a_L}{\gamma} - \frac{2a}{\gamma-1}$$

$$\Rightarrow u_L + \frac{2a_L}{\gamma-1} - \frac{x}{t} = a\left(1 + \frac{2}{\gamma-1}\right) = a\,\frac{\gamma+1}{\gamma-1}$$

$$\Rightarrow \quad a = \frac{\gamma-1}{\gamma+1}\left(u_L + \frac{2a_L}{\gamma-1} - \frac{x}{t}\right)$$

$$\Rightarrow u = \frac{x}{t} + a = \frac{x}{t} + \frac{\gamma-1}{\gamma+1}\left(u_L + \frac{2a_L}{\gamma-1} - \frac{x}{t}\right) =$$

$$= \frac{x}{t}\left(1 - \frac{\gamma-1}{\gamma+1}\right) + \frac{\gamma-1}{\gamma+1}\left(u_L + \frac{2a_L}{\gamma-1}\right)$$

$$= \frac{2}{\gamma+1}\,\frac{x}{t} + \frac{\gamma-1}{\gamma+1}\left(u_L + \frac{2a_L}{\gamma-1}\right)$$

$$= \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}u_L + a_L + \frac{x}{t}\right)$$

Now using again that $p = C\rho^\gamma$ and $a = \sqrt{\frac{\gamma p}{\rho}}$,

we get once again $a = a_L\left(\frac{p}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}$

$$\Rightarrow \quad \frac{p}{p_L} = \left(\frac{a}{a_L}\right)^{\frac{2\gamma}{\gamma-1}}$$

$$\Rightarrow \quad p = p_L\left(\frac{a}{a_L}\right)^{\frac{2\gamma}{\gamma-1}} = p_L\left[\frac{1}{a_L}\left(\frac{\gamma-1}{\gamma+1}\right)\left(u_L + \frac{2a_L}{\gamma-1} - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}}$$

$$= p_L\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{1}{a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}}$$

$$\Rightarrow \quad \rho = \rho_L\left(\frac{p}{p_L}\right)^{1/\gamma} = \rho_L\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{1}{a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}}$$

To summarize:

$$\boxed{\underline{W_{L\,fan}} = \begin{cases} \rho = \rho_L\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{1}{a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}u_L + a_L + \frac{x}{t}\right) \\ p = p_L\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{1}{a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases}}$$

3) Numerical Solution for Pressure

3.1) Behaviour of f

To find the pressure p^* in the star region, we need to solve

$$f(p, \underline{W}_L, \underline{W}_R) \equiv f_L(p, \underline{W}_L) + f_R(p, \underline{W}_R) + \Delta u = 0$$

with
$$f_{L,R} = \begin{cases} (p - p_{L,R})\left[\frac{A_{L,R}}{p + B_{L,R}}\right]^{1/2} & \text{if } p > p_{L,R} \text{ shock} \\ \frac{2 a_{L,R}}{(\gamma - 1)}\left[\left(\frac{p}{p_{L,R}}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] & \text{if } p \leq p_{L,R} \text{ rarefaction} \end{cases}$$

and $A_{L,R} = \frac{2}{(\gamma+1) \rho_L}$, $B_{L,R} = \frac{\gamma - 1}{\gamma + 1} p_L$

any standard technique can be used. The behaviour of the pressure function $f(p)$ plays a fundamental role in finding its roots numerically.

Let $K = (L, R)$

i) Check for monotonicity:

i.i) shocks

$$\frac{d f_K}{dp} = f'_K = \left[\frac{A_K}{p + B_K}\right]^{1/2} + (p - p_K)\frac{1}{2}\left[\frac{A_K}{p + B_K}\right]^{-1/2} \frac{A_K}{(p + B_K)^2}$$

$$= \left[\frac{A_K}{p + B_K}\right]^{1/2} + \frac{1}{2}\frac{(p - p_K) A_K^{1/2}}{(p + B_K)^{3/2}} = \left[\frac{A_K}{p + B_K}\right]^{1/2}\left(1 + \frac{p - p_K}{2(p + B_K)}\right)$$

