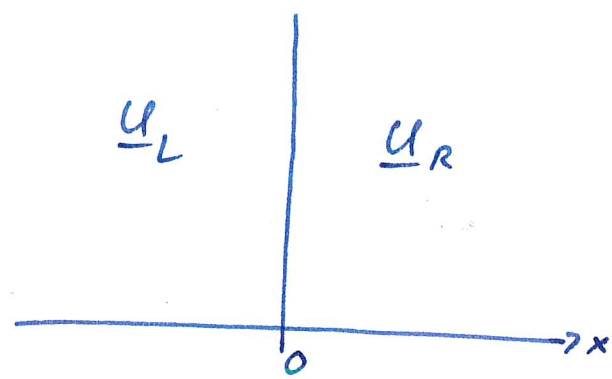
Introduction to Riemann Solvers and Numerical Fluid Dynamics

Following E. F. Toro's "Riemann Solvers and Numerical Methods for Fluid Dynamics"

1) Introduction

In scope of this introduction, we will concern ourselves with numerically solving equations of fluid dynamics. In particular, we will look at one-dimensional Euler equations for ideal gases. Extending the problem to multiple dimensions is relatively straightforward, and solving the problems for non-ideal gases generally follow the same steps, so there is no black magic required to obtain the solutions for more complicated cases.

At the heart of Eulerian (not co-moving) numerical fluid dynamics is the solution to a specific initial value problem called the "Riemann Problem".



For a hyperbolic system of conservation laws of the form

$$\underline{U}_t + \underline{F}(\underline{U})_x = 0$$

with $\underline{U}$ being the "state vector" and $\underline{F}(\underline{U})$ the "flux vector" which depends on $\underline{U}$ the Riemann

Problem is defined as: (in 1D):

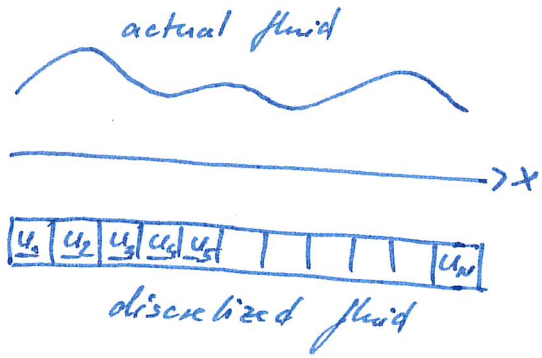
$$\underline{U}(x, t=0) = \begin{cases} \underline{U}_L & \text{if } x < 0 \\ \underline{U}_R & \text{if } x > 0 \end{cases}$$

Unfortunately, there is no exact closed-form solution to the Riemann problem for the Euler equations. However, it is possible to devise iterative schemes whereby the solution can be computed numerically. To solve full fluid dynamics problems, this calculation needs to be repeated many many times, making the solution quite expensive. For that reason, people have developed approximate Riemann solvers, which we also will have a look at.

So how is the solution to this particular physical problem used to solve any kind of fluid dynamical initial conditions?

Well, consider how we may represent a fluid in a computer simulation. We can't represent it as a continuum, due to the nature of our computing devices, it must be in some discretized form.

We may represent it as particles, an approach with which we will not concern ourselves in this introduction. Another option is to discretize the physical volume that we want to simulate into (smallest possible) cells.



Then by assuming that the State $\underline{U}_i$ within each cell i is constant at a fixed time t, we actually end up with a collection of Riemann problems, where $\underline{U}_L = \underline{U}_i$ and $\underline{U}_R = \underline{U}_{i+1}$. How exactly the Solution of the Riemann problem is used to advance the simulation will be a topic at later times, when we look at Godunov Schemes.

1.1) Comment on Notation

$\underline{U} = (u_1, u_2, u_3)^T$; Underline means vector

$\underline{U}_t = \frac{\partial \underline{U}}{\partial t}$; partial derivatives

$\underline{\underline{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$; two underlines = matrix

1.2) Euler Equations

The Euler equations are a system of non-linear hyperbolic conservation laws that govern the dynamics of a compressible material, for which the effects of body forces, viscous stresses and heat flux are neglected.

