Higher Order and TVD schemes for Non-Linear Systems

1. Weighted Average Flux (WAF) Schemes

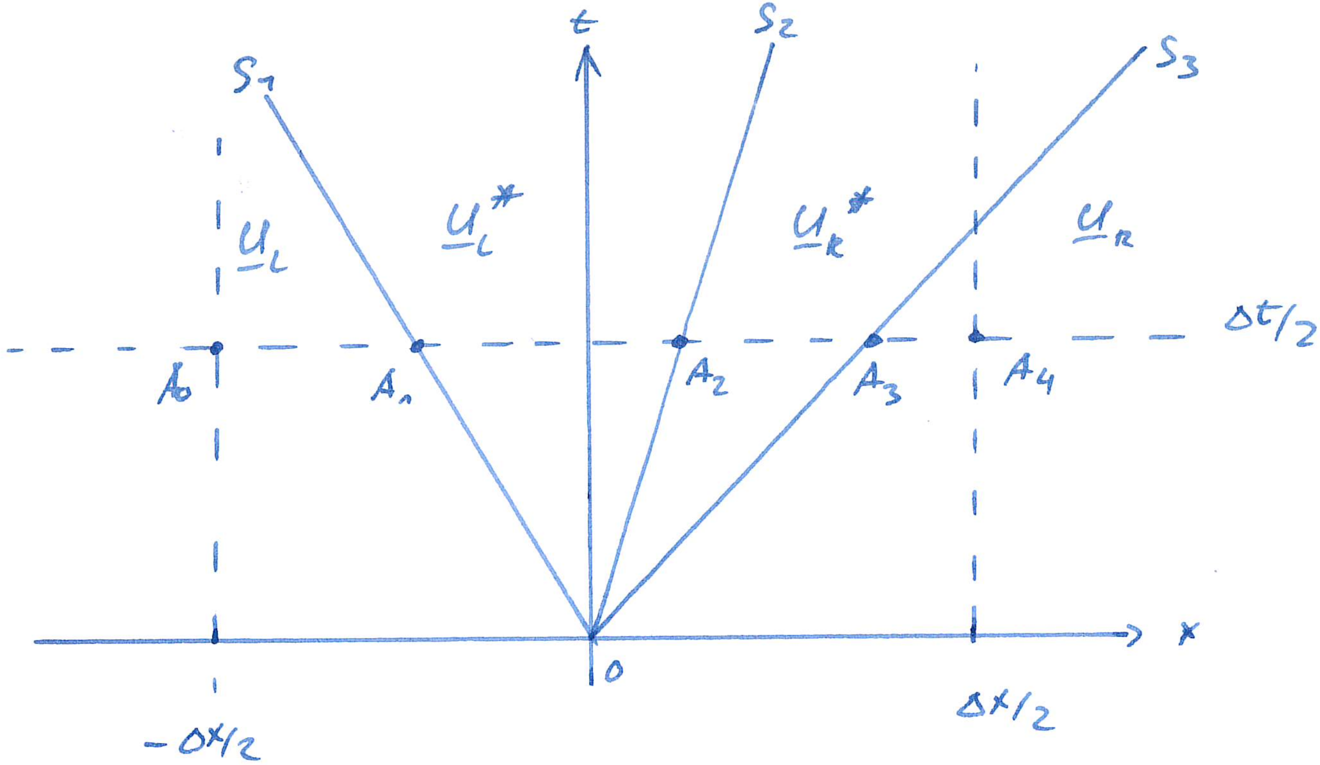
1.1 The original version of WAF

The simplest WAF flux is given as

$$\underline{F}_{i+1/2} = \frac{1}{\Delta x} \int_{-1/2\,\Delta x}^{1/2\,\Delta x} \underline{F}\left(\underline{U}_{i+1/2}\left(x, \frac{\Delta t}{2}\right)\right) dx$$

where $\underline{U}_{i+1/2}(x,t)$ is the solution of the Riemann problem with piecewise constant data $\underline{U}_i^n$, $\underline{U}_{i+1}^n$ at the interface position $i+1/2$. $\underline{U}_{i+1/2}$ is given by the Riemann solver.

