Approximate Riemann Solvers

1) Introduction

The method of Godunov and its high-order extensions require the solution of the Riemann problem. In a practical computation this is solved billions of times, making the Riemann problem solution the single most demanding task in the numerical method.

Approximate, non-iterative solutions have the potential to provide the necessary items of information for numerical purposes. There are essentially two ways of extracting approximate information from the solution of the Riemann problem to be used in Godunov-type methods:

- find approximation to the numerical flux
- find approximation to a state and then evaluate the physical flux function at this state.

We're going to look at the latter option and approximate Riemann solvers that do not need an iteration process. We provide an approximate solution for the state required to evaluate the Godunov flux.

The approximations can be used directly in the first-order Godunov method and its high-order extensions; some of the approximations however are exceedingly simple, but not accurate enough to produce robust numerical methods.

2) Problem Overview

We want to solve numerically the general Initial Boundary Value Problem:

PDEs: $\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$

ICs: $\underline{U}(x,0) = \underline{U}^{(0)}(x)$

BCs: $\underline{U}(0,t) = \underline{U}_l(t)$; $\underline{U}(L,t) = \underline{U}_r(t)$

using the explicit conservative formula

$$\underline{U}_i^{n+1} = \underline{U}_i^n + \frac{\Delta t}{\Delta x}\left[\underline{F}_{i-\frac{1}{2}} - \underline{F}_{i+\frac{1}{2}}\right]$$

with the Godunov intercell flux

$$\underline{F}_{i\pm\frac{1}{2}} = \underline{F}\left(\underline{U}_{i\pm\frac{1}{2}}(0)\right)$$

Here $\underline{U}_{i+\frac{1}{2}}(0)$ is the similarity solution $\underline{U}_{i\pm\frac{1}{2}}(x/t)$ of the Riemann problem

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ \underline{U}_R & \text{if } x>0 \end{cases}$$

For the x-split Riemann problem, the conserved variables are:

$$\underline{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{bmatrix}$$

and the primitive variables are:

$$\underline{W} = \begin{bmatrix} \rho \\ u \\ v \\ w \\ p \end{bmatrix}$$

In our solution procedure, we split the task of solving the complete Riemann problem into three subproblems:

- 1) The star values p_*, u_*, ρ_{L*}, ρ_{R*}
- 2) The solution for the tangential velocity components v and w throughout the wave structure
3) The solution for ρ, u, and p inside sonic rarefactions

The star region variables are a bit more difficult to derive, but the tangential velocity components and the solution inside rarefactions can be dealt with assuming the star states are known.

2.1 Tangential Velocity Components

In the exact solution, the values of the tangential velocity components v and w do not change across the non-linear waves, but do change, discontinuously, across the middle wave. Thus, given an approximate solution u_* for the normal velocity component in the star region, the solution for v and w is

$$v(x,t),\ w(x,t) = \begin{cases} v_L,\ w_L & \text{if } \frac{x}{t} \leq u_* \\ v_R,\ w_R & \text{if } \frac{x}{t} > u_* \end{cases}$$

Any passive scalar quantity $q(x,y,z,t)$ advected with the fluid will have this property.

2.2) Sonic Rarefactions

As per usual. Set $\frac{x}{t} = 0$ in

$$\underline{W}_{Lfan} = \begin{cases} \rho = \rho_L \left[\frac{2}{\gamma+1} + \frac{(\gamma-1)}{(\gamma+1)a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{(\gamma+1)}\left[a_L + \frac{(\gamma-1)}{2} u_L + \frac{x}{t}\right] \\ \rho = \rho_L \left[\frac{2}{(\gamma+1)} + \frac{(\gamma-1)}{(\gamma+1)a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases}$$

and analogously for $\underline{W}_{Rfan}$:

$$\underline{W}_{Rfan} = \begin{cases} \rho = \rho_R \left[\frac{2}{(\gamma+1)} - \frac{(\gamma-1)}{(\gamma+1)a_R}\left(u_R - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{(\gamma+1)}\left[-a_R + \frac{\gamma-1}{2} u_R + \frac{x}{t}\right] \\ p = p_R \left[\frac{2}{\gamma+1} - \frac{(\gamma-1)}{(\gamma+1)a_R}\left(u_R - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases}$$

3.) Approximations based on the exact solver

The exact Riemann solver is based on the pressure equation

$$f(p) = f_L(p, \underline{W}_L) + f_R(p, \underline{W}_R) + \Delta u = 0$$

$$\Delta u = u_R - u_L$$

with
$$f_K = \begin{cases} (p - p_K)\left[\frac{A_K}{p + B_K}\right]^{1/2} & \text{if } p > p_K \text{ (shock)} \\ \frac{2a_K}{(\gamma-1)}\left[\left(\frac{p}{p_K}\right)^z - 1\right] & \text{if } p \leq p_K \text{ (rarefl.)} \end{cases}$$

$$z = \frac{\gamma-1}{2\gamma}, \quad A_K = \frac{2}{(\gamma+1)S_K}, \quad B_K = \left(\frac{\gamma-1}{\gamma+1}\right)p_K, \quad K = L, R$$