The set of variables to describe the state and the flow of a fluid can be chosen:

$\underline{U} = (\rho, \rho u, E)$ conserved variables

$\underline{W} = (\rho, u, p)$ primitive variables

with $E = \rho\left(\frac{1}{2}u^2 + e\right)$ total energy

$e = \frac{p}{(\gamma-1)\rho}$ specific internal energy of ideal gases

$\gamma = \frac{c_p}{c_v} = 1.4$ adiabatic exponent / specific heat ratio; $\frac{f+2}{f} = 1.4$ for diatomic gases

$a = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\frac{\gamma p}{\rho}}$ sound speed

The equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$

$$\frac{\partial}{\partial t}(E) + \frac{\partial}{\partial x}[(E + p)u] = 0$$

with $\underline{U} = (\rho, \rho u, E)^T$ and $\underline{F} = (\rho u, \rho u^2 + p, u(E+p))$

we can write:

$$\underline{U}_t + \underline{F}_x = 0$$

It turns out that it is easier to solve the Riemann problem using primitive variables instead of conserved ones. By expressing conservative variables with primitive ones and using the equations to simplify each other by inserting the previous equation into the next we get the following relations:

$$\rho_t + u\rho_x + \rho u_x = 0$$

$$u_t + uu_x + \frac{1}{\rho} p_x = 0$$

$$p_t + \rho a^2 u_x + up_x = 0$$

2. The Riemann Problem and Solution

2.1 Some definitions

Def Quasi-linear equations

An equation of the form

$$\underline{U}_t + \underline{\underline{A}}\,\underline{U}_x + \underline{B} = 0$$

is called

- linear, with constant coefficients, if $\underline{\underline{A}}, \underline{B} = \text{const}$
- linear, with variable coefficients, if $\underline{\underline{A}} = \underline{\underline{A}}(x,t)$, $\underline{B} = \underline{B}(x,$
- homogeneous: $\underline{B} = 0$
- quasi-linear: $\underline{\underline{A}} = \underline{\underline{A}}(\underline{U})$

Def Conservation laws

Conservation laws are systems of PDEs that can be written in the form

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

Def Jacobian Matrix

The Jacobian of the flux $\underline{F}(\underline{U})$ is the matrix

$$a_{ij}(\underline{U}) = \frac{\partial F_i}{\partial u_j}$$

$\Rightarrow$ From that it follows that conservation laws can be written in quasi-linear form:

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{U}_t + \underbrace{\frac{\partial \underline{F}}{\partial \underline{U}}}_{\text{Jacobian}} \frac{\partial \underline{U}}{\partial x} = \underline{U}_t + \underline{\underline{A}}(\underline{U})\underline{U}_x = 0$$

Def Hyperbolic system

A system $\underline{U}_t + \underline{\underline{A}}\,\underline{U}_x + \underline{B} = 0$ is said to be hyperbolic at the point (x, t) if $\underline{\underline{A}}$ has m real eigenvalues $\lambda_1, \dots, \lambda_m$ and a corresponding set of m linearly independent right eigenvectors $\underline{K}^{(1)}, \dots, \underline{K}^{(m)}$.

If the λ_i are all distinct, the system is called strictly hyperbolic.

2.2 Notions on hyperbolic conservation laws

By assuming hyperbolicity, the eigenvalues λ_i and the eigenvectors $K^{(i)}$ exist for the matrix $\underline{\underline{A}}$. This is the case for the Euler equations.