The solution of the Riemann problem gives three waves of speeds S_1, S_2, and S_3, respectively, separating the initial states $\underline{U}_L$ and $\underline{U}_R$ into four states $\underline{U}_L$, $\underline{U}_L^*$, $\underline{U}_R^*$, $\underline{U}_R$:



Let us define

$$\underline{U}^{(1)} = \underline{U}_L$$
$$\underline{U}^{(2)} = \underline{U}_L^*$$
$$\underline{U}^{(3)} = \underline{U}_R^*$$
$$\underline{U}^{(4)} = \underline{U}_R$$

and assume $S_1 < S_2 < S_3$.

Then the WAF flux can be written as

$$\underline{F}_{i+1/2} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \underline{F}(\underline{U}(x, \Delta t/2))dx = \frac{1}{\Delta x} \sum_{k=1}^{N+1} \int_{A_{k-1}}^{A_k} \underline{F}(\underline{U}^k(x, \Delta t/2))dx$$

Let us find expressions for the distances $\overline{A_{k-1} A_k}$.

If $S_1 < 0$:

$$\overline{A_0 A_1} = \frac{\Delta x}{2} - |S_1| \frac{\Delta t}{2} = \frac{\Delta x}{2} + S_1 \frac{\Delta t}{2}$$

$$= \frac{\Delta x}{2}\left(1 + \frac{S_1 \Delta t}{\Delta x}\right) = \frac{\Delta x}{2}(1 + C_1)$$

If $S_1 > 0$:

$$\overline{A_0 A_1} = \frac{\Delta x}{2} + |S_1| \frac{\Delta t}{2} = \frac{\Delta x}{2} + S_1 \frac{\Delta t}{2} = \frac{\Delta x}{2}(1 + C_1)$$

If $S_2 > 0$, $S_1 > 0$:

$$\overline{A_1 A_2} = |S_2| \frac{\Delta t}{2} - |S_1| \frac{\Delta t}{2}$$

$$= (S_2 - S_1) \frac{\Delta t}{2} = (S_2 - S_1) \frac{\Delta t}{\Delta x} \frac{\Delta x}{2}$$

$$= (C_2 - C_1) \frac{\Delta x}{2}$$

If $S_2 > 0$, $S_1 < 0$:

$$\overline{A_1 A_2} = |S_2| \frac{\Delta t}{2} + |S_1| \frac{\Delta t}{2} = S_2 \frac{\Delta t}{2} - S_1 \frac{\Delta t}{2}$$

$$= (C_2 - C_1) \frac{\Delta x}{2}$$

If $S_2 < 0$, $S_1 < 0$:

$$\overline{A_1 A_2} = |S_1| \frac{\Delta t}{2} - |S_2| \frac{\Delta t}{2} = -S_1 \frac{\Delta t}{2} + S_2 \frac{\Delta t}{2}$$

$$= (C_2 - C_1) \frac{\Delta x}{2}$$

If $S_3 > 0, \; S_2 > 0$:

$$\overline{A_2 A_3} = |S_3|\frac{\Delta t}{2} - |S_2|\frac{\Delta t}{2} = S_3 \frac{\Delta t}{\Delta x}\frac{\Delta x}{2} - S_2 \frac{\Delta t}{\Delta x}\frac{\Delta x}{2}$$

$$= (c_3 - c_2)\frac{\Delta x}{2}$$

If $S_3 > 0, \; S_2 < 0$:

$$\overline{A_2 A_3} = |S_3|\frac{\Delta t}{2} + |S_2|\frac{\Delta t}{2} = S_3\frac{\Delta t}{2} - S_2\frac{\Delta t}{2} = (c_3 - c_2)\frac{\Delta x}{2}$$

If $S_3 < 0, \; S_2 < 0$:

$$\overline{A_2 A_3} = |S_2|\frac{\Delta t}{2} - |S_3|\frac{\Delta t}{2} = -S_2\frac{\Delta t}{2} + S_3\frac{\Delta t}{2} = (c_3 - c_2)\frac{\Delta x}{2}$$

If $S_3 > 0$:

$$\overline{A_3 A_4} = \frac{\Delta x}{2} - |S_3|\frac{\Delta t}{2} = \frac{\Delta x}{2} - S_3\frac{\Delta t}{2}$$

$$= \frac{\Delta x}{2} - S_3\frac{\Delta t}{\Delta x}\frac{\Delta x}{2} = \frac{\Delta x}{2}(1 - c_3)$$