3.1) A Two-Rarefaction Riemann Solver (TRRS)

If one assumes a-priori that both non-linear waves are rarefactions, then we get

$$f(p) = \frac{2a_L}{(\gamma-1)}\left[\left(\frac{p}{p_L}\right)^z - 1\right] + \frac{2a_R}{(\gamma-1)}\left[\left(\frac{p}{p_R}\right)^z - 1\right] + \Delta u = 0$$

$$\Rightarrow p^z\left(\frac{a_L}{p_L^z} + \frac{a_R}{p_R^z}\right) - a_L - a_R = -\frac{\gamma-1}{2}\Delta u = \frac{1-\gamma}{2}\Delta u$$

$$p^z = \frac{\frac{1-\gamma}{2}\Delta u + a_L + a_R}{\frac{a_L}{p_L^z} + \frac{a_R}{p_R^z}}$$

$$\Rightarrow \rho_* = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/\rho_L^2 + a_R/\rho_R^2}\right]^{1/2}$$

Then using the rarefaction wave relations:

$$u_* = u_L - \frac{2a_L}{(\gamma-1)}\left[\left(\frac{\rho_*}{\rho_L}\right)^2 - 1\right]$$
$$= u_R + \frac{2a_R}{(\gamma-1)}\left[\left(\frac{\rho_*}{\rho_R}\right)^2 - 1\right]$$

We can eliminate ρ_* from this equation:

$$u_* = u_R + \frac{2a_R}{\gamma-1}\left[\left(\frac{\rho_*}{\rho_R}\right)^2 - 1\right]$$

$$= u_R + \frac{2a_R}{\gamma-1}\left[\frac{1}{\rho_R^2}\left(\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R-u_L)}{a_L/\rho_L^2 + a_R/\rho_R^2}\right) - 1\right]$$

$$= u_R + \frac{2a_R}{\gamma-1}\left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R-u_L) - a_L\rho_R^2/\rho_L^2 - a_R}{a_L\rho_R^2/\rho_L^2 + a_R}\right]$$

$$= u_R + \frac{2a_R}{\gamma-1}\left[\frac{a_L(1-(\rho_R/\rho_L)^2) - \frac{\gamma-1}{2}(u_R-u_L)}{a_L(\rho_R/\rho_L)^2 + a_R}\right]$$

$$= u_R + \frac{\frac{2a_L}{\gamma-1}(1-(\rho_R/\rho_L)^2) - (u_R-u_L)}{a_L/a_R\,(\rho_R/\rho_L)^2 + 1}$$

$$= \frac{\frac{2a_L}{\gamma-1}(1+(\rho_R/\rho_L)^2) - u_R + u_L + u_R + u_R\frac{a_L}{a_R}\left(\frac{\rho_R}{\rho_L}\right)^2}{a_L/a_R\,(\rho_R/\rho_L)^2 + 1}$$

$$= \frac{\frac{2a_La_R}{\gamma-1}\left(\left(\frac{\rho_L}{\rho_R}\right)^2 + 1\right) + u_La_R\left(\frac{\rho_L}{\rho_R}\right)^2 + u_Ra_L}{a_L + a_R\left(\frac{\rho_R}{\rho_L}\right)^2}$$

$$\boxed{u_* = \frac{\frac{2}{\gamma-1}\left(1+\left(\frac{\rho_L}{\rho_R}\right)^2\right) + \frac{u_L}{a_L}\left(\frac{\rho_L}{\rho_R}\right)^2 + \frac{u_R}{a_R}}{1/a_R + \left(\frac{\rho_L}{\rho_R}\right)^2/a_L}}$$

Computing $\rho_* = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/\rho_L^2 + a_R/\rho_R^2}\right]^{1/2}$ requires the evaluation of three fractional powers.

A more efficient method is to first evaluate u_* and then evaluate ρ_* from

$$u_* = u_R + \frac{2a_R}{\gamma-1}\left[\left(\frac{\rho_*}{\rho_R}\right)^2 - 1\right]$$

$$\Rightarrow \left(\frac{\rho_*}{\rho_R}\right)^2 - 1 = \frac{\gamma-1}{2a_R}\left[u_* - u_R\right]$$

$$\Rightarrow \boxed{\rho_* = \rho_R\left[\frac{\gamma-1}{2a_R}(u_* - u_R) + 1\right]^{1/2}}$$

Being consistent with the two-rarefactions assumption we get $S_{*R,L}$:

$$S_{*R} = S_R\left(\frac{\rho_*}{\rho_R}\right)^{1/\gamma}, \quad S_{*L} = S_L\left(\frac{\rho_*}{\rho_L}\right)^{1/\gamma}$$

An improved version of the two-rarefaction solution is obtained by using exact relations for given p_* or u_*. Suppose we calculated p_*, then

$$u_* = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[f_R(p_*) - f_L(p_*)\right]$$

where f_R and f_L are evaluated according to the exact relations by comparing p_* with p_L and p_R. S_{*L} and S_{*R} can also be obtained using either shock or rarefaction relations.

