The Method of Godunov for Non-Linear Schemes

The essential ingredient of Godunov's method is the solution of the Riemann problem, which may be the exact solution or some suitable approximation to it.

1) Bases of Godunov's Method

Consider the general initial-boundary value problem for non-linear systems of hyperbolic conservation laws:

PDEs: $\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$

ICs: $\underline{U}(x,0) = \underline{U}^{(0)}(x)$

BCs: $\underline{U}(0,t) = \underline{U}_l(t), \quad \underline{U}(L,t) = \underline{U}_r(t)$

In order to admit discontinuous solutions we must use one of the integral forms of the conservation laws:

$$\int_{x_1}^{x_2} \underline{U}(x,t_2)\,dx = \int_{x_1}^{x_2} \underline{U}(x,t_1)\,dx + \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_1,t))\,dt - \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_2,t))\,dt$$

which follows from integrating the PDE:

$$\underline{U}_t + \underline{F}(\underline{U})_x = 0 \Rightarrow \iint_{xt} \underline{U}_t\,dt\,dx + \iint_{xt} \underline{F}(\underline{U})_x\,dt\,dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} (\underline{U}(t_2) - \underline{U}(t_1))\,dx + \int_{t_1}^{t_2} (\underline{F}(x_2) - \underline{F}(x_1))\,dt = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \underline{U}(x,t_2)\,dx = \int_{x_1}^{x_2} \underline{U}(t_1)\,dx + \int_{t_1}^{t_2} \underline{F}(x_1)\,dt - \int_{t_1}^{t_2} \underline{F}(x_2)\,dt$$

We discretize the spatial domain $[0, L]$ into M computing cells of regular size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{L}{M}$ with $i = 1, \dots, M$.

For a given cell i:

$$\text{centre} \quad x_i = (i - \tfrac{1}{2}) \Delta x$$

$$\text{boundaries} \quad x_{i-\frac{1}{2}} = (i-1) \Delta x$$

$$x_{i+\frac{1}{2}} = i \Delta x$$

The Godunov method first assumes a piece-wise constant distribution of the data. This is formally realized by defining cell-averages:

$$\underline{U}_i^n = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \underline{\tilde{U}}(x, t^n)\, dx$$

Once the piece-wise constant distribution of data has been established, the next step in the godunov method is to solve the IVP for the original conservation laws but with the modified initial data. Effectively this generates local Riemann problems $RP(\underline{U}_i^n, \underline{U}_{i+1}^n)$ with data $\underline{U}_i$ (left side) and $\underline{U}_{i+1}^n$ (right side) centred at the intercell boundary positions $x_{i+\frac{1}{2}}$.

The solution of $RP(\underline{U}_i^n, \underline{U}_{i+1}^n)$ is a similarity solution and depends on the ratio $\bar{x}/\bar{t}$; The solution is denoted as $\underline{U}_{i+\frac{1}{2}}(\bar{x}/\bar{t})$, where $(\bar{x}, \bar{t})$ are local coordinates for the local Riemann problem

For a time step Δt that is sufficiently small to avoid wave interaction, one can define a global solution $\underline{\tilde{U}}(x,t)$ in the strip $0 \leq x \leq L$, $t^n \leq t \leq t^{n+1}$ in terms of the local solutions as follows:

$$\underline{\tilde{U}}(x,t) = \underline{U}_{i+\frac{1}{2}}(\bar{x}/\bar{t}) \quad , x \in [x_i, x_{i+1}]$$

where $\quad \bar{x} = x - x_{i-\frac{1}{2}} \quad , \quad \bar{t} = t - t^n$

$$x \in [x_i, x_{i+1}], \quad t \in [t^n, t^{n+1}]$$

$$\bar{x} \in \left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right], \quad \bar{t} \in [0, \Delta t]$$

Having found a solution $\underline{\tilde{U}}(x,t)$ in terms of solutions $\underline{U}_{i+\frac{1}{2}}(\bar{x},\bar{t})$ to local Riemann problems, the Godunov method advances the solution to a time $t^{n+1} = t^n + \Delta t$ by defining a new set of average values $\{\underline{U}_i^{n+1}\}$.