shocks occur for $p > p_K$

$\Rightarrow f'_K > 0 \quad \forall p > p_K$

i.ii) Rarefactions

$$\frac{df_K}{dp} = f_K' = \frac{d}{dp}\left[\frac{2a_K}{\gamma-1}\left[\left(\frac{p}{p_K}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]\right]$$

$$= \frac{2a_K}{\gamma-1}\,\frac{\gamma-1}{2\gamma}\left[\frac{1}{p_K^{\frac{\gamma-1}{2\gamma}}}\, p^{\frac{\gamma-1}{2\gamma}-1}\right] =$$

$$= \frac{a_K}{\gamma}\,\frac{1}{p_K}\left(\frac{p}{p_K}\right)^{\frac{\gamma-1}{2\gamma}-1} = \frac{a_K}{\gamma p_K}\left(\frac{p}{p_K}\right)^{-\frac{(\gamma+1)}{2\gamma}}$$

$$= \sqrt{\frac{\gamma p_K}{\rho_K}}\,\frac{1}{\gamma p_K}\left(\frac{p}{p_K}\right)^{-\frac{(\gamma+1)}{2\gamma}} = \frac{1}{\sqrt{\rho_K \gamma p_K}}\left(\frac{p}{p_K}\right)^{-\frac{(\gamma+1)}{2\gamma}}$$

$$= \frac{1}{\rho_K a_K}\left(\frac{p}{p_K}\right)^{-\frac{(\gamma+1)}{2\gamma}}$$

which is $>0 \; \forall \, p$

$\Rightarrow$ f_K' is $>0 \; \forall p$ and therefore monotone.

ii) Check for concavity

ii.i) shocks

$$f''_K = \frac{d}{d\rho}\left[\left(\frac{A_K}{B_K+\rho}\right)^{1/2}\left[1-\frac{\rho-\rho_K}{2(B_K+\rho)}\right]\right]$$

$$= A_K^{1/2}(B_K+\rho)^{-3/2}\cdot\frac{-1}{2}\left[1-\frac{\rho-\rho_K}{2(B_K+\rho)}\right] +$$

$$+\left(\frac{A_K}{B_K+\rho}\right)^{1/2}\cdot\left[-\frac{2(B_K+\rho)-(\rho-\rho_K)\cdot 2}{4(B_K+\rho)^2}\right]$$

$$= \left(\frac{A_K}{B_K+\rho}\right)^{1/2}\left[\frac{-1}{2(B_K+\rho)}\;\frac{-\rho+\rho_K}{4(B_K+\rho)^2}-\frac{B_K+\rho-\rho+\rho_K}{2(B_K+\rho)^2}\right]$$

$$= \left(\frac{A_K}{B_K+\rho}\right)^{1/2}\left[\frac{-2B_K-2\rho-\rho+\rho_K-2B_K-2\rho_K}{4(B_K+\rho)^2}\right]$$

$$= -\left(\frac{A_K}{B_K+\rho}\right)^{1/2}\left[\frac{4B_K+3\rho+\rho_K}{4(B_K+\rho)^2}\right]$$

ii.ii) Rarefactions

$$f''_K = \frac{d}{d\rho}\left[\frac{1}{\varsigma_K a_K}\left(\frac{\rho}{\rho_K}\right)^{-\frac{(\gamma+1)}{2\gamma}}\right] =$$

$$= \frac{1}{\varsigma_K a_K \rho_K^{-\frac{(\gamma+1)}{2\gamma}}}\;\frac{-(\gamma+1)}{2\gamma}\;\rho^{-\frac{(\gamma+1)}{2\gamma}-1}$$

$$= -\frac{(\gamma+1)}{2\gamma\,\varsigma_K a_K\rho_K}\left(\frac{\rho}{\rho_K}\right)^{\frac{-\gamma-1-2\gamma}{2\gamma}}$$