The Two-Rarefaction solution is generally very robust.

3.2) A Two-Shock Riemann Solver (TSRS)

Assuming both non-linear waves are shocks:

$$f(p) = (p-p_L)\, g_L(p) + (p-p_R)\, g_R(p) + u_R - u_L = 0$$

with $$g_K(p) = \left[\frac{A_K}{p+B_K}\right]^{1/2}$$

Unfortunately this approximation does not lead to a closed-form solution. Further approximations must be constructed.

Obvious approximations to the two-shock approximation involve quadratic equations. These do not generally lead to robust schemes and have problems:

- non-uniqueness of solutions
- complex solutions

An alternative approach is to assume an estimate for p_0 for the solution of the pressure. Then insert them in $g_K(p)$ and use it in the pressure function:

$$(p_* - p_L)\, g_L(p_0) + (p_* - p_R)\, g_R(p_0) + u_R - u_L = 0$$

$$\Rightarrow \quad p_*(g_L + g_R) - p_L g_L - p_R g_R + u_R - u_L = 0$$

$$\boxed{p_* = \frac{g_L(p_0)\, p_L + g_R(p_0)\, p_R - (u_R - u_L)}{g_L(p_0) + g_R(p_0)}}$$

Being consistent with the two-shock assumption:

$$u_* = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[(p_* - p_R)\,g_R(p_0) - (p_* - p_L)\,g_L(p_0)\right]$$

Solutions for $\rho_{*L,R}$ are obtained with shock relations

$$\rho_{*L,R} = \rho_{L,R}\left[\frac{\dfrac{p_*}{p_{L,R}} + \dfrac{\gamma-1}{\gamma+1}}{\dfrac{\gamma-1}{\gamma+1}\dfrac{p_*}{p_{L,R}} + 1}\right]$$

For the choice for the pressure estimate p_0, we propose

$$p_0 = \max(0, p_{PVRS})$$

where p_{PVRS} is the solution for pressure given by the primitive variable Riemann solver:

$$p_{0*} = \frac{1}{2}(p_L + p_R) + \frac{1}{2}(u_L - u_R)(\bar{\rho}\,\bar{a})$$

$$\bar{\rho} = \frac{1}{2}(\rho_L + \rho_R), \quad \bar{a} = \frac{1}{2}(a_L + a_R)$$

The approximation to the star values is more efficient than the TRRS and is more accurate for a wider range of flow conditions, except for near vacuum conditions when TRRS is very accurate or indeed exact. It can be improved by using true wave relations whenever possible.

4.) The HLL and HLLC Riemann Solvers

4.1) Introduction

For the purpose of computing a Godunov flux, Harten, Lax and van Leer presented an approach for solving the Riemann problem approximately. The resulting Riemann solvers have become known as HLL Riemann solvers. In this approach an approximation for the intercell flux is obtained directly. The central idea is to assume for the solution a wave configuration that consists of two waves separating three constant states.
Assuming that the wave speeds are given by some algorithm, application of the integral form of the conservation laws gives a closed form approximate expression for the flux.

One difficulty with these schemes is the assumptions of a two-wave configuration. This is correct only for hyperbolic systems of two equations. For larger systems, such as the Euler equations, the two-wave assumption is incorrect.

As a consequence, the resolution of physical features such as contact surfaces, shear waves and material interfaces, can be very inaccurate.

A different approach to remedy the problem of intermediate waves in the HLL approach is the HLLC solver (C stands for Contact). HLLC is a three-wave model. It is a complete Riemann solver for the Euler equations, but insufficient for hyperbolic systems with four or more equations.

4.2) Integral Relations

Consider the exact solution containing the whole wave structure that is contained in the control volume $[x_L, x_R] \times [0, T]$ with

$$x_L \leq T S_L, \quad x_R \geq T S_R$$

where S_L and S_R are the fastest signal velocities perturbing the initial data states $\underline{U}_L$ and $\underline{U}_R$ respectively, and T is a chosen time.