We assume that the eigenvalues are ordered:

$$\lambda_1(\underline{U}) < \lambda_2(\underline{U}) < \dots < \lambda_m(\underline{U}).$$

We can then write for every component i of the conservation law:

$$\frac{\partial u_i}{\partial t} + \lambda(u)\frac{\partial u_i}{\partial x} = 0$$

"Characteristics" may be defined as curves $x(t)$ in the $t-x$ plane along which the PDE becomes an ODE. Consider for example the curves $x = x(t)$ satisfying $\frac{\partial x}{\partial t} = \lambda(u)$

Then

$$\frac{du}{dt} \overset{\text{per definition}}{=} \frac{\partial u}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial u}{\partial x} \overset{\text{along char.}}{=} \frac{\partial u}{\partial t} + \lambda(u)\frac{\partial u}{\partial x} \overset{\text{conserv. law}}{=} 0$$

$\Rightarrow$ u is constant along these characteristic curves. Therefore also the slope $\lambda(u)$ must be constant, and the characteristic curves are straight lines.

Along each curve, we have $u(x,t) = u_0(x_0, t=0)$ and $\lambda(u) = \lambda(u_0(x_0))$. The λ correspond to the eigenvalues of the Jacobian $\frac{\partial F}{\partial u}$ and are also called the "characteristic speeds".

We can use these characteristics to construct the solution for the components u_i of the conservation law: $\lambda(u) = \lambda(u_0(x_0))$

$$\frac{dx}{dt} = \lambda(u), \quad x(0) \equiv x_0$$

$$\Rightarrow x(t) = x_0 + \lambda(u_0)\, t$$

but also: $x_0 = x(t) - \lambda(u_0)t$ depends on (x,t)!

$\Rightarrow$ depending on how far ahead in the future you want the solution for, and at which place, you'll have to pick a different initial state u_0!

Since $u(x,t) = u_0(x_0, t=0)$ along characteristics, we may obtain a solution by enforcing the characteristic condition implicitly:

$$u(x,t) = u_0(x - \lambda(u_0(x_0))\, t)$$

It can easily be verified that this is indeed a solution by inserting it into the conservation law equation.

What does this solution tell us?

In fact, it can already tell us the general shape of the solution: For every characteristic speed λ_i, we expect an (associated) wave, separating the state $\underline{u}$ over all space in unit constant states.

The remaining questions are:

- what type of waves can we have?
- how do we distinguish which wave we currently have for some specific λ_i?

It turns out that for the Riemann problem we can only have three types of waves: shocks, rarefactions and contact waves.

2.3 Waves in the solution of the Riemann problem

2.3.1 Wave types

To distinguish between wave types, let's first define some things which will help us.

Every characteristic speed $\lambda_i(\underline{U})$ defines a <u>characteristic field.</u>

A λ_i-characteristic field is said to be <u>linearly degenerate</u> if

$$\nabla \lambda_i(\underline{U}) \cdot \underline{K}^{(i)}(\underline{U}) = 0 \quad \forall \underline{U} \in \mathbb{R}^m$$

$$\nabla \lambda_i \equiv \left(\frac{\partial \lambda_i}{\partial u_1}, \frac{\partial \lambda_i}{\partial u_2}, \dots, \frac{\partial \lambda_i}{\partial u_m} \right)$$

A λ_i-characteristic field is said to be <u>genuinely non-linear</u> if

$$\nabla \lambda_i(\underline{U}) \cdot \underline{K}^{(i)}(\underline{U}) \neq 0 \quad \forall \underline{U} \in \mathbb{R}^m$$

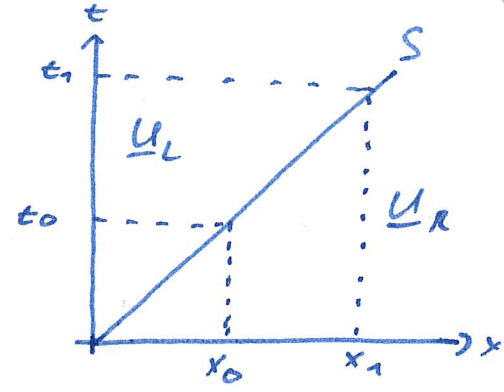
<u>Generalized Riemann Invariants</u> are relations that hold true for certain waves across the wave structure, meaning for both states left and right of the wave. For a general quasi-linear hyperbolic system

$$\underline{W}_t + \underline{\underline{A}}(\underline{W}) \underline{W}_x = \underline{0}$$

with $\underline{w} = [w_1, w_2, \ldots, w_m]^T$, and eigenvalues and eigenvectors λ_i, $\underline{k}^{(i)} = [k_1^{(i)}, k_2^{(i)}, \ldots, k_m^{(i)}]^T$.
For the wave associated with λ_i, the GRI are