$$\varsigma_K a_K = \varsigma_K\sqrt{\frac{\rho_K\gamma}{\varsigma_K}} = \sqrt{\varsigma_K\rho_K\gamma}$$

$$\Rightarrow \varsigma_K\rho_K\gamma = \varsigma_K^2 a_K^2$$

$$= -\frac{(\gamma+1)}{2\varsigma_K^2 a_K^3}\left(\frac{\rho}{\rho_K}\right)^{-\frac{(3\gamma-1)}{2\gamma}}$$

$$a_K^4 = \frac{\gamma^2\rho_K^2}{\varsigma_K^2}$$

$$= -\frac{(\gamma+1)a_K}{2\gamma^2\rho_K^2}\left(\frac{\rho}{\rho_K}\right)^{-\frac{(3\gamma-1)}{2\gamma}}$$

Clearly $f_K'' < 0 \quad \forall p > 0$

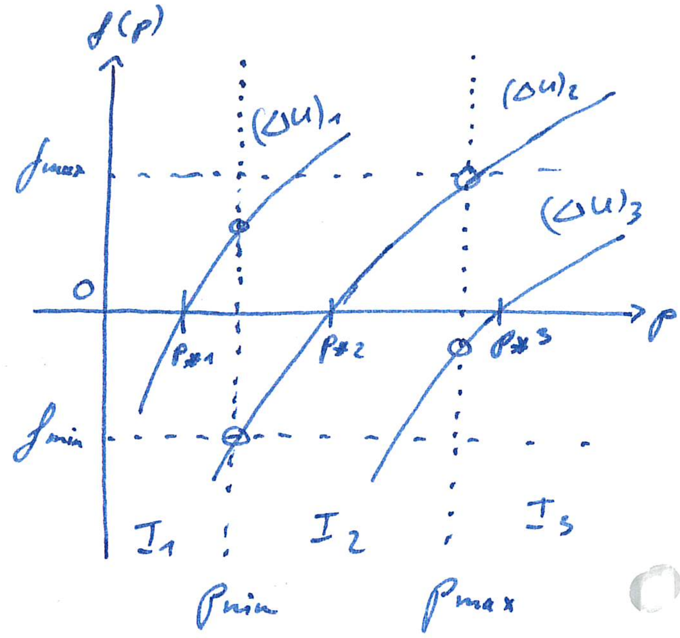
$f(p)$ is monotone and concave

Define $p_{min} = \min(p_L, p_R)$

$p_{max} = \max(p_L, p_R)$

$f_{min} = f(p_{min})$

$f_{max} = f(p_{max})$



For given p_L, p_R, it is so that determines the value of p_*. We can identify three intervals:

p_* in $I_1 = (0, p_{min})$ if $f_{min} > 0$ and $f_{max} > 0$
$p_* < p_{L,R} \to 2$ rarefactions

p_* in $I_2 = [p_{min}, p_{max}]$ if $f_{min} \leq 0$ and $f_{max} \geq 0$
$p_{min} < p_* < p_{max}$: 1 shock, 1 rarefaction

p_* in $I_3 = (p_{max}, \infty)$ if $f_{min} < 0$ and $f_{max} < 0$
$p_* > p_{L,R}$: 2 shocks

To obtain a positive solution, we require $f(0) < 0$. Since it's monotone and convex, the only possible p_* will be > 0. At $p_* = 0$, we have rarefactions, so we have:

$$f(\rho, \underline{W}_L, \underline{W}_R) = f_L(\rho, \underline{W}_L) + f_R(\rho, \underline{W}_R) + \Delta u$$
$$= \frac{2a_L}{\gamma-1}\left[\left(\frac{\rho}{\rho_L}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] + \frac{2a_R}{\gamma-1}\left[\left(\frac{\rho}{\rho_R}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] + \Delta u$$

Then $f(0) = \frac{2a_L}{\gamma-1}[0-1] + \frac{2a_R}{\gamma-1}[0-1] + \Delta u$

$$= -\frac{2a_L}{\gamma-1} - \frac{2a_R}{\gamma-1} + \Delta u \overset{!}{<} 0$$

$$\Rightarrow \boxed{(\Delta u)_{crit} = \frac{2}{\gamma-1}(a_L + a_R) > \Delta u = u_R - u_L}$$

If this condition is violated, vacuum is created by the non-linear waves. Then we need another method for the solution.

