Stability and Convergence of Advection Schemes

A) Numerical Diffusion

The diffusion equation for some constant diffusion coefficient D is

$$\partial_t q - D\partial_x^2 q = 0$$

We may discretize it as follows:

$$\frac{q_i^{n+1} - q_i}{t_{n+1} - t_n} - D \frac{\overbrace{\left(\overbrace{\frac{q_{i+1}^n - q_i^n}{x_{i+1} - x_i}}^{\partial_x q|_{i+1/2}} - \overbrace{\frac{q_i^n - q_{i-1}^n}{x_i - x_{i-1}}}^{\partial_x q|_{i-1/2}} \right)}^{\Delta \partial_x q}}{\underbrace{\frac{x_{i+1} - x_{i-1}}{2}}_{\Delta x}}$$

$$= \frac{q_i^{n+1} - q_i^n}{t_{n+1} - t_n} - D \frac{2}{x_{i+1} - x_{i-1}} \left(\frac{q_{i+1}^n - q_i^n}{x_{i+1} - x_i} - \frac{q_i^n - q_{i-1}^n}{x_i - x_{i-1}} \right)$$

$$= 0.$$

For constant grid spacing, we have

$$\frac{x_{i+1} - x_{i-1}}{2} = x_i - x_{i-1} = x_{i+1} - x_i = \Delta x$$

$$\Rightarrow \frac{q_i^{n+1} - q_i^n}{\Delta t} - D \frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{\Delta x^2} = 0$$

Hence

$$q_{i+1}^n - 2q_i^n + q_{i-1}^n$$

is a discrete way of writing the diffusion term.

For an advection equation, in principle numeric diffusion is unavoidable.

Consider the upwind / upstream differencing scheme for $u > 0$:

$$\partial_t q + u \, \partial_x q = 0$$

$$\rightarrow \quad \frac{q_i^{n+1} - q_i^n}{t_{n+1} - t_n} + u \, \frac{q_i^n - q_{i-1}^n}{x_i - x_{i-1}} = 0$$

$$\Rightarrow \quad q_i^{n+1} = q_i^n - u \frac{\Delta t}{\Delta x} \left(q_i^n - q_{i-1}^n \right)$$

Now we can re-write the finite difference:

$$\frac{q_i - q_{i-1}}{\Delta x} = \frac{1}{2} \cdot \frac{2q_i - 2q_{i-1}}{\Delta x} + \underbrace{\frac{1}{2} \frac{q_{i+1} - q_{i+1}}{\Delta x}}_{\text{adding } +0}$$

$$= \frac{1}{2} \frac{1}{\Delta x} \left(2q_i - 2q_{i-1} + q_{i+1} - q_{i+1} \right)$$

$$= \frac{1}{2\Delta x} \left(q_{i+1} - q_{i-1} - q_{i+1} + 2q_i - q_{i-1} \right)$$

$$= \frac{1}{2\Delta x} \left(q_{i+1} - q_{i-1} \right) - \frac{1}{2\Delta x} \left(q_{i+1} - 2q_i + q_{i-1} \right)$$

$$= \frac{q_{i+1} - q_{i-1}}{2\Delta x} - \Delta x \frac{q_{i+1} - 2q_i + q_{i-1}}{2\Delta x^2}$$

$$= \frac{q_{i+1} - q_{i-1}}{2\Delta x} - \frac{D}{u} \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2}$$

with $D = \frac{\Delta x\, u}{2}$

$\Rightarrow$ The upstream difference can be regarded to be the same as the center difference scheme supplemented with a diffusion term.

The pure centered difference scheme is unconditionally unstable, so in a way the diffusion term makes it stable.

A different way of demonstrating the diffusiveness is as follows (see also "Modified equations", chapter 8.6 in R. J. LeVeque):

We discretize the advection equation as follows:

$$\partial_t q + u \partial_x q = 0$$

$$\longrightarrow \quad \frac{q_i^{n+1} - q_i^n}{\Delta t} + u \frac{q_i^n - q_{i-1}^n}{\Delta x} = 0$$

We can now Taylor-expand the terms q_i^{n+1} and q_{i-1}^n:

$$q_i^{n+1} = q_i^n + \Delta t \frac{\partial q}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 q}{\partial t^2} + \mathcal{O}(\Delta t^3)$$

$$q_{i-1}^n = q_i^n - \Delta x \frac{\partial q}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(\Delta x^3)$$