The integral form of the conservation laws is:

$$\int_{x_L}^{x_R} \underline{U}(x,T)\,dx = \int_{x_L}^{x_R} \underline{U}(x,0)\,dx + \int_0^T \underline{F}(\underline{U}(x_L,t))\,dt - \int_0^T \underline{F}(\underline{U}(x_R,t))\,dt$$

Since $\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ \underline{U}_R & \text{if } x>0 \end{cases}$

$$\Rightarrow \int_{x_L}^{x_R} \underline{U}(x,0)\,dx = \int_{x_L}^{0} \underline{U}(x,0)\,dx + \int_0^{x_R} \underline{U}(x,0)\,dx$$

$$= \int_{x_L}^{0} \underline{U}_L\,dx + \int_0^{x_R} \underline{U}_R\,dx = \underline{U}_R \cdot x_R - \underline{U}_L\, x_L$$

Since $t \le T$ and $x_L \le T S_L$, $x_R \ge T S_R$:

$$\int_0^T \left[\underline{F}(\underline{U}(x_L,t))\,dt - \underline{F}(\underline{U}(x_R,t))\,dt \right] =$$

$$= \int_0^T \left[\underline{F}(\underline{U}_L) - \underline{F}(\underline{U}_R) \right] dt$$

$$= T(\underline{F}_L - \underline{F}_R)$$

$$\Rightarrow \boxed{\int_{x_L}^{x_R} \underline{U}(x,T)\,dx = x_R \underline{U}_R - x_L \underline{U}_L + T(\underline{F}_L - \underline{F}_R)}$$

"consistency condition"

Now split the left-hand-side integral:

$$\int_{x_L}^{x_R} \underline{U}(x,T)\,dx = \int_{x_L}^{TS_L} \underline{U}(x,T)\,dx + \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx + \int_{TS_R}^{x_R} \underline{U}(x,T)\,dx$$

$$= \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx + (TS_L - x_L)\underline{U}_L + (x_R - TS_R)\underline{U}_R$$

Using the consistency condition

$$\int_{x_L}^{x_R} \underline{U}(x,T)\,dx = x_R \underline{U}_R - x_L \underline{U}_L + T(\underline{F}_L - \underline{F}_R)$$

$$= \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx + (TS_L - x_L)\underline{U}_L + (x_R - TS_R)\underline{U}_R$$

$$\Rightarrow \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx = \underline{U}_R(x_R - x_R + TS_R) - \underline{U}_L(x_L + TS_L - x_L) + T(\underline{F}_L - \underline{F}_R)$$

$$= T(S_R \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R)$$

Divide by width of the wave system of the Riemann problem's solution, $T(S_R - S_L)$:

$$\boxed{\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx = \frac{S_R \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R}{S_R - S_L}}$$

$\Rightarrow$ The integral average of the exact solution of the Riemann problem between the slowest and the fastest signals at time T is a known constant, provided that the speeds S_L, S_R are known.

We denote this constant value as

$$\boxed{\underline{U}^{hll} = \frac{S_R \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R}{S_R - S_L}}$$

Now we apply the integral form of the conservation laws to the left portion of the control volume $[x_L, 0] \times [0, T]$, and the right portion $[0, x_R] \times [0, T]$ respectively:

$$\int_{x_L}^{0} \underline{U}(x, T)\, dx = \int_{x_L}^{TS_L} \underline{U}(x, T)\, dx + \int_{TS_L}^{0} \underline{U}(x, T)\, dx$$

$$= (TS_L - x_L)\underline{U}_L + \int_{TS_L}^{0} \underline{U}(x, T)\, dx$$

$$= \int_{x_L}^{0} \underline{U}(x, 0)\, dx + \int_{0}^{T} \underline{F}(\underline{U}(x_L, t))\, dt - \int_{0}^{T} \underline{F}(\underline{U}(0, t))\, dt$$

$$= -x_L \underline{U}_L + T(\underline{F}_L - \underline{F}_{0L})$$

$$\Rightarrow T\underline{F}_{0L} = -TS_L \underline{U}_L - \int_{TS_L}^{0} \underline{U}(x, T)\, dx + T\underline{F}_L$$

$$\Rightarrow \boxed{\underline{F}_{0L} = \underline{F}_L - S_L \underline{U}_L - \frac{1}{T}\int_{TS_L}^{0} \underline{U}(x, T)\, dx}$$

For the right part:

$$\int_0^{x_R} \underline{u}(x,T) = \int_0^{TS_R} \underline{u}(x,T)\,dx + \int_{TS_R}^{x_R} \underline{u}(x,T)\,dx$$

$$= \int_0^{TS_R} \underline{u}(x,T)\,dx + (x_R - TS_R)\,\underline{u}_R$$

$$= \int_0^{x_R} \underline{u}(x,0)\,dx + \int_0^T \underline{F}(\underline{u}(0,t))\,dt - \int_0^T F(\underline{u}(x_R,t))\,dt$$

$$= x_R \underline{u}_R + \int_0^T \underline{F}(\underline{u}(0,t))\,dt - T\underline{F}_R$$

$$= x_R \underline{u}_R + T(\underline{F}_{0R} - \underline{F}_R)$$

$$\Rightarrow \boxed{\underline{F}_{0R} = \underline{F}_R - S_R \underline{u}_R + \frac{1}{T}\int_0^{TS_R} \underline{u}(x,T)\,dx}$$