3.2) Iterative Scheme for Finding the Pressure

Given the simple behaviour of the pressure function f and the availability of analytical expressions for the derivative of $f(p)$, we use a Newton-Raphson iterative procedure to find the root of $f(p)=0$.

Suppose a guess value p_0 for the true solution p_* is available. Then the approximate value of $f(p)$ at $p_0+\delta$ is

$$f(p_0+\delta) = f(p_0) + \delta f'(p_0) + O(\delta^2)$$

If $p_0+\delta$ is a solution of $f(p)=0$ then

$$f(p_0) + \delta f'(p_0) = 0$$

then the corrected value $p_1 = p_0 + \delta = p_0 - \dfrac{f(p_0)}{f'(p_0)}$

The iteration procedure is stopped when the relative pressure change

$$\frac{|p_{(k)} - p_{(k-1)}|}{\frac{1}{2}|p_k + p_{k-1}|} < \varepsilon$$

What we need now is a guess value for the pressure, p_0.

i) Two-Rarefaction approximation:

$$p_{TR} = \left[\frac{a_L + a_R - \frac{1}{2}(\gamma - 1)(u_R - u_L)}{a_L / p_L^{\frac{\gamma-1}{2\gamma}} + a_R / p_R^{\frac{\gamma-1}{2\gamma}}} \right]^{\frac{2\gamma}{\gamma-1}}$$

exact function if both non-linear waves are rarefactions

ii) Primitive Variable Linearisation

$$p_0 = \max(\varepsilon, p_{PV})$$

$$p_{PV} = \frac{1}{2}(p_L + p_R) - \frac{1}{8}(u_R - u_L)(\rho_L + \rho_R)(a_L + a_R)$$

iii) Two-Shock-approximation

$$p_0 = \max(\varepsilon, p_{TS})$$

$$p_{TS} = \frac{g_L(\hat{p})p_L + g_R(\hat{p})p_R - \Delta u}{g_L(\hat{p}) + g_R(\hat{p})}$$