Inserting these expansions into the discretized equation gives

$$\frac{1}{\Delta t} \left[q_i^n + \Delta t \frac{\partial q}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 q}{\partial t} + \mathcal{O}(\Delta t^3) - q_i^n \right] +$$

$$+ \frac{u}{\Delta x} \left[q_i^n - q_i^n + \Delta x \frac{\partial q}{\partial x} - \frac{\Delta x^2}{2} \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(\Delta x^3) \right] = 0$$

$$= \frac{\partial q}{\partial t} + \frac{1}{2} \Delta t \frac{\partial^2 q}{\partial t} + u \frac{\partial q}{\partial x} - \frac{u}{2} \Delta x \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$$

$$= 0$$

We can re-write this equation as

$$\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = -\frac{1}{2} \Delta t \frac{\partial^2 q}{\partial t^2} + u \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2}$$

We recognize the left hand side as our initial analytical equation, which should be zero. Hence the entire right hand side is the highest order error term that we have:

$$Err = -\frac{1}{2}\Delta t \frac{\partial^2 q}{\partial t^2} + u\frac{\Delta x}{2}\frac{\partial^2 q}{\partial x^2}$$

Using the analytical advection equation again, we can express $\frac{\partial^2 q}{\partial t^2}$ as a function of $\frac{\partial^2 q}{\partial x^2}$:

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0 \qquad (1)$$

$$\Rightarrow \frac{\partial^2 q}{\partial t^2} = -u\frac{\partial^2 q}{\partial t\,\partial x} \qquad \left|\ \frac{\partial}{\partial t}(1)\right.$$

and $$\frac{\partial^2 q}{\partial x\,\partial t} = -u\frac{\partial^2 q}{\partial x^2} \qquad \left|\ \frac{\partial}{\partial x}(1)\right.$$

$$\Rightarrow \frac{\partial^2 q}{\partial t^2} = -u\frac{\partial^2 q}{\partial t\partial x} = u^2\frac{\partial^2 q}{\partial x^2}$$

This allows us to write the error as

$$Err = -\frac{1}{2}\Delta t\frac{\partial^2 q}{\partial t^2} + u\frac{\Delta x}{2}\frac{\partial^2 q}{\partial x^2}$$

$$= -\frac{1}{2}\Delta t\, u^2\frac{\partial^2 q}{\partial x^2} + u\frac{\Delta x}{2}\frac{\partial^2 q}{\partial x^2}$$

$$= \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right)\frac{\partial^2 q}{\partial x^2}$$

We recognize in parentheses the C_{cfl} number:

$$C_{cfl} = \frac{u \Delta t_{max}}{\Delta x} \leq 1$$

Hence

$$Err = \frac{u \Delta x}{2} (1 - C_{cfl}) \frac{\partial^2 q}{\partial x^2}$$

which is again a diffusion term with

$$D = \frac{u \Delta x}{2} (1 - C_{cfl})$$

$$\Rightarrow D \propto \Delta x$$

$$D \propto (1 - C_{cfl})$$

So the diffusivity decreases with

- smaller grid spacing Δx
- higher time step size C_{cfl}

2) Courant - Friedrichs - Lewy Condition

No matter how stable an explicit method is, it cannot work for arbitrarily large time steps Δt.

If $u\Delta t > \Delta x$, then within a single time step the function is advected over a larger distance than the grid spacing. In a first order upstream scheme the q_i^{n+1} depends only on the old q_{i-1}^n and q_i^n values, it doesn't include information about q_{i-2}^n, but with such a large Δt it should have included it. The algorithm doesn't know about q_{i-2}^n and therefore it will produce something that is clearly not a solution.

To keep a numerical algorithm stable, the time step has to obey the CFL condition:

" The domain of dependence of q_i^{n+1} of the algorithm at $t = t_n + \Delta t$ should include the domain of dependence at time $t = t_n$.

Nothing is allowed to flow more than 1 grid spacing with 1 time step:

$$\Delta t \leq \frac{\Delta x}{u}$$

The CFL condition is a necessary but not sufficient condition for any explicit differencing method.

Typically one defines the Courant number

$$C_{cfl} \equiv \frac{u \Delta t}{\Delta x} \leq 1$$

which can be set according to your task I guess.