Setting $\underline{F}_{0R} = \underline{F}_{0L}$:

$$\underline{F}_R - S_R \underline{u}_R + \frac{1}{T}\int_0^{TS_R} \underline{u}(x,T)\,dx = \underline{F}_L - S_L \underline{u}_L - \frac{1}{T}\int_{TS_L}^{0} \underline{u}(x,T)\,dx$$

$$\Rightarrow \frac{1}{T}\int_{TS_L}^{TS_R} \underline{u}(x,T)\,dx = \underline{F}_L - \underline{F}_R + S_R \underline{u}_R - S_L \underline{u}_L$$

Which then gives us for the entire integral

$\int_{x_L}^{x_R} \underline{u}(x,T)\,dx$:

$$\int_{x_L}^{x_R} \underline{U}(x,T)\,dx = \int_{x_L}^{TS_L} \underline{U}(x,T)\,dx + \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx + \int_{TS_R}^{x_R} \underline{U}(x,T)\,dx$$

$$= (TS_L - x_L)\underline{U}_L + T(\underline{F}_L - \underline{F}_R + S_R\underline{U}_R - S_L\underline{U}_L) + (x_R - TS_R)\underline{U}_R$$

$$= x_R\underline{U}_R - x_L\underline{U}_L + T(\underline{F}_L - \underline{F}_R)$$

$\Rightarrow$ So by setting $\underline{F}_{0R} = \underline{F}_{0L}$, we get the consistency condition again.

4.3) The HLL Approximate Riemann Solver

Harten, Lax and van Leer proposed the following Riemann solver:

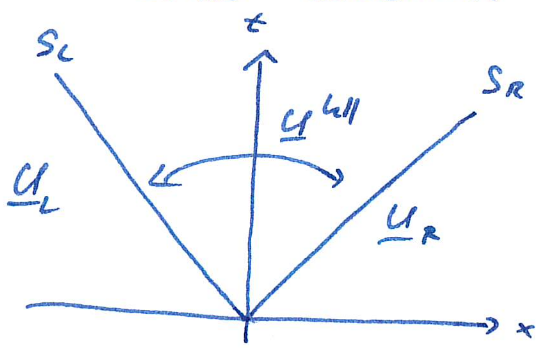
$$\underline{\hat{U}}(x,t) = \begin{cases} \underline{U}_L & \text{if } \frac{x}{t} \le S_L \\ \underline{U}^{hll} & \text{if } S_L \le \frac{x}{t} \le S_R \\ \underline{U}_R & \text{if } \frac{x}{t} \ge S_R \end{cases}$$

when $\underline{U}^{hll}$ is the constant state wave vector

$$\underline{U}^{hll} = \frac{S_R \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R}{S_R - S_L} = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx$$

and the integral average of the exact solution.

This approximation consists of just three constant states separated by two waves; the star region consists of a single constant state.



The corresponding flux $\underline{F}^{hll}$ along the t-axis is found from the relations

$$\underline{F}_{0R} = \underline{F}_R - S_R \underline{U}_R + \frac{1}{T} \int_0^{TS_R} \underline{U}(x,T)\,dx$$

$$= \underline{F}_{0L} = \underline{F}_L - S_L \underline{U}_L - \frac{1}{T} \int_0^{TS_L} \underline{U}(x,T)\,dx$$

with the exact integrand $\underline{U}(x,T)$ replaced with the approximate solution $\underline{\tilde{U}}(x,T)$. Note that we do not take $\underline{F}^{hll} = \underline{F}(\underline{U}^{hll})$. Explicitly for the case $S_L \le 0 \le S_R$, we have:

$$\underline{F}^{hll} = \underline{F}_L - S_L \underline{U}_L - \frac{1}{T}\int_{TS_L}^{0} \underline{\tilde{U}}(x,T)\,dx \qquad \Big|\ \underline{\hat{U}}(x,T) = \underline{U}^{hll} \text{ for } TS_L \le x \le TS_R$$

$$= \underline{F}_L - \underline{S}_L \underline{U}_L - \frac{1}{T}\underline{U}^{hll}(0 - TS_L)$$

$$\boxed{= \underline{F}_L + S_L(\underline{U}^{hll} - \underline{U}_L)}$$

$$= \underline{F}_R - S_R \underline{U}_R + \frac{1}{T}\int_{0}^{TS_R} \underline{\tilde{U}}(x,T)\,dx$$

$$= \underline{F}_R - S_R \underline{U}_R + \frac{1}{T}\underline{U}^{hll}(TS_R - 0)$$

$$\boxed{= \underline{F}_R + S_R(\underline{U}^{hll} - \underline{U}_R)}$$

Note that these two relations are also obtained from applying Rankine-Hugoniot conditions across the left and right waves respectively.