$$g_K(p) = \left(\frac{A_K}{p + B_K}\right)^{1/2}$$

$\hat{p}$ = some other guess

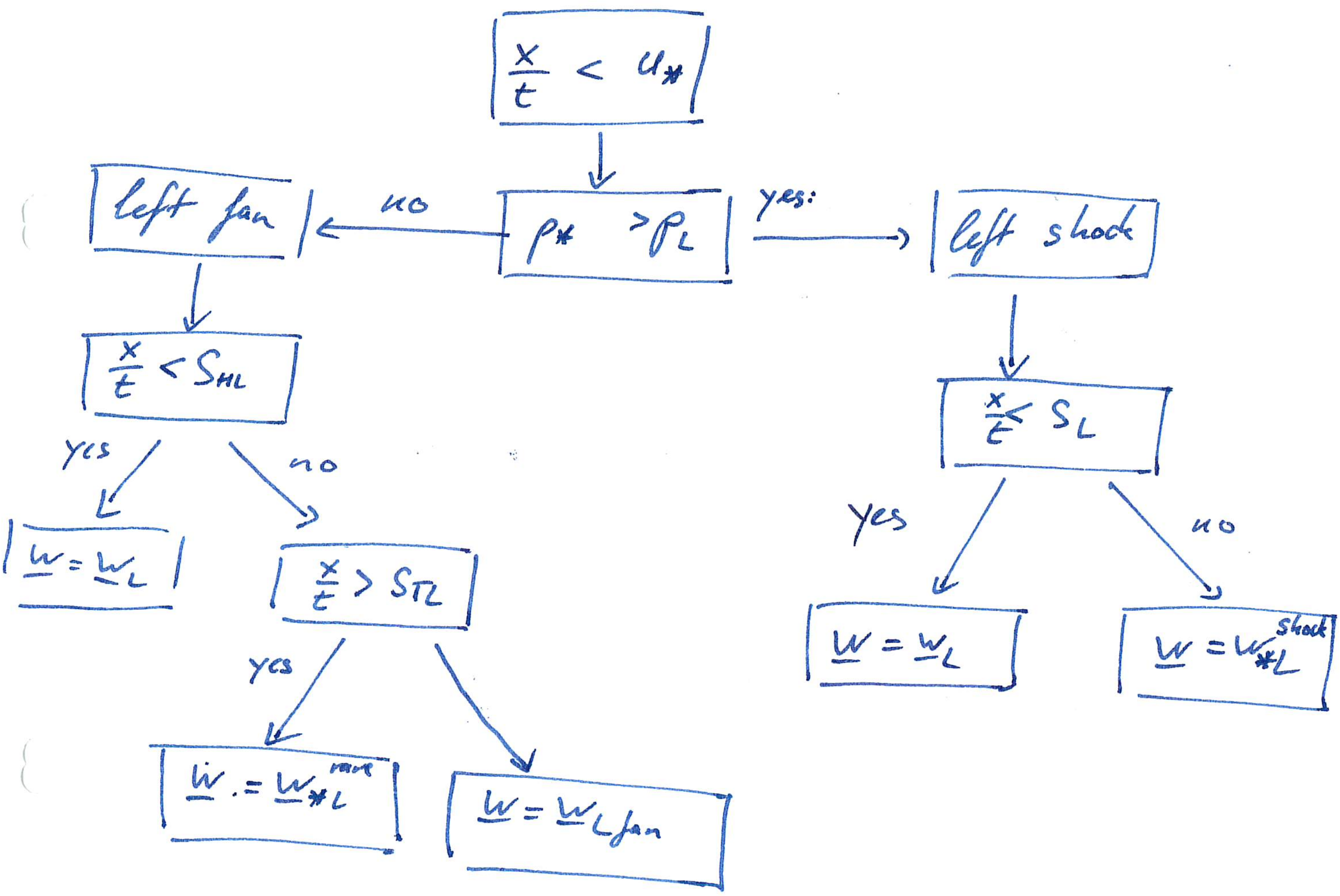
iv) Arithmetic mean

$$p_0 = \frac{1}{2}(p_L + p_R)$$

4) Sampling the Solution

We want to evaluate the solution at a general point (x, t).

Flowchart for left side:



S_{HL} : Head of left rarefaction wave

S_{TL} : Tail of left rarefaction wave

S_L : Speed of shock wave

The complete solution formulas are: for a given p^*, u^*

i) Left shock:

$$\rho_{*L} = \rho_L \left(\frac{\frac{p^*}{p_L} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p^*}{p_L} + 1} \right)$$

$$S_L = u_L - \frac{Q_L}{\rho_L} = u_L - a_L \left[\frac{\gamma+1}{2\gamma} \frac{p^*}{p_L} + \frac{\gamma-1}{2\gamma} \right]^{\frac{1}{2}}$$

ii) Left Rarefaction:

$$\rho_{*L} = \rho_L \left(\frac{p^*}{p_L} \right)^{1/\gamma}$$

$$a_{*L} = a_L \left(\frac{p^*}{p_L} \right)^{\frac{\gamma-1}{2\gamma}}$$

$$S_{HL} = u_L - a_L \,, \quad S_{TL} = u_* - a_{*L}$$

iii) Inside the rarefaction

$$\rho = \rho_L \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)a_L} \left(u_L - \frac{x}{t} \right) \right]^{\frac{2}{\gamma-1}}$$

$$u = \frac{2}{\gamma+1} \left[a_L + \frac{\gamma-1}{2} u_L + \frac{x}{t} \right]$$

$$p = p_L \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)a_L} \left(u_L - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma-1}}$$