2) The Godunov Scheme

The first version of Godunov's method defines new average values $\underline{U}_i^{n+1}$ at time $t^{n+1} = t_n + \Delta t$ via the integrals

$$\underline{U}_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{U}(x, t^{n+1})\, dx$$

within each cell.

To perform the averaging, we need to make the assumption that no wave interaction takes place within cell I_i in the chosen time Δt.

This is satisfied by imposing

$$\Delta t \leq \frac{\frac{1}{2}\Delta x}{S^n_{max}}$$

where S^n_{max} denotes the maximum wave velocity present throughout the domain at time t^n.

A consequence of this restriction is that only two Riemann problem solutions affect cell I_i, namely the right travelling waves of $\underline{U}_{i-\frac{1}{2}}(x/t)$ and the left travelling waves of $\underline{U}_{i+\frac{1}{2}}(x/t)$.

$$\Rightarrow \underline{U}_i^{n+1} = \frac{1}{\Delta x} \int_0^{\frac{1}{2}\Delta x} \underline{U}_{i-\frac{1}{2}}\left(\frac{x}{\Delta t}\right) dx + \frac{1}{\Delta x} \int_{-\frac{1}{2}\Delta x}^{0} \underline{U}_{i+\frac{1}{2}}\left(\frac{x}{\Delta t}\right) dx$$

This version of Godunov's method can be implemented as a practical computational scheme, however it has two main drawbacks:

- The CFL-like condition is computationally somewhat restrictive on Δt
- The evaluation of the integrals is possible, but can be involved.

The Godunov method can be written in conservative form:

$$\underline{U}_i^{n+1} = \underline{U}_i^n + \frac{\Delta t}{\Delta x}\left[\underline{F}_{i-\frac{1}{2}} - \underline{F}_{i+\frac{1}{2}}\right]$$

with intercell numerical flux

$$\underline{F}_{i+\frac{1}{2}} = \underline{F}(\underline{U}_{i+\frac{1}{2}}(0))$$

if $\Delta t \leq \frac{\Delta x}{S^n_{max}}$

Proof: In $\underline{U}_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \underline{\tilde{U}}(x, t^{n+1})\, dx$, which are the average values, $\underline{\tilde{U}}$ is an exact solution of the conservation laws: $\underline{\tilde{U}}(x,t) = \underline{U}_{i+\frac{1}{2}}(\bar{x}/\bar{t})$

We can therefore apply the integral form

$$\int_{x_1}^{x_2} \underline{U}(x, t_2)\, dx = \int_{x_1}^{x_2} \underline{U}(x, t_1)\, dx + \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_1, t))\, dt - \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_2, t))\, dt$$

The conservation law can be applied to any control volume $[x_1, x_2] \times [t_1, t_2]$. In particular we can apply it to the case in which

$$x_1 = x_{i-1/2}, \qquad x_2 = x_{i+1/2}$$
$$t_1 = t^n, \qquad t_2 = t^{n+1}$$

which gives us

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \underline{\tilde{U}}(x, t^{n+1})dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{\tilde{U}}(x, t^n)dx + \int_0^{\Delta t} \underline{F}(\underline{\tilde{U}}(x_{i-\frac{1}{2}}, t))dt -$$
$$- \int_0^{\Delta t} \underline{F}(\underline{\tilde{U}}(x_{i+1/2}, t))\, dt$$