If $S_3 < 0$:

$$\overline{A_3 A_4} = \frac{\Delta x}{2} + |S_3|\frac{\Delta t}{2} = \frac{\Delta x}{2} - S_3\frac{\Delta t}{2} = \frac{\Delta x}{2}(1 - c_3)$$

In summary:

$$\overline{A_0A_1} = \frac{\Delta x}{2}(1+c_1)$$

$$\overline{A_1A_2} = (c_2 - c_1)\frac{\Delta x}{2}$$

$$\overline{A_2A_3} = (c_3 - c_2)\frac{\Delta x}{2}$$

$$\overline{A_3A_4} = \frac{\Delta x}{2}(1 - c_3) \qquad \text{with } c_k = \frac{S_k \Delta t}{\Delta x}$$

Regardless of the sign of the wave speed S_k.

We may define

$$\beta_k = \frac{\overline{A_{k-1}A_k}}{\Delta x}$$

with

$$\beta_k = \frac{1}{2}(c_k - c_{k-1})$$

where $c_0 = -1$, $c_5 = c_{N+1} = 1$

for N = number of waves in the problem.

This gives us the WAF flux

$$\underline{F}_{i+1/2} = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \underline{F}(\underline{U}(x, \Delta t/2))\,dx$$

$$= \frac{1}{\Delta x}\sum_{k=1}^{N+1}\int_{A_{k-1}}^{A_k} \underline{F}(\underline{U}(x, \Delta t/2))\,dx$$

$$= \frac{1}{\Delta x}\sum_{k=1}^{N+1} \overline{A_{k-1}A_k}\,\underline{F}(\underline{U}^{(k)}) = \sum_{k=1}^{N+1}\beta_k \underline{F}^{(k)}$$

If the left or right wave is a rarefaction, we need to adapt the integral slightly.

Rarefactions are bound by the head and tail characteristics. For a right rarefaction, we have

$$S_3 = S_{HR} = u_{i+1}^n + a_{i+1}^n \qquad \text{head}$$

$$\hat{S}_3 = S_{TL} = u_* + a_{*R} \qquad \text{tail}$$

Inside the fan, we have the relations

$$\rho_{fan,R} = \rho_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \frac{1}{a_R} \left(u_R - \frac{x}{t} \right) \right]^{\frac{2}{\gamma-1}}$$

$$u_{fan,R} = \frac{2}{\gamma+1} \left[\frac{\gamma-1}{2} u_L - a_R + \frac{x}{t} \right]$$

$$p_{fan,R} = p_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \frac{1}{a_R} \left(u_R - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma-1}}$$

Let us express the equations for simplicity as

$$\rho_{fan,R} = \rho_R \left[a + bx \right]^c$$

$$u_{fan,R} = d + ex$$

$$p_{fan,R} = p_R \left[f + gx \right]^h$$

And integrate the flux for each component individually:

$$\underline{F}_{fan,R} = \begin{pmatrix} \rho_{fan,R}\, u_{fan,R} \\ \rho_{fan,R}\, u_{fan,R} + p_{fan,R} \\ (E_{fan,R} + p_{fan,R})\, u_{fan,R} \end{pmatrix} \equiv \begin{pmatrix} F_\rho \\ F_{\rho u} \\ F_E \end{pmatrix}$$

So we have

$$F_S = \varrho_{fan,R}\, u_{fan,R} = \varrho_R [a+bx]^c (d+ex)$$

Then

$$\int_T^H F_S\, dx = \varrho_R \frac{(a+bx)^{c+1}\left[-ae + b(c+2)d + b(c+1)ex\right]}{b^2 (c+1)(c+2)}$$

$$F_{Su} = \varrho_{fan,R}\, u_{fan,R}^2 + p_{fan,R} =$$

$$= \varrho_R (a+bx)^c (d+ex)^2 + p_R [f+gx]^h$$

Then

$$\int_T^H F_{Su}\, dx = \frac{\varrho_R (a+bx)^{c+1} \left(2a^2e^2 - 2abe((c+3)d + (c+1)ex) + b^2((c^2+5c+6)d^2 + 2(c^2+4c+3)dex + (c^2+3c+2)e^2x^2\right)}{b^3(c+1)(c+2)(c+3)} +$$