Fully written out, we have

$$\underline{F}^{hll} = \underline{F}_L + S_L\left(\underline{U}^{hll} - \underline{U}_L\right) =$$

$$= \underline{F}_L + S_L\left(\frac{S_R\underline{U}_R - S_L\underline{U}_L + \underline{F}_L - \underline{F}_R}{S_R - S_L} - \underline{U}_L\right)$$

$$= \underline{F}_L + S_L\left(\frac{S_R\underline{U}_R - S_L\underline{U}_L + \underline{F}_L - \underline{F}_R - S_R\underline{U}_L + S_L\underline{U}_L}{S_R - S_L}\right)$$

$$= \frac{(S_R - S_L)\underline{F}_L + S_L S_R\underline{U}_R - S_L^2\underline{U}_L + S_L\underline{F}_L - S_L\underline{F}_R - S_R S_L\underline{U}_L + S_L^2\underline{U}_L}{S_R - S_L}$$

$$= \frac{S_L S_R\underline{U}_R - S_L S_R\underline{U}_L + S_R\underline{F}_L - S_L\underline{F}_R}{S_R - S_L}$$

$$\boxed{= \frac{S_R\underline{F}_L - S_L\underline{F}_R + S_L S_R\left(\underline{U}_R - \underline{U}_L\right)}{S_R - S_L}}$$

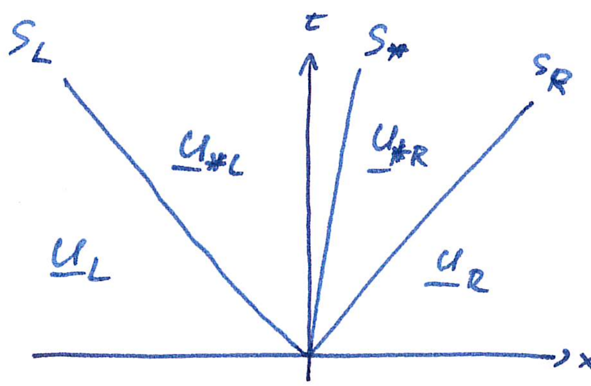
Given an algorithm to compute the speeds S_L, S_R, we have an approximate intercell flux to produce an approximate Godunov method:

$$\underline{F}^{hll}_{i+\frac{1}{2}} = \begin{cases} \underline{F}_L & \text{if } 0 \le S_L \\ \underline{F}^{hll} & \text{if } S_L \le 0 \le S_R \\ \underline{F}_R & \text{if } S_R \le 0 \end{cases}$$

A shortcoming of the HLL scheme is exposed by contact discontinuities, shear waves, material interfaces, or any type of intermediate waves.

4.4) The HLLC Approximate Riemann Solver

Is a modification of the HLL scheme, whereby the missing contact wave in the Euler equations are restored.



Now, in addition to the slowest and fastest signal speeds S_L and S_R we include a middle wave of speed S_*.

The HLL relation still holds:

$$\int_{x_L}^{x_R} \underline{U}(x,T)\,dx = \int_{x_L}^{TS_L} \underline{U}(x,T)\,dx + \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx + \int_{TS_R}^{x_R} \underline{U}(x,T)\,dx$$

$$= \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx + (TS_L - x_L)\underline{U}_L + (x_R - TS_R)\underline{U}_R$$

$$= \int_{x_L}^{x_R} \underline{U}(x,0)\,dx + \int_0^T \underline{F}(\underline{U}(x_L,t))\,dt - \int_0^T \underline{F}(\underline{U}(x_R,t))\,dt$$

$$= x_R U_R - x_L U_L + T(\underline{F}_L - \underline{F}_R)$$

$$\Rightarrow \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx = x_R \underline{U}_R - x_L \underline{U}_L + T(\underline{F}_L - \underline{F}_R) - (TS_L - x_L)\underline{U}_L - (x_R - TS_R)\underline{U}_R$$

$$= T\left(S_R \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R\right)$$

$$\Rightarrow \boxed{\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx = \frac{S_L \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R}{S_R - S_L} \equiv \underline{U}^{hll}}$$

even if variations of the integrand across the wave speed are allowed. Now split the lhs of the integral:

$$\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \underline{U}(x,T)\,dx = \frac{1}{T(S_R - S_L)} \left[\int_{TS_L}^{TS_*} \underline{U}(x,T)\,dx + \int_{TS_*}^{TS_R} \underline{U}(x,T)\,dx \right]$$

We define the integral averages:

$$\underline{U}_{*L} \equiv \frac{1}{T(S_* - S_L)} \int_{TS_L}^{TS_*} \underline{U}(x,T)\,dx$$

$$\underline{U}_{*R} \equiv \frac{1}{T(S_R - S_*)} \int_{TS_*}^{TS_R} \underline{U}(x,T)\,dx$$

which gives us

$$\boxed{\underline{U}^{hll} = \frac{S_* - S_L}{S_R - S_L} \underline{U}_{*L} + \frac{S_R - S_*}{S_R - S_L} \underline{U}_{*R}}$$