In multiple dimensions, following the same physical arguments as before, one would expect the CFL condition to be in the form

$$\Delta t_{max} = C_{CFL} \min \left\{ \frac{\Delta x_i}{|u_{i,max}|} , \quad i \leq ndim \right\}$$

However, doing a proper stability analysis reveals that a more restrictive condition needs to be imposed, namely

$$\Delta t_{max} = C_{CFL} \left(\sum_{i=1}^{ndim} \frac{|u_{i,max}|}{\Delta x_i} \right)^{-1}$$

or explicitly in 2D:

$$\Delta t_{max} = C_{CFL} \left(\frac{|u_{x,max}|}{\Delta x} + \frac{|u_{y,max}|}{\Delta y} \right)^{-1}$$

3) Local Truncation Error and Order of the algorithm

Let $q_e(x,t)$ be an exact solution to the advection equation. Let q_i^n be a discrete solution. We represent the numerical algorithm by a transport operator T:

$$q_i^{n+1} = T[q_i^n]$$

For upstream differencing:

$$T[q_i^n] = q_i^n - \frac{\Delta t}{\Delta x} u (q_i^n - q_{i-1}^n)$$

We can define a One Step Error OSE:

$$OSE \equiv T[q_{e,i}^n] - q_{e,i}^{n+1}$$

we can again Taylor-expand the terms $q_{e,i}^{n+1}$ and q_{i-1}^n:

$$q_{e,i}^{n+1} = q_{e,i}^n + \frac{\partial q_i^n}{\partial t} \Delta t + \frac{1}{2} \Delta t^2 \frac{\partial^2 q_i^n}{\partial t^2} + \mathcal{O}(\Delta t^3)$$

$$q_{e,i-1}^n = q_{e,i}^n - \frac{\partial q}{\partial x} \Delta x + \frac{1}{2} \Delta x^2 \frac{\partial^2 q_i^n}{\partial x^2} + \mathcal{O}(\Delta x^3)$$

This gives us for the OSE:

$$OSE = T[q_{e,i}^n] - q_{e,i}^{n+1}$$

$$= q_{e,i}^n - \frac{\Delta t}{\Delta x} u (q_{e,i}^n - q_{e,i-1}^n) - q_{e,i}^{n+1}$$

$$= q_{e,i}^n - u\frac{\Delta t}{\Delta x}\left[q_{e,i}^n - q_{e,i}^n + \frac{\partial q}{\partial x}\Delta x - \frac{1}{2}\Delta x^2 \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(\Delta x^3)\right]$$

$$- \left[q_{e,i}^n + \frac{\partial q}{\partial t}\Delta t + \frac{1}{2}\Delta t^2 \frac{\partial^2 q}{\partial t^2} + \mathcal{O}(\Delta t^3)\right]$$

$$= -u\left[\frac{\partial q}{\partial x}\Delta t - \frac{\Delta t}{2}\frac{\partial^2 q}{\partial x^2}\Delta x + \mathcal{O}(\Delta x^2)\right] -$$

$$- \left[\Delta t \frac{\partial q}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 q}{\partial t^2} + \mathcal{O}(\Delta t^3)\right]$$

$$= -\Delta t\left[u\left(\frac{\partial q}{\partial x} - \frac{1}{2}\Delta x \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(\Delta x^2)\right) +\right.$$

$$\left. + \frac{\partial q}{\partial t} + \frac{1}{2}\Delta t \frac{\partial^2 q}{\partial t^2} + \mathcal{O}(\Delta t^2)\right]$$

Ignoring higher order terms, we have

$$OSE = -\Delta t\left[\underbrace{\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x}}_{=0,\ \text{advection equation}} + \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)\right]$$

$$= \Delta t\left[\mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)\right]$$

With smaller Δt, the OSE gets smaller by a factor Δt^2, however you'll need to do more steps to get to a fixed end time T, thus introducing proportionally more One Step Errors.

So it is convenient to define the local truncation error LTE:

$$LTE = \frac{1}{\Delta t} OSE$$

For this particular scheme:

$$LTE = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$$

In general, an algorithm is called consistent with the PDE when the LTE behaves as

$$LTE = \sum_{k=0,l} \mathcal{O}(\Delta t^k \Delta x^{l-k}).$$

For this method, $l = 1$, and

$$LTE = \sum_{k=0,1} \mathcal{O}(\Delta t^k \Delta x^{l-k}) = \mathcal{O}(\Delta t^0 \Delta x^1) + \mathcal{O}(\Delta t^1 \Delta x^0)$$
$$= \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)$$

This is, however, not the end of the story. As it turns out, this is only valid for smooth functions $q(x,0)$. So what if we have an initial condition that has a discontinuity?