$$+ \frac{p(f+gx)^{h+1}}{gh+g}$$

$$F_E = (E_{fan,s} + p_{fan})\, u_{fan,R}$$

$$E_{fan,s} = \frac{1}{2} \rho_{fan} u_{fan}^2 + \frac{p_{fan}}{\gamma - 1}$$

$$= \frac{1}{2} \rho [a + b_x]^c (d + ex)^2 + \frac{p[f + gx]^h}{\gamma - 1}$$

$$\Rightarrow F_E = \left(\frac{1}{2} \rho [a + b_x]^c (d + ex)^2 + \frac{p[f + gx]^h}{\gamma - 1} + p[f + gx]^h \right) \cdot$$

$$\cdot (d + ex)$$

and

$$\int F_E \, dx =$$

The same can be done for left rarefactions, with slightly different coefficients a, b, c, d, e, f, g, h. Rarefactions just add a new speed $\hat{S}_1$ or $\hat{S}_3$, and a new coefficient β_k to the integral.
However, computational experience suggests that for the purpose of evaluating the integral, one may lump together the rarefaction state with the closest constant state in the direction of the t-axis, or in other words: just pretend it's a constant state like the others with only the head velocity S_1 or S_3.

The special treatment of rarefactions is not needed when HLL type Riemann solvers are employed, since those directly give you the flux.

1.2 TVD version of WAF Schemes

We arrived at the expression

$$\underline{F}_{i+1/2} = \sum_{k=1}^{N+1} \beta_k \underline{F}^{(k)}$$

$$\beta_k = \frac{1}{2}(c_k - c_{k-1})$$

$$c_0 = -1, \quad c_{N+1} = 1, \quad c_k = \frac{S_k \Delta t}{\Delta x}$$

for N = number of waves in the problem

for the WAF flux.

We can rearrange this expression to

$$\underline{F}_{i+1/2} = \sum_{k=1}^{N+1} \beta_k \underline{F}^{(k)} = \frac{1}{2}\sum_{k=1}^{N+1} (c_k - c_{k-1}) \underline{F}^{(k)}$$

$$= \frac{1}{2}(c_1 - c_0)\underline{F}^{(1)} + \frac{1}{2}(c_2 - c_1)\underline{F}^{(2)} + \dots$$

$$+ \frac{1}{2}(c_{N+1} - c_N)\underline{F}^{(N+1)}$$

$$= -\frac{1}{2}c_0 \underline{F}^{(1)} + \frac{1}{2}c_{N+1}\underline{F}^{N+1} + \frac{1}{2}c_1 \underline{F}^{(1)} - \frac{1}{2}c_N \underline{F}^{(N+1)} +$$

$$+ \frac{1}{2}(c_2 - c_1)\underline{F}^{(2)} + \frac{1}{2}(c_3 - c_2)\underline{F}^{(3)} + \dots$$

$$= \frac{1}{2}\left(\underset{\underset{c_0=-1}{\uparrow}}{\underline{F}^{(1)}} + \underset{\underset{c_{N+1}=1}{\uparrow}}{\underline{F}^{(N+1)}}\right) - \sum_{k=1}^{N} \frac{1}{2} c_k \left(\underline{F}^{(k+1)} - \underline{F}^{(k)}\right)$$

$$= \frac{1}{2}\left(\underline{F}_i + \underline{F}_{i+1}\right) - \frac{1}{2}\sum_{k=1}^{N} c_k \left(\underline{F}^{(k+1)} - \underline{F}^{(k)}\right)$$

$$= \frac{1}{2}\left(\underline{F}_i + \underline{F}_{i+1}\right) - \frac{1}{2}\sum_{k=1}^{N} c_k \Delta \underline{F}_{i+1/2}^{(k)}$$

The extension from linear advection to non-linear hyperbolic conservation laws to make a method TVD is somewhat empirical, but is found to work well in practice.

For the purpose of applying a TVD constraint to non-linear systems, a valuable, though empirical, observation is that the solution to the complete system may be characterized by jumps in a single quantity q across each of the three waves.

We therefore require the computation of three limiter functions $\phi_{i+1/2}(r)$ per interval boundary.