iv) Right shock:

$$\rho_{*R} = \rho_R \left[\frac{\frac{p_*}{p_R} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_R} + 1} \right]$$

$$S_R = u_R + \frac{Q_R}{\rho_R} = u_R + a_R \left[\frac{\gamma+1}{2\gamma}\frac{p_*}{p_R} + \frac{\gamma-1}{2\gamma} \right]^{1/2}$$

v) Right Rarefaction

$$\rho_{*R} = \rho_R \left(\frac{p_*}{p_R} \right)^{1/\gamma}$$

$$a_{*R} = a_R \left(\frac{p_*}{p_R} \right)^{\frac{\gamma-1}{2\gamma}}$$

$$S_{HR} = u_R + a_R, \qquad S_{TR} = u_* + a_{*R}$$

vi) Inside Right Rarefaction:

$$\rho = \rho_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{(\gamma+1)a_R}\left(u_R - \frac{x}{t}\right) \right]^{\frac{2}{\gamma-1}}$$

$$u = \frac{2}{\gamma+1}\left[-a_R + \frac{\gamma-1}{2}u_R + \frac{x}{t} \right]$$

$$p = p_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{(\gamma+1)a_R}\left(u_R - \frac{x}{t}\right) \right]^{\frac{2\gamma}{\gamma-1}}$$

5) The Riemann Problem in the Presence of Vacuum

Vacuum is characterized by the condition $\rho = 0$, and therefore $E = \rho(\frac{1}{2}u^2 + e) = 0$.

In the presence of vacuum, the structure of the solution of the Riemann problem is different from that of the conventional case: There is no star region. Using very small values for ρ also doesn't work well.

A shock wave cannot be adjacent to a vacuum region. Consider a left non-vacuum state $\underline{W}_L$ and an adjacent right vacuum state $\underline{W}_0$ at $t=0$. Assuming the states are connected through a shock with speed S and applying Rankine-Hugoniot:

$$\rho_L u_L - \rho_0 u_0 = S(\rho_L - \rho_0)$$

$$\rho_L u_L^2 + p_L - (\rho_0 u_0^2 + p_0) = S(\rho_L u_L - \rho_0 u_0)$$

$$u_L(E_L + p_L) - u_0(E_0 + p_0) = S(E_L - E_0)$$

assume u_0 is finite. We know that $\rho_0 = E_0 = 0$

$$\Rightarrow \quad \rho_L u_L = S \rho_L$$

$$\rho_L u_L^2 + p_L - p_0 = S \rho_L u_L$$

$$u_L(E_L + p_L) - u_0 p_0 = S E_L$$

$\Rightarrow \quad u_L = S$

$\Rightarrow \quad \rho_L S^2 + p_L - p_0 = \rho_L S^2 \quad \Rightarrow p_L = p_0$

$\Rightarrow \quad u_L E_L + u_L p_L - u_0 p_L = u_L E_L \quad \Rightarrow u_L = u_0$

$\Rightarrow$ Since $p_L = p_0$, it can't be a shock wave.

With $u_L = u_0$ and $p_L = p_0$, a contact discontinuity. The wave separates a region of material from a region of no material and is therefore a boundary. The front has the velocity u_0.

We denote $\underline{W}_0 = (\rho_0, u_0, p_0)^T = (0, u_0, 0)^T$
for an ideal gas, $a(\rho=0, p=0) = 0$, but this is not necessarily generally the case, it depends on the equation of state.

We consider three cases:)

i) <u>Vacuum Right State</u>

$$\underline{W}(x,0) = \begin{cases} \underline{W}_L \neq \underline{W}_0 & \text{if } x < 0 \\ \underline{W}_0 & \text{if } x > 0 \end{cases}$$

The structure of the solution consists of a left rarefaction wave and a contact wave that coalesces with the tail of the rarefaction.