We can again write down our solution up to first order using Taylor expansions:

$$q_i^{n+1} = q_i^n - u\frac{\Delta t}{\Delta x}\left(q_i^n - q_{i-1}^n\right)$$

$$q_i^{n+1} = q_i^n + \frac{\partial q_i^n}{\partial t}\Delta t + \frac{1}{2}\Delta t^2 \frac{\partial^2 q_i^n}{\partial t^2} + \mathcal{O}(\Delta t^3)$$

$$q_{i-1}^n = q_i^n - \frac{\partial q_i^n}{\partial x}\Delta x + \frac{1}{2}\Delta x^2 \frac{\partial^2 q_i^n}{\partial x^2} + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow q_i^{n+1} - q_i^n = q_i^n + \frac{\partial q_i^n}{\partial t}\Delta t + \frac{1}{2}\Delta t^2 \frac{\partial^2 q_i^n}{\partial t^2} + \mathcal{O}(\Delta t^3) - q_i^n$$

$$= \Delta t\left(\frac{\partial q_i^n}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 q_i^n}{\partial t^2} + \mathcal{O}(\Delta t^2)\right)$$

$$= u\frac{\Delta t}{\Delta x}\left(q_{i-1}^n - q_i^n\right)$$

$$= u\frac{\Delta t}{\Delta x}\left(-\frac{\partial q_i^n}{\partial x}\Delta x + \frac{1}{2}\Delta x^2 \frac{\partial^2 q_i^n}{\partial x^2} + \mathcal{O}(\Delta x^3)\right)$$

We re-write it as:

$$0 = \Delta t \left[\frac{\partial q_i^n}{\partial t} + \frac{\Delta t}{2} \frac{\partial^2 q_i^n}{\partial t^2} + u \frac{\partial q_i^n}{\partial x} - \frac{u}{2} \Delta x \frac{\partial^2 q_i^n}{\partial x^2} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) \right]$$

$$= \Delta t \left[\frac{\partial q_i^n}{\partial t} + u \frac{\partial q_i^n}{\partial x} + \frac{\Delta t}{2} \frac{\partial^2 q_i^n}{\partial t^2} - \frac{u \Delta x}{2} \frac{\partial^2 q_i^n}{\partial x^2} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) \right]$$

We use the advection equation to relate the second derivatives:

$$\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = 0 \qquad [1]$$

$$\Rightarrow \frac{\partial^2 q}{\partial t^2} = -u \frac{\partial^2 q}{\partial t \partial x} \qquad \Big| \frac{\partial}{\partial t} [1]$$

and $$\frac{\partial^2 q}{\partial x \partial t} = -u \frac{\partial^2 q}{\partial x^2} \qquad \Big| \frac{\partial}{\partial x} [1]$$

$$\Rightarrow \frac{\partial^2 q}{\partial t^2} = u^2 \frac{\partial^2 q}{\partial x^2}$$

Inserting that into our equation gives us

$$\frac{\partial q_i^n}{\partial t} + u \frac{\partial q_i^n}{\partial x} = -\frac{1}{2} \left[\Delta t u^2 \frac{\partial^2 q}{\partial x^2} - u \Delta x \frac{\partial^2 q_i^n}{\partial x^2} \right] + \mathcal{O}(2)$$

$$= \frac{u \Delta x}{2} \left(1 - \frac{u \Delta t}{\Delta x} \right) \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(2)$$