The TVD modification of the WAF flux is

$$\underline{F}_{i+1/2} = \frac{1}{2}(\underline{F}_i + \underline{F}_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} \mathrm{sign}(c_k)\, \phi_{i+1/2}^{(k)} \Delta \underline{F}_{i+1/2}^{(k)}$$

where

$$\phi_{i+1/2}^{(k)} = \phi_{i+1/2}(r^{(k)})$$

$$r^{(k)} = \begin{cases} \dfrac{\Delta q_{i-1/2}^{(k)}}{\Delta q_{i+1/2}^{(k)}} & \text{if } c_k > 0 \\[2ex] \dfrac{\Delta q_{i+3/2}^{(k)}}{\Delta q_{i+1/2}^{(k)}} & \text{if } c_k < 0 \end{cases}$$

We select one single quantity q which is known to change across every wave. For the Euler equations, the choice

q = density or q = internal energy

give satisfactory results.

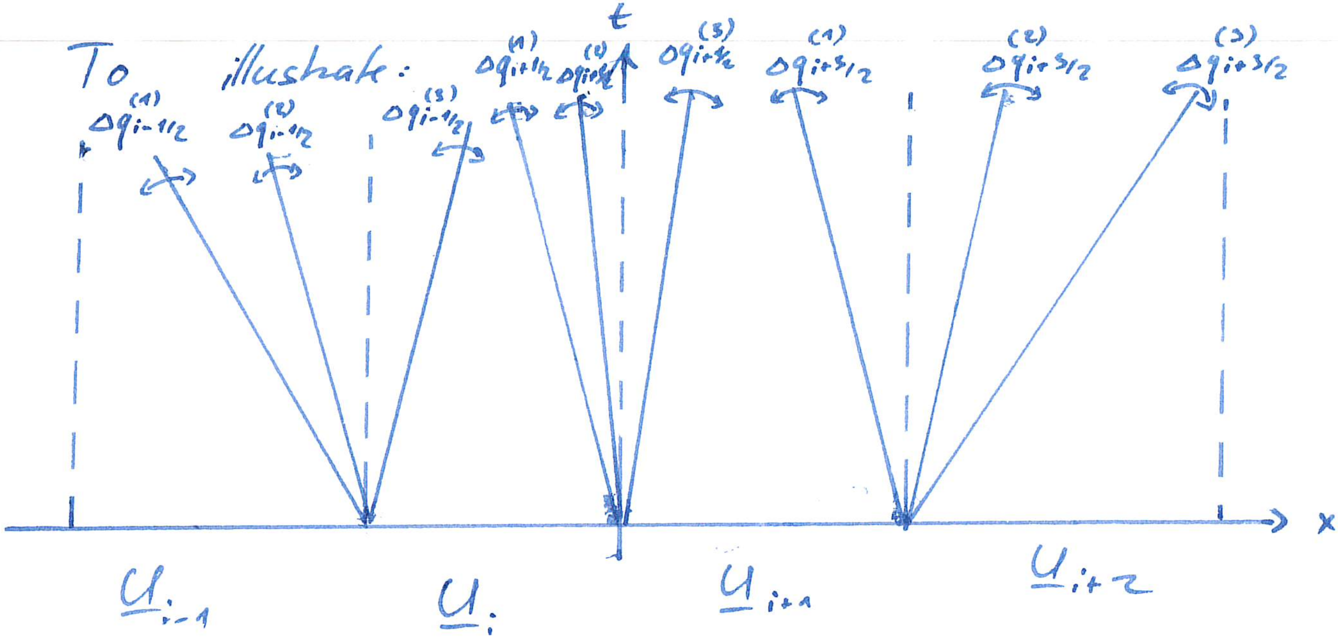
$\Delta q^{(k)}_{i+1/2}$, $\Delta q^{(k)}_{i-1/2}$, and $\Delta q^{(k)}_{i+3/2}$ however aren't the jumps in e.g. density over the three waves emerging out of the local solution of the Riemann problem. They are the jumps over waves in neighbouring Riemann problems:

- $\Delta q^{(k)}_{i+1/2}$ is the jump between wave k and $k+1$ of the solution of the Riemann problem with initial states $\underline{U}_L = \underline{U}_i$, $\underline{U}_R = \underline{U}_{i+1}$

- $\Delta q^{(k)}_{i-1/2}$ is the jump between wave k and $k+1$ of the Riemann solution with $\underline{U}_L = \underline{U}_{i-1}$, $\underline{U}_R = \underline{U}_i$