Across the contact wave: $\Delta u = \Delta p = 0$

Since we chose an EoS of the form $p = p(\rho)$, the primitive variables of the contact wave are the same as in the vacuum:

$$\Delta p = 0 \Rightarrow p_{contact} = p_0 \;;\; p = p(\rho) \Rightarrow \rho_{contact} = \rho_0$$
$$\Delta u = 0 \Rightarrow u_{contact} = u_0$$

Now applying our rules over the rarefaction wave:

$$u_0 + \frac{2a_0}{\gamma - 1} = u_L + \frac{2a_L}{\gamma - 1} \overset{a_0 = 0}{=} u_0$$

$\Rightarrow$ The speed of the front is

$$S_{*L} \equiv u_0 = u_L + \frac{2a_L}{\gamma - 1}$$

This gives us the total solution:

$$\underline{W} \equiv \underline{W}_{L0} = \begin{cases} \underline{W}_L & \text{if } \frac{x}{t} \leq u_L - a_L \\ \underline{W}_{L fan} & \text{if } u_L - a_L < \frac{x}{t} < S_{*L} \\ \underline{W}_0 & \text{if } \frac{x}{t} \geq S_{*L} \end{cases}$$

ii) Left Vacuum State

Following the same argumentation:

$$\underline{W} \equiv \underline{W}_{R0} = \begin{cases} \underline{W}_0 & \text{if } \frac{x}{t} \leq S_{*R} \\ \underline{W}_{R\,fan} & \text{if } S_{*R} < \frac{x}{t} < u_R + a_R \\ \underline{W}_R & \text{if } x/t \geq u_R + a_R \end{cases}$$

with $S_{*R} = u_R - \dfrac{2a_R}{\gamma - 1}$

iii) Generation of Vacuum

This case has general data $\underline{W}_L = (\rho_L, u_L, p_L)^T \neq \underline{W}_0$ and $\underline{W}_R = (\rho_R, u_R, p_R)^T \neq \underline{W}_0$ but has combinations of particle and sound speeds that are such that vacuum is generated as part of the interaction between $\underline{W}_L$ and $\underline{W}_R$.

The structure of the solution are two rarefaction waves and two contact waves of speeds S_{*L} and S_{*R} that enclose the generated vacuum state.

Full solution: $$\underline{W}(x,t) = \begin{cases} \underline{W}_{L0} & \text{if } \frac{x}{t} \leq S_{*L} \\ \underline{W}_0 & \text{if } S_{*L} < \frac{x}{t} < S_{*R} \\ \underline{W}_{R0} & \text{if } \frac{x}{t} \geq S_{*R} \end{cases}$$

where $\underline{W}_{R0}$, $\underline{W}_{L0}$ are the solutions of the previous solution

For the solution to be applicable, we must have $S_{*L} \le S_{*R}$, which implies

$$(\Delta u_{crit}) \equiv \frac{2a_L}{\gamma-1} + \frac{2a_R}{\gamma-1} \le u_R - u_L$$

which is consistent with the pressure positivity condition: If the pressure positivity condition is violated, we will get a vacuum.

The full solution for the vacuum generating case is:

$$\underline{W}(x,t) = \begin{cases} \underline{W}_{L0} & \text{if } \frac{x}{t} \le S_{*L} \quad \begin{cases} \underline{W}_L & \text{if } \frac{x}{t} \le u_L - a_L \\ \underline{W}_{L fan} & \text{if } u_L - a_L < \frac{x}{t} < S_{*L} \\ \underline{W}_0 & \text{if } \frac{x}{t} \ge S_{*L} \end{cases} \\ \underline{W}_0 & \text{if } S_{*L} \le \frac{x}{t} < S_{*R} \\ \underline{W}_{R0} & \text{if } \frac{x}{t} \ge S_{*R} \quad \begin{cases} \underline{W}_0 & \text{if } x \le S_{*R} \\ \underline{W}_{R fan} & \text{if } S_{*R} < \frac{x}{t} < u_R + a_R \\ \underline{W}_R & \text{if } \frac{x}{t} \ge u_R + a_R \end{cases} \end{cases}$$