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \cdots = \frac{dw_m}{k_m^{(i)}}$$

Lastly, across (infinitely) thin waves we can use the Rankine-Hugoniot conditions which are otherwise also very useful.



Consider a situation where two states, $\underline{U}_L$ and $\underline{U}_R$, are separated by an infinitely thin wave of speed S. (The wave is assumed to be a jump discontinuity.)

For any conservation law

$$\underline{U}_t + \underline{F}(\underline{U})_x = 0$$

we can integrate:

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1} [\underline{U}_t + \underline{F}(\underline{U})_x]\, dx\, dt = 0$$

Then:

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1}\left[\underline{U}_t + \underline{F}(\underline{U})_x\right]dx\,dt = \int_{x_0}^{x_1}\left[\underline{U}(t_1,x) - \underline{U}(t_0,x)\right]dx + \int_{t_0}^{t_1}\left[\underline{F}(t,x_1) - \underline{F}(t,x_0)\right]dt$$

Then $\forall\, x \in [x_0, x_1] : \underline{U}(t_1,x) = \underline{U}_L \;;\; \underline{U}(t_0,x) = \underline{U}_R$

and $\forall\, t \in [t_0, t_1] : \underline{F}(t,x_1) = \underline{F}_R \;;\; \underline{F}(t,x_0) = \underline{F}_L$

$$\Rightarrow \int_{x_0}^{x_1}\left[\underline{U}(t_1,x) - \underline{U}(t_0,x)\right]dx + \int_{t_0}^{t_1}\left[\underline{F}(t,x_1) - \underline{F}(t,x_0)\right]dt =$$

$$= -\left[\underline{U}_R - \underline{U}_L\right](x_1 - x_0) + \left[\underline{F}_R - \underline{F}_L\right](t_1 - t_0) = 0$$

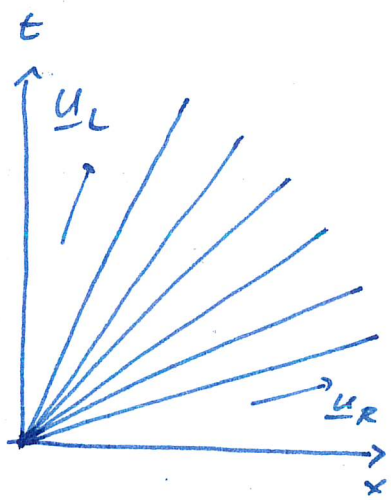
$$\Rightarrow \boxed{\left(\underline{F}_R - \underline{F}_L\right) = \frac{x_1 - x_0}{t_1 - t_0}\left(\underline{U}_R - \underline{U}_L\right) = S\left(\underline{U}_R - \underline{U}_L\right)}$$

These definitions and relations give us enough to discern and solve for the three wave types occuring in the solution of the Riemann problem:

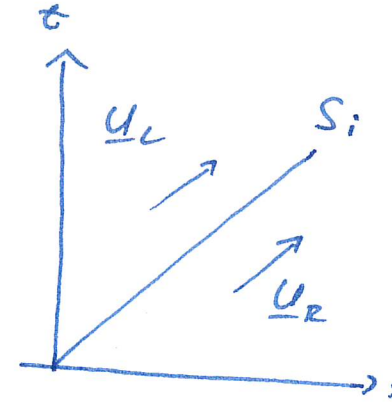
i) Rarefaction wave

$\underline{U}_L$ and $\underline{U}_R$ are connected through a smooth transition in a genuinely non-linear field i: The GRI apply, and the characteristics must diverge:

$\lambda_i(\underline{U}_L) < \lambda_i(\underline{U}_R)$



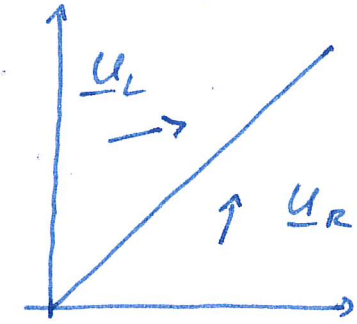
ii) Contact Wave



$\underline{U}_L$, $\underline{U}_R$ are connected through a jump discontinuity of speed S_i in a linearly degenerate field i. The GRI apply, as well as the Rankine-Hugeniot jump conditions.

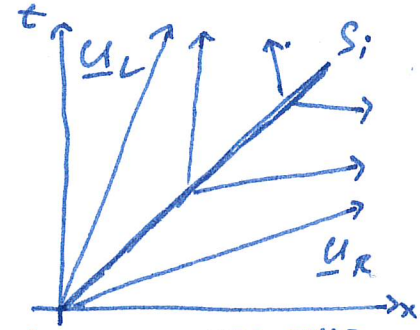
iii) Shock Wave

$\underline{U}_L$, $\underline{U}_R$ are connected through a single jump discontinuity in a genuinely non-linear field i. The Rankine-Hugeniot jump conditions apply and furthermore, we need the entropy condition $\lambda_i(\underline{U}_L) > S_i > \lambda_i(\underline{U}_R)$.



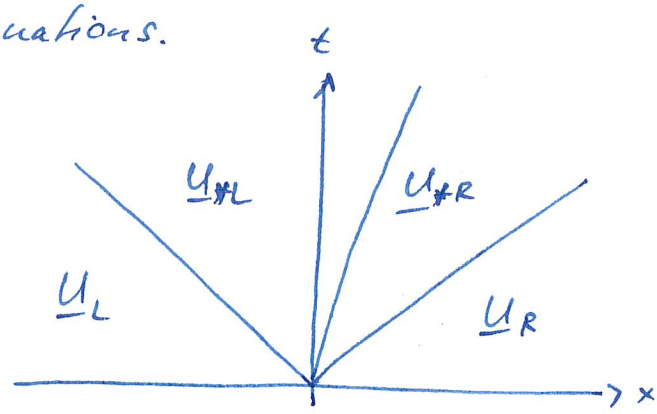
$\lambda_i(\underline{U}_L) < S_i < \lambda_i(\underline{U}_R)$

Otherwise, you'd get unphysical solutions for shock waves, albeit mathematically valid. If we head for a shock, the characteristics diverge from the discontinuity. Characteristics carry information about the solution; Diverging characteristics like they appear when the entropy condition is violated generate new information to be carried, which is unphysical!



WHY WE NEED THE ENTROPY CONDITION.

With the wave types known, as well as relations across the waves and conditions for a wave type to occur, we can now derive practical relations between states for the Riemann problem of the Euler equations.



We know that we will have $m+1$ constant states, with $m=3$ for 1D Euler equations, with the initial conditions

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ \underline{U}_R & \text{if } x>0 \end{cases}$$

we denote the two new states, which will arise for $t>0$ between $\underline{U}_L$ and $\underline{U}_R$, as $\underline{U}_{*L}$ and $\underline{U}_{*R}$, called the "star states".

The middle wave is always associated to a linearly degenerate field, and therefore always a contact wave.

The other two waves are always associated to a genuinely non-linear field, and can therefore be either rarefactions or shocks, depending on the initial conditions.

2.3.2 Contact Wave Relations

We get that across the contact wave:

$$\left.\begin{array}{l} p = \text{const} \\ u = \text{const} \end{array}\right\} \Rightarrow \begin{array}{l} p_{*L} = p_{*R} \\ u_{*L} = u_{*R} \end{array}$$

A contact wave is merely a jump discontinuity in the density.