- $\Delta q^{(k)}_{i+3/2}$ is the jump between wave k and $k+1$ of the Riemann solution with $\underline{U}_L = \underline{U}_{i+1}$, $\underline{U}_R = \underline{U}_{i+2}$

To illustrate:



2. The MUSCL-Hancock Scheme

2.1 The Basic Scheme

The MUSCL-Hancock approach achieves a second order extension of the Godunov first order upwind method if the intercell flux $F_{i+1/2}$ is computed according to the following steps:

1) Data reconstruction

Data cell average values $\underline{U}_i^n$ are locally replaced by piece-wise linear functions in each cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ as follows:

$$\underline{U}_i(x) = \underline{U}_i^n + \frac{x - x_i}{\Delta x} \Delta_i \quad , \quad x \in [0, \Delta x]$$

Δ_i is a suitably chosen slope vector of $\underline{U}_i(x)$ in cell I_i.

The values of $\underline{U}_i(x)$ at cell boundaries are given by

$$\underline{U}_L = \underline{U}_i^n - \frac{1}{2}\Delta_i \, , \qquad \underline{U}_R = \underline{U}_i^n + \frac{1}{2}\Delta_i$$

2) Evolution

For each cell I_i, the boundary extrapolated values are evolved by a time $\frac{1}{2}\Delta t$ according to

$$\bar{\underline{U}}_i^L = \underline{U}_i^L + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\underline{F}(\underline{U}_i^L) - \underline{F}(\underline{U}_i^R)\right]$$

$$\bar{\underline{U}}_i^R = \underline{U}_i^R + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\underline{F}(\underline{U}_i^L) - \underline{F}(\underline{U}_i^R)\right]$$

Note that this evolution step is entirely contained to each cell I_i. At each intercell position $i+1/2$ there are two fluxes, $\underline{F}(\underline{U}_i^R)$ and $\underline{F}(\underline{U}_{i+1}^L)$, which are in general distinct.

These are not true intercell fluxes, but only intermediate steps.

3) The Riemann Problem

To compute the actual intercell flux $\underline{F}_{i+1/2}$ one now solves the conventional problem with data

$$\underline{U}_L \equiv \underline{\bar{U}}_i^R ; \quad \underline{U}_R = \underline{\bar{U}}_{i+1}^L$$

to obtain the solution $\underline{U}_{i+1/2}(x/t)$.

The intercell flux is then computed exactly as in the first order Godunov method:

$$\underline{F}_{i+1/2} = \underline{F}(\underline{U}_{i+1/2}(0))$$

2.2 TVD Version of the Scheme

The TVD version of the scheme is obtained by replacing the slopes Δ_i by limited slopes $\bar{\Delta}_i$.

There are essentially two ways of achieving this:

1) Limited slopes

Select limited slopes directly from forcing equivalence of the schemes with conventional flux limiter methods for the model scalar equations. Limited slopes are obtained from

$$\bar{\Delta}_i = \begin{cases} \max\left[0, \min(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \min(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right] & \text{if } \Delta_{i+1/2} > 0 \\ \min\left[0, \max(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \max(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right] & \text{if } \Delta_{i+1/2} < 0 \end{cases}$$

for particular values of the parameter β. $\beta = 1$ gives the minmod limiter, and $\beta = 2$ gives the superbee limiter.

2) Use of slope limiters

Find a slope limiter ξ_i such that

$$\bar{\Delta}_i = \xi_i \Delta_i$$

Some conventional slope limiters are

Minmod: $$\xi(r) = \begin{cases} 0 & r \le 0 \\ r & 0 \le r \le 1 \\ \min(1, \xi_R(r)) & r \ge 1 \end{cases}$$

Superbee: $$\xi(r) = \begin{cases} 0 & r \le 0 \\ 2r & 0 \le r \le 1/2 \\ 1 & 1/2 \le r \le 1 \\ \min\{r, \xi_R(r), 2\} & r \ge 1 \end{cases}$$

van Leer: $$\xi(r) = \begin{cases} 0 & r \le 0 \\ \min\left\{\frac{2r}{1+r}, \xi_R(r)\right\} & r \ge 0 \end{cases}$$

with $$\xi_R(r) = \frac{2}{1-\omega+(1+\omega)r}$$