The HLLC Riemann solver is given as follows:

$$\hat{\underline{U}}(x,t) = \begin{cases} \underline{U}_L & \text{if } \frac{x}{t} \le S_L \\ \underline{U}_{*L} & \text{if } S_L \le \frac{x}{t} \le S_* \\ \underline{U}_{*R} & \text{if } S_* \le \frac{x}{t} \le S_R \\ \underline{U}_R & \text{if } S_R \le \frac{x}{t} \end{cases}$$

By applying the Rankine-Hugoniot conditions across each wave, we get:

$$\underline{F}_{*L} = \underline{F}_L + S_L(\underline{U}_{*L} - \underline{U}_L)$$
$$\underline{F}_{*R} = \underline{F}_{*L} + S_*(\underline{U}_{*R} - \underline{U}_{*L})$$
$$\underline{F}_{*R} = \underline{F}_R + S_R(\underline{U}_{*R} - \underline{U}_R)$$

These are three equations for the four unknowns $\underline{U}_{*L,R}$, $\underline{F}_{*L,R}$, so we need more restraints. Obvious conditions to impose are those satisfied by the exact solution:

$$p_{*L} = p_{*R} = p_*$$
$$u_{*L} = u_{*R} = u_*$$

In addition, it's entirely justified and convenient to set

$$S_* = u_*$$

Thus: Using the first two components of $\underline{U}_{*L,R}$, $\underline{F}_{*L,R}$:

$$\underline{F}_{*L,R} = \underline{F}_{L,R} + S_{L,R}\left(\underline{U}_{*L,R} - \underline{U}_{L,R}\right)$$

with $\underline{F} = (\rho u, \rho u^2 + p, u(E+p))$ and $\underline{U} = (\rho, \rho u, E)$

$$\Rightarrow \begin{cases} \rho_* u_* = \rho u + S(\rho_* - \rho) \\ \rho_* u_*^2 + p_* = \rho u^2 + p + S(\rho_* u_* - \rho u) \end{cases}$$

$$\Rightarrow p_* = p + \rho u^2 - \rho_* u_*^2 + S(\rho_* u_* - \rho u)$$

$$= p + \rho u^2 - (\rho u + S(\rho_* - \rho))u_* + S^2(\rho_* - \rho)$$

$$= p + \rho u(u - u_*) + (\rho_* - \rho)(S^2 - S u_*)$$

$$= p + \rho u (u - u_*) + (\rho_* - \rho) S (S - u_*)$$

Again using $\rho_* u_* = \rho u + S(\rho_* - \rho)$:

$$\rho_* (u_* - S) = \rho(u - S)$$

$$\Rightarrow (\rho_* - \rho) S (S - u_*) = S\rho_* (S - u_*) - S\rho(S - u_*))$$

$$= -S\rho(u - S) - S\rho(S - u_*)$$

$$= S\rho(S - u - S + u_*) = S\rho(u_* - u)$$

$$\Rightarrow p_* = p + \rho u (u - u_*) + S\rho(u_* - u)$$

$$= p + \rho(u - u_*)(u - S)$$

$$\boxed{p_* = p + \rho(S_* - u)(S - u)}$$

with $p = p_{L,R}$, $u = u_{L,R}$, $S = S_{L,R}$, $S_* = u_*$

By demanding $p_{*L} = p_{*R}$:

$$p_L + \rho_L (S_L - u_L)(S_* - u_L) = p_R + \rho_R (S_R - u_R)(S_* - u_R)$$

$$S_* \left[\rho_L (S_L - u_L) - \rho_R (S_R - u_R) \right] = p_R + \rho_R - u_R \rho_R (S_R - u_R) +$$
$$+ u_L \rho_L (S_L - u_L) - p_L$$

$$\Rightarrow \boxed{S_* = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L (S_L - u_L) - \rho_R (S_R - u_R)}}$$

Thus we only need estimates for S_L, S_R.

Since $p_{*L,R} = p_*$ and $u_{*L,R} = u_* = S_*$, the remaining unknown for the star states is $\rho_{*L,R}$, which we can get from the first component of $\underline{U}$ and $\underline{F}$ using the Rankine-Hugoniot conditions:

$$\underline{F}_{*K} = \underline{F}_K + S_K (\underline{U}_{*K} - \underline{U}_K)$$

$$\Rightarrow \rho_{*K} u_{*K} = \rho_K u_K + S_K (\rho_{*K} - u_K)$$

$$\Rightarrow \rho_{*K} (u_{*K} - S_K) = \rho_K (u_K - S_K)$$

$$\Rightarrow \boxed{\rho_{*K} = \rho_K \frac{u_K - S_K}{u_{*K} - S_K} = \rho_K \frac{u_K - S_K}{S_* - S_K}}$$

With this, the full star state is known:

$$\underline{U}_{*K} = \rho_{*K} \begin{bmatrix} 1 \\ u_{*K} \\ v_K \\ w_K \\ \frac{E_{*K}}{\rho_*} \end{bmatrix}$$