2.3.3 Shock Wave Relations

All three quantities ρ, u, p change across a shock wave. We get the following relations. For a left shock wave: ($p_* > p_L$)

$$\rho_{*L} = \frac{\frac{p_*}{p_L} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_L} + 1}\,\rho_L$$

$$u_* = u_L - \frac{p_* - p_L}{\sqrt{\frac{p_* + B_L}{A_L}}}$$

$$A_L \equiv \frac{2}{(\gamma+1)\rho_L}$$

$$B_L \equiv \frac{\gamma-1}{\gamma+1}\,p_L$$

and the shock speed S_L:

$$S_L = u_L - a_L\left[\frac{\gamma+1}{2\gamma}\frac{p_*}{p_L} + \frac{\gamma-1}{2\gamma}\right]^{1/2}$$

For a right shock wave: ($p_* > p_R$)

$$s_{*L} = \frac{\frac{p_*}{p_R} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_R} + 1} s_R$$

$$u_* = u_R + \frac{p_* - p_R}{\sqrt{\frac{p_* - p_R}{A_R}}}$$

$$A_R \equiv \frac{2}{(\gamma+1)\rho_R}$$

$$B_R \equiv \frac{\gamma-1}{\gamma+1}\rho_R$$

and the right shock speed

$$s_R = u_R + a_R\left[\frac{(\gamma+1)}{2\gamma}\frac{p_*}{p_R} + \frac{\gamma-1}{2\gamma}\right]^{1/2}$$

The solution for p_* will be discussed later.

2.3.4 Rarefaction Wave Relations

Across rarefactions, entropy is conserved. We obtain:

$$s_{*L,R} = s_{L,R}\left(\frac{p_*}{p_{L,R}}\right)^{1/\gamma}$$

$$u_* = u_L + \frac{2a_L}{\gamma-1}\left[1 - \left(\frac{p_{*L}}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}\right] \quad \text{left facing wave } (p_* \le p_L)$$

$$u_* = u_R - \frac{2a_R}{\gamma-1}\left[1 - \left(\frac{p_{*R}}{p_R}\right)^{\frac{\gamma-1}{2\gamma}}\right] \quad \text{right facing wave } (p_* \le p_R)$$

Rarefaction waves are smooth transitions, not infinitesimally thin jump discontinuities. The waves are enclosed by the head and tail, which are characteristic speeds given by

$$S_{HL} = u_L - a_L, \quad S_{TL} = u_* - a_{*L}, \quad a_{*L} = a_L \left(\frac{p_*}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}$$

$$S_{HR} = u_R + a_R, \quad S_{TR} = u_* + a_{*R}, \quad a_{*R} = a_R \left(\frac{p_*}{p_R}\right)^{\frac{\gamma-1}{2\gamma}}$$

for left- or right facing waves, respectively. The solution for p_* will be discussed later.

2.3.5 Solution Inside the Rarefaction wave

In the cases where we want to find the solution inside the rarefaction fan, i.e. $S_{HL} \leq \frac{x}{t} \leq S_{TL}$ if $p_* < p_L$ or $S_{HR} \geq \frac{x}{t} \geq S_{TL}$ if $p_* < p_R$

Using the Generalized Riemann Invariants, we obtain:

$$s_{fan} = s_{L,R}\left[\frac{2}{\gamma+1} \pm \frac{\gamma-1}{\gamma+1}\frac{1}{a_{L,R}}\left(u_{L,R} - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}}$$

$$u_{fan} = \frac{2}{\gamma+1}\left[\frac{\gamma-1}{2}u_{L,R} \pm a_{L,R} + \frac{x}{t}\right]$$

$$p_{fan} = p_{L,R}\left[\frac{2}{\gamma+1} \pm \frac{\gamma-1}{\gamma+1}\frac{1}{a_{L,R}}\left(u_{L,R} - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}}$$

upper sign for left facing wave (index L)