Assuming the condition $\Delta t \leq \dfrac{\Delta x}{S^n_{max}}$, we have

$$\underline{\tilde{U}}(x_{i-1/2}, t) = \underline{U}_{i-\frac{1}{2}}(\bar{x}/\bar{t} = 0) = \text{const}$$
$$\underline{\tilde{U}}(x_{i+\frac{1}{2}}, t) = \underline{U}_{i+\frac{1}{2}}(\bar{x}/\bar{t} = 0) = \text{const}$$

when $\underline{U}_{i\pm 1/2}(\frac{x}{t} = 0)$ is the solution of the Riemann problem along the ray $x/t = 0$. Remember that $\underline{\tilde{U}}(x, t) = \underline{U}_{i+\frac{1}{2}}(\bar{x}/\bar{t})$ is the exact solution to the piece-wise constant problem $\underline{U}(x, t^n) = U_i^n = \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{U}(x, t^n)dx$ therefore $\underline{\tilde{U}}(x_{i\pm\frac{1}{2}}, t) = \text{const}.$

Now divide by Δx:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \hat{\underline{U}}(x,t^{n+1})dx = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \hat{\underline{U}}(x,t^{n})dx + \frac{\Delta t}{\Delta x}\left[\underline{F}(\underline{U}_{i-1/2}(0)) - \underline{F}(\underline{U}_{i+1/2}(0))\right]$$

Using $\underline{U}_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \hat{\underline{U}}(x,t^n)\,dx$ and $\underline{F}_{i\pm 1/2} = \underline{F}(\underline{U}_{i\pm 1/2})$:

$$\underline{U}^{n+1} = \underline{U}^{n} + \frac{\Delta t}{\Delta x}\left[\underline{F}_{i-1/2} - \underline{F}_{i+1/2}\right]$$

- The CFL condition is more generous than the one in the first method
- The condition remains valid even if wave interaction takes place in time Δt within cell I_i under the assumption that no wave acceleration takes place as a consequence of wave interaction.

3) Godunov's Method for Euler Equations

Data $\{\underline{U}_i^n\}$ at time level n is assumed. To march the solution to level $n+1$, we need to compute the intercell fluxes $\underline{F}_{i\pm 1/2}$.

3.1) Evaluation of the intercell fluxes

$$\underline{F}_{i\pm 1/2} = \underline{F}(\underline{U}_{i\pm 1/2}(0))$$

We therefore require the solution $\underline{U}_{i\pm 1/2}(x/t)$ of the Riemann problem $RP(\underline{U}_i^n, \underline{U}_{i+1}^n)$ evaluated at $\frac{x}{t}=0$

In practice we use the solution in terms of the primitive variables $\underline{W}_{i\pm 1/2}$.
For sampling the solution, there are two situations $u_* \geq 0$ and $u_* \leq 0$, each of which has five cases.

Or simply said: just use a Riemann solver to find $\underline{W}_{i\pm 1/2}$, then compute $\underline{F}(\underline{W}_{i\pm 1/2})$

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3.2 Time step size

$$\Delta t = \frac{C_{CFL}\,\Delta x}{S^n_{max}}$$

C_{CFL} = Courant/CFL coefficient; $0 < C_{CFL} \le 1$

For time-dependent, 1D Euler equations a reliable choice is

$$S^n_{max} = \max_i \{ |S^L_{i+1/2}|, |S^R_{i-1/2}| \}$$

where $S^L_{i+1/2}$, $S^R_{i+1/2}$ are the wave speeds of the left and right non-linear waves (shocks, rarefactions). For rarefactions, one selects the speed of the head.

For multi-dimensional problems however, this scheme for estimating the maximum wave speed is unsuitable.

A popular alternative is

$$S^n_{max} = \max_i \{ |u_i^n| + a_i^n \}$$

this might underestimate the shock speed, but can be amended by choice of C_{CFL}

3.3 Boundary Conditions

i) Reflective Boundaries

To get a fixed, reflective, impermeable wall:

Create a fictitious state $\underline{w}^n_{M+1}$ with

$$\rho^n_{M+1} = \rho^n_M, \quad u^n_{M+1} = -u^n_M, \quad p^n_{M+1} = p^n_M$$