Lastly, we also add the Courant number

$$C_{cfl} = \frac{u \Delta t}{\Delta x} \leq 1$$

to get

$$\frac{\partial q_i^n}{\partial t} + u \frac{\partial q_i^n}{\partial x} = \frac{u \Delta x}{2} (1 - C_{cfl}) \frac{\partial^2 q_i^n}{\partial x^2}$$

$$\equiv \beta \frac{\partial^2 q_i^n}{\partial x^2}$$

with $\beta \equiv \frac{1}{2} u \Delta x (1 - C_{cfl})$

The solution of this advection-diffusion equation is

$$q^n = q^0 \operatorname{erfc}\left(\frac{x - ut}{\sqrt{4 \beta t}} \right)$$

with

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} \, dz$$

The exact solution of the advection equation is

$$q_e(x,t) = q_e(x = x - ut,\ t = 0)$$

Now let us simplify by introducing a coordinate transform

$$x' = x - ut$$

then $$q_e(x,t) = q_e(x', 0) = q_0(x')$$

and $$q^n(x,t) = q_0 \operatorname{erfc}\left(\frac{x'}{\sqrt{4\beta t}}\right)$$

Suppose now our initial conditions were a step function:

$$q_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Then we can compute the error at time t to be

$$Err = \| q_e(x,t) - q^n(x,t) \|_1$$

$$= \int_{-\infty}^{\infty} |(q_e(x,t) - q^n(x,t))| \, dx$$

$$= \int_{-\infty}^{\infty} \left| \left(q_0(x';t) - q_0(x';t)\, \mathrm{erfc}\left(\frac{x'}{\sqrt{4\beta t}}\right)\right)\right| dx'$$

$$= \int_{-\infty}^{0} \left| q_0(x';t)\left(1 - \mathrm{erfc}\left(\frac{x'}{\sqrt{4\beta t}}\right)\right)\right| dx'$$ | integral from 0 to ∞ $=0$ because of initial condition

$$= -\int_{0}^{-\infty} \left| 1 - \mathrm{erfc}\left(\frac{x'}{\sqrt{4\beta t}}\right)\right| dx'$$ | switched integration limits

$$= \sqrt{4\beta t} \int_{0}^{\infty} \left| 1 - \mathrm{erfc}(-y)\right| dy$$ | $y = \frac{-x'}{\sqrt{4\beta t}} \to dy = \frac{-dx'}{\sqrt{4\beta t}}$

$$= \sqrt{4\beta t} \int_{0}^{\infty} \left| \mathrm{erf}(-y)\right| dy$$ | $\mathrm{erfc} = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-z^2} dz$

| $\mathrm{erf} = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-z^2} dz$

$= 1 - \mathrm{erfc}$

$$= \sqrt{4\beta t} \int_{0}^{\infty} \left| -\mathrm{erf}(y)\right| dy$$ | $\mathrm{erf}(x)$ is odd

$$= \sqrt{4\beta t} \int_{0}^{\infty} \mathrm{erf}(y)\, dy$$ | $\mathrm{erf}(x) \geq 0$ for $x \geq 0$

$$= C_1 \sqrt{\beta t}$$ | we don't care about the actual integral

Inserting $\beta = \frac{u\Delta x}{2}(1 - C_{CFL}) \equiv c\Delta x$

We get

$$Err = C_1\sqrt{\beta t} = C_2\sqrt{\Delta x\, t}$$

Suppose we have fixed Δx and Δt, such that $t = N\Delta t$, then for a fixed number of steps N:

$$Err = C_3\sqrt{\Delta t}$$

More generally, let us write

$$\beta = \frac{u\Delta x}{2}(1 - C_{CFL}) = \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right)$$

$$= \frac{u\Delta x}{2} - \frac{u^2}{2}\Delta t \equiv a\Delta x + b\Delta t$$

and again

$$t = N\Delta t$$

Then we end up with

$$Err = C_1 \sqrt{\beta t} = C_1 \sqrt{(a\Delta x + b\Delta t) N \Delta t}$$

for variable Cfl, or

$$Err = C_2 \sqrt{\Delta x \, N \, \Delta t}$$

for fixed Cfl.

$\Rightarrow$ In order to test convergence of your method, you must be very careful which values you hold fixed to obtain the required power laws!