2.3.6 Solution for the pressure in the Star Region

The solution for p^* can be obtained using the fact that pressure and particle velocity are constant across the star region. For both shocks and rarefactions on either side, we have equations for u^* depending on the outer states and p^*; By setting $u^*_{\text{left wave}} - u^*_{\text{right wave}} = 0$ which must hold, we get:

$$f(p, \underline{w}_L, \underline{w}_R) \equiv f_L(p, w_L) + f_R(p, \underline{w}_R) + (u_R - u_L) = 0$$

with

$$f_{L,R} = \begin{cases} (p - p_{L,R}) \left[\dfrac{A_{L,R}}{p + B_{L,R}} \right]^{1/2} & \text{if } p > p_{L,R} \text{ (shock)} \\ \dfrac{2a_{L,R}}{(\gamma - 1)} \left[\left(\dfrac{p}{p_{L,R}} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right] & \text{if } p \leq p_{L,R} \text{ (rarefaction)} \end{cases}$$

and $A_{L,R} = \dfrac{2}{(\gamma+1)\rho_{L,R}}, \qquad B_{L,R} = \dfrac{\gamma - 1}{\gamma + 1} p_{L,R}$

we need to find the p^* such that $f(p^*, \underline{w}_L, \underline{w}_R)$ is solved.

Since we have an analytic function and the first derivative with respect to ρ can be easily computed, the Newton-Raphson method can be used:

Suppose we have some initial guess for ρ_0. Then the approximate value of $f(\rho)$ at $\rho_0+\Delta\rho$ is

$$f(\rho_0+\Delta\rho) = f(\rho_0) + \Delta\rho f'(\rho_0) + \mathcal{O}(\Delta^2)$$
$$\approx f(\rho_0) + \Delta\rho f'(\rho_0)$$

If $\rho_0+\Delta\rho$ is a solution for $f(\rho)=0$, then we have

$$f(\rho_0) + \Delta\rho f'(\rho_0) = 0$$
$$\Rightarrow \Delta\rho = -\frac{f(\rho_0)}{f'(\rho_0)}$$

This gives us the corrected value $\rho_1 = \rho_0 + \Delta\rho$
$$= \rho_0 - \frac{f(\rho_0)}{f'(\rho_0)}$$

This process can be re-iterated until it converges, i.e. when the relative pressure change