To find the expression for $\frac{E_{*K}}{\rho_{*K}}$, we once again use the Rankine-Hugoniot conditions:

$$\underline{F}_{*K} = \underline{F}_K + S_K(\underline{U}_{*K} - \underline{U}_K)$$

for energy:

$$u_*(E_* + p_*) = u_K(E_K + p_K) + S_K(E_* - E_K) \overset{u_* = S_*}{=} S_*(E_* + p_*)$$

$$E_*(S_* - S_K) = E_K(u_K - S_K) + u_K p_K - S_* p_*$$

$$= E_K(u_K - S_K) + u_K p_K - S_*\left(p_K + \rho_K(S_* - u_K)(S_K - u_K)\right)$$

$$= E_K(u_K - S_K) + p_K(u_K - S_*) - \rho_K S_*(S_* - u_K)(S_K - u_K)$$

$$\Rightarrow E_* = E_K \frac{u_K - S_K}{S_* - S_K} + p_K \frac{u_K - S_*}{S_* - S_K} - \rho_K S_* \frac{(S_* - u_K)(S_K - u_K)}{(S_* - S_K)}$$

$$\Rightarrow \frac{E_*}{\rho_*} = \frac{E_*}{\rho_K \frac{u_K - S_K}{S_* - S_K}} = \frac{E_*}{\rho_K} \frac{S_* - S_K}{u_K - S_K} =$$

$$= \frac{E_K}{\rho_K} + \frac{p_K}{\rho_K} \frac{u_K - S_*}{u_K - S_K} + S_* \frac{S_* - u_K}{1}$$

$$= \boxed{\frac{E_K}{\rho_K} + (S_* - u_K)\left(S_* + \frac{p_K}{\rho_K(S_K - u_K)}\right)}$$

giving us:

$$\underline{U}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*}\right) \begin{bmatrix} 1 \\ S_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K)\left[S_* + \frac{p_K}{\rho_K(S_K - u_K)}\right] \end{bmatrix}$$

The flux can be computed with

$$\boxed{\underline{F}_{*K} = \underline{F}_K + S_K\left(\underline{U}_{*K} - \underline{U}_K\right)}$$

and now depends only on S_K.

The choice of the HLLC flux is given by

$$\underline{F}^{hllc}_{i+1/2} = \begin{cases} \underline{F}_L & \text{if } 0 \leq S_L \\ \underline{F}_{*L} & \text{if } S_L \leq 0 \leq S_* \\ \underline{F}_{*R} & \text{if } S_* \leq 0 \leq S_R \\ \underline{F}_R & \text{if } S_R \leq 0 \end{cases}$$

4.5) Wave-Speed Estimates

4.5.1) Direct Wave Speed Estimates

Davis:

$$S_L = u_L - a_L \,,$$
$$S_R = u_R + a_R$$

and

$$S_L = \min\{u_L - a_L,\ u_R - a_R\}$$
$$S_R = \max\{u_L + a_L,\ u_R + a_R\}$$

not recommended.

Roe average eigenvalues:

$$S_L = \hat{u} - \hat{a}$$
$$S_R = \hat{u} + \hat{a}$$

$$\hat{u} = \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_R} + \sqrt{\rho_L}} \,, \quad \hat{a} = \left[(\gamma-1)\left(\hat{H} - \tfrac{1}{2}\hat{u}^2\right)\right]^{1/2}$$

$$\hat{H} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \,, \quad H = \frac{E+p}{\rho}$$

Einfeldt:

$$S_L = \bar{u} - \bar{d} \,, \quad S_R = \bar{u} + \bar{d}$$

$$\bar{d}^2 = \frac{\sqrt{\rho_L}\, a_L^2 + \sqrt{\rho_R}\, a_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}} + \eta_2 (u_R - u_L)^2$$

$$\eta_2 = \frac{1}{2} \frac{\sqrt{\rho_L \rho_R}}{(\sqrt{\rho_L} + \sqrt{\rho_R})^2}$$

Also from Davis:

$$S_L = -S^+, \quad S_R = S^+$$

$$S^+ = \max\{|u_L| + a_L, |u_R| + a_R\}$$

or $S^+ = S^n_{max}$

5.2) Pressure based wave speed estimates

Suppose we have an estimate for p_*.
Then we choose the following wave speeds:

$$S_L = u_L - a_L q_L, \quad S_R = u_R + a_R q_R$$

where $q_K = \begin{cases} 1 & \text{if } p_* \le p_K \text{ (rarefaction)} \\ \left[1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_*}{p_K} - 1\right)\right]^{1/2} & \text{if } p_* > p_K \text{ (shock)} \end{cases}$

Estimates for p_* are for example

$$p_* = p_{PVRS} = \frac{1}{2}(p_L + p_R) - \frac{1}{2}(u_R - u_L)\,\bar{\rho}\,\bar{a}$$

$$\bar{\rho} = \frac{1}{2}(\rho_L + \rho_R), \quad \bar{a} = \frac{1}{2}(a_L + a_R)$$