4) Lax-Richtmyer Stability Analysis

Let us again represent the numerical algorithm by a transport operator T:

$$q_i^{n+1} \equiv T[q_i^n] \overset{!}{=} q_i^n - u\frac{\Delta t}{\Delta x}(q_i^n - q_{i-1}^n)$$

And we also use again the discrete values of the exact solution q_e:

$$q_{e,i}^n = q_e(x = x_i,\ t = t_n)$$

Finally, let us define a norm $\|\cdot\|$ by which we can measure the magnitude of the error. The p-norm is given by

$$\|E\|_p = \left(\int_{-\infty}^{\infty} |E(x)|^p\,dx\right)^{1/p}$$

$$= \left(\Delta x \sum_{i=-\infty}^{\infty} |E(x_i)|^p\right)^{1/p}$$

In each time step undertaken the discrete solution acquires an error. At any timestep n, we can write the accumulated error as

$$E_i = q_i^n - q_{e,i}^n$$

So when we apply T to q_i^n:

$$q_i^{n+1} = T[q_i^n] = T[q_{e,i}^n + E_i^n]$$

and

$$E_i^{n+1} = q_i^{n+1} - q_{e,i}^{n+1} \qquad \text{by definition of } E_i$$

$$= T[q_{e,i}^n + E_i^n] - q_{e,i}^{n+1}$$

$$= T[q_{e,i}^n + E_i^n] - q_{e,i}^{n+1} + \left(T[q_{e,i}^n] - T[q_{e,i}^n]\right)$$

$$= T[q_{e,i}^n + E_i^n] - T[q_{e,i}^n] + \underbrace{\Delta t \, LTE[q_{e,i}^n]}_{OSE \equiv q[q_{e,i}^n] - q_{e,i}^{n+1}}$$

We want to show that a contractive operator

$$\| T[P] - T[Q] \| \leq \| P - Q \| \Leftrightarrow T[\,] \text{ contractive}$$

then the numerical method is stable.

Assume $T[\,]$ is contractive. Then

$$\|E_i^{n+1}\| = \|T[q_{e,i}^n + E_i^n] - T[q_{e,i}^n] + \Delta t\, LTE[q_{e,i}^n]\|$$

$$\leq \|T[q_{e,i}^n + E_i^n] - T[q_{e,i}^n]\| + \|\Delta t\, LTE[q_{e,i}^n]\|$$

$$\leq \|q_{e,i}^n + E_i - q_{e,i}^n\| + \|\Delta t\, LTE[q_{e,i}^n]\|$$

$$= \|E_i^n\| + \|\Delta t\, LTE[q_{e,i}^n]\|$$

We take the error at $t=0$ to be zero, as at that point nothing has been done to the initial conditions. Then we apply the inequality recursively to get

$$\|E^N\| \leq \Delta t \sum_{n=1}^{N} \|LTE[q_e^n]\|$$

The LTE is always defined on the true solution $q_{e,i}^n$. We expect the true solution to be well-behaved. Therefore we expect that $\|LTE[q_e^n]\|$ is a bounded number, and define

$$M_{LTE} = \max_{1\leq n\leq N} \|LTE[q_e^n]\|$$

$$\Rightarrow \|E^N\| \leq \Delta t \sum_{n=1}^{N} \|LTE[q_e^n]\| \leq \Delta t\, N M_{LTE} = t\, M_{LTE}$$

$\Rightarrow$ the final global error $\|E^N\|$ is bounded if $T[\,]$ is contractive.

Sometimes the stability requirement is loosened a bit: If

$$\| T[P] - T[Q] \| \leq (1 + \alpha \Delta t) \| P - Q \|$$

then

$$\| E^{n} \| \leq (1 + \alpha \Delta t) \| E^{n} \| + \Delta t \, \| LTE[q_e^{n}] \|$$

$$\Rightarrow \| E^{N} \| \leq t \, M_{LTE} \, e^{\alpha t}$$

with t being the total simulation time, independent on Δt. Otherwise, exponentials are clear signs of instability.

$\Rightarrow$ As long as the LTE is linear or higher in Δt, one can always find a small enough Δt such that

$$\| E^{N} \| \ll \| q_e^{N} \|.$$

This leads to a fundamental theorem of numerical integration, the Lax Equivalence Theorem:

$$\text{Consistency} + \text{Stability} \longrightarrow \text{Convergence}$$

5) Von Neumann Stability Analysis

We apply the Von Neumann Stability analysis on linear advection.

We use the advection equation

$$\partial_t q + u \partial_x q = 0$$

with the solution $q(x,t) = q(x - ut)$

and look at what's happening in Fourier space:

$$q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{q}(k) e^{ikx} dk$$

$$\Rightarrow \quad q(x-ut) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{q}(k) e^{ik(x-ut)} dk$$

$$\Rightarrow \quad \tilde{q}(k,t) = \tilde{q}(k, t=0)\, e^{-iukt}$$

which is just a phase rotation in Fourier space.

$\Rightarrow$ The true operator $T_e[\,]$ is merely a complex number: $T_e = e^{-iuk\Delta t}$.

Formally, we need to let the numerical operator $T[\,]$ act on the Fourier transforms of $q_{e,i}^n + \varepsilon_i^n$ and $q_{e,i}^n$ and subtract them. For a linear operator however we can do the analysis directly on $\hat{\tilde{q}}_e^n(k)$ and check if the amplitude is ≤ 1.

We can directly derive the CTE in terms of an amplitude error and a phase error.

If the amplitude of $T \leq 1$ $\forall k, \Delta x$, and $\Delta t \leq C_{cfl} \frac{\Delta x}{u}$, then $\hat{q}(k,t)$ is not growing exponentially and therefore the error is not growing exponentially.

If the algorithm has amplitude $\ll 1$, then it is very difusive.

We can also analyze the phase error to see if the algorithm transports each mode with the correct speed.

Let us try this on the example of the centered difference scheme.

Suppose we have the analytical solution

$$q(x_i, t) = A e^{ik(x_i - ut)}$$

and assume

$$q_i^0 = q(x = x_i, 0) = e^{ikx_i}$$

Then let us insert that into the centered difference scheme:

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{2\Delta x} u \left(q_{i+1}^n - q_{i-1}^n \right)$$

$$\Rightarrow q_i^1 = e^{ikx_i} - \frac{\Delta t}{2\Delta x} u \left(e^{ik(x_i+\Delta x)} - e^{ik(x_i-\Delta x)} \right)$$

$$= e^{ikx_i} \left(1 - \frac{\Delta t}{2\Delta x} u \left[e^{ik\Delta x} - e^{-ik\Delta x} \right] \right)$$

$$= e^{ikx_i} \left(1 - i \frac{\Delta t}{\Delta x} u \sin (k\Delta x) \right)$$

$$\equiv e^{ikx_i} \left(1 - i \varepsilon \sin \beta \right)$$

where we defined

$$\varepsilon = u \frac{\Delta t}{\Delta x}, \quad \text{and}$$

$$\beta = k \Delta x$$

From this it follows

$$q^{n+1} = T[q_i^n] = (1 - i \varepsilon \sin \beta) q^n$$

$$\Rightarrow T = (1 - i \varepsilon \sin \beta)$$

T is most easily analysed by computing

a) the squared magnitude R:

$$R = T^* \cdot T = Re(T)^2 + Im(T)^2$$

$$= 1 + \varepsilon^2 \sin^2 \beta$$

b) Phase ϕ

$$\tan \phi = \frac{Im\, T}{Re\, T} = - \varepsilon \sin \beta$$

Now compare this to the analytic solution:

a) $R_e = \cos^2(k(x-u\Delta t)) + \sin^2(k(x-u\Delta t)) = 1$

b) $\tan \phi = \dfrac{\sin(k(x-u\Delta t))}{\cos(k(x-u\Delta t))} = \tan(k(x-u\Delta t))$

a) $\dfrac{R}{R_e} = 1 + \varepsilon^2 \sin^2 \beta > 1$

The amplitude of the scheme always gets bigger. The scheme is unconditionally unstable as no (positive) values for Δt, Δx make it stable.

b) The phase also has an error.

Adding artificial viscosity can stabilize an algorithm.

If we for example solve

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{2\Delta x} u \left(q_{i+1}^n - q_{i-1}^n\right) + D\frac{\Delta t}{\Delta x^2}\left(q_{i+1}^n - 2q_i^n + q_{i-1}^n\right)$$

$$\equiv q_i^n - \frac{\varepsilon}{2}\left(q_{i+1}^n - q_{i-1}^n\right) + \nu\left(q_{i+1}^n - 2q_i^n + q_{i-1}^n\right)$$

with $\varepsilon \equiv u\frac{\Delta t}{\Delta x}$, $\nu = D\frac{\Delta t}{\Delta x^2}$

then doing the same analysis before shows that we can obtain a stable algorithm for

$$\nu \geq \varepsilon^2/2$$

If we choose precisely enough viscosity such that the algorithm is stable, $\nu = 1/2\, \varepsilon^2$, we end up with the upwind differencing Lax-Wendroff method, i.e. piecewise linear advection.