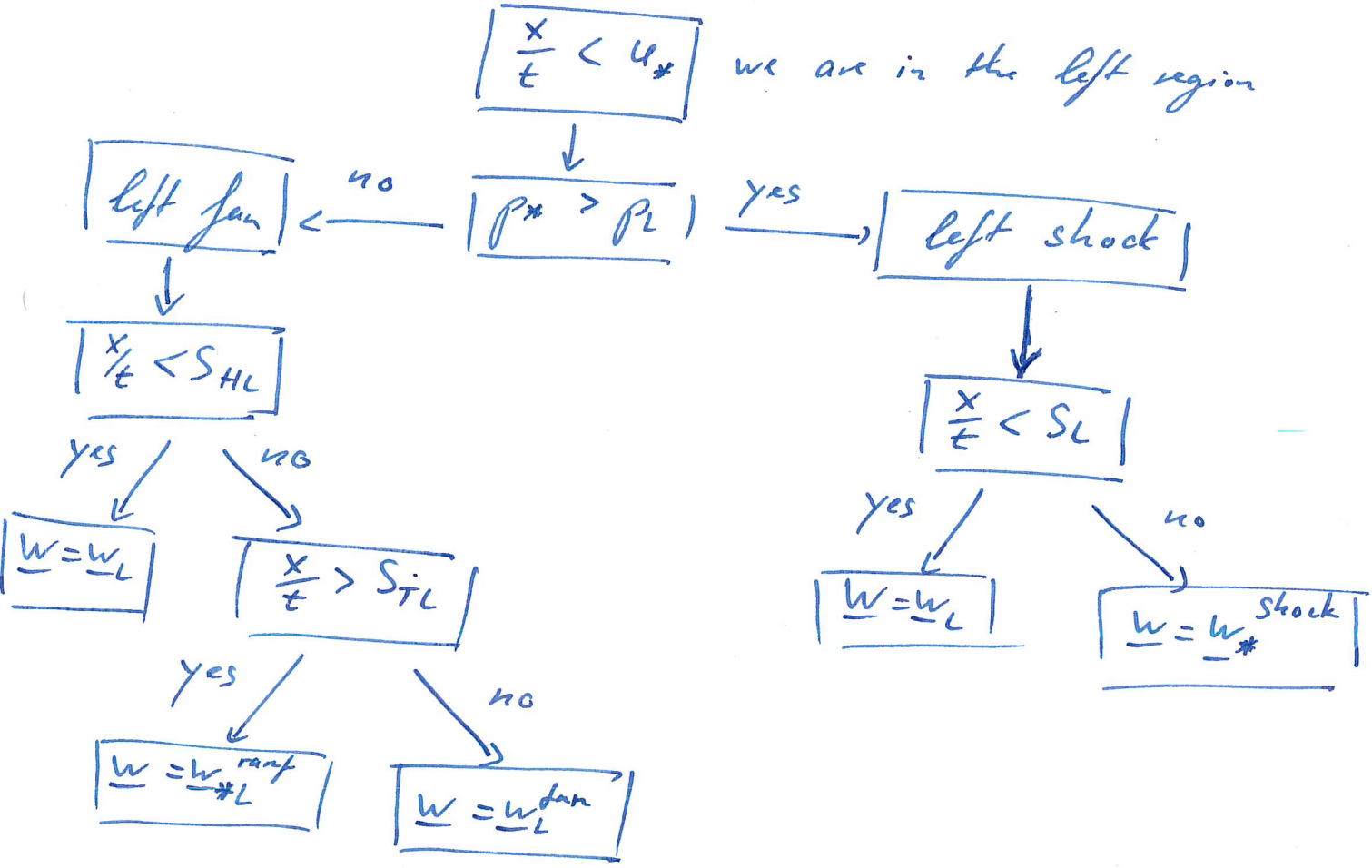
$$\frac{|p(k) - p(k-1)|}{\frac{1}{2}|p(k) + p(k+1)|} < \varepsilon$$

where ε is some tolerance, typically 10^{-6}.

2.4 Sampling the Solution

Now we have everything we need to solve the Riemann problem numerically, and we want to do so at a general point (x, t).

Assuming we have computed all the star region state variables, what is left to do is to determine in which case the point (x,t) is located: For the left side of the solution, we have:



Analogualy, such a flow chart can be created for the right side of the wave.

3) The Method of Godunov

3.1 Bases of Godunov's Method

Consider the general initial value boundary problem for non-linear systems of hyperbolic conservation laws:

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{0}$$

$$\underline{U}(x,0) = \underline{U}^{(0)}(x)$$

$$\underline{U}(0,t) = \underline{U}_l(t), \quad \underline{U}(L,t) = \underline{U}_r(t)$$

In order to admit discontinuous solutions, we must use the integral form of the conservation law:

$$\int_{x_1}^{x_2} \underline{U}(x,t_2)\,dx = \int_{x_1}^{x_2} \underline{U}(x,t_1)\,dx + \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_1,t))\,dt - \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_2,t))\,dt$$

We discretize the spatial domain $[0, L]$ into M computing cells of regular size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{L}{M}$ with $i = 1, \ldots, M$.

For any given cell i:

cell centre $\quad x_i = (i - \frac{1}{2}) \Delta x$

boundaries $\quad x_{i+\frac{1}{2}} = i \Delta x$

$\quad x_{i-\frac{1}{2}} = (i-1) \Delta x$

The Godunov method assumes a piece-wise constant distribution of data. This is formally realized by

$$\underline{U}_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}_{ini}(x, t^n)\, dx$$

given the general initial data $\underline{U}_{ini}(x, t^n)$.

To solve the IVP for the original conservation laws, but with the modified initial data, is effectively to solve local Riemann problems with data $\underline{U}_i$ (left side) and $\underline{U}_{i+1}$ (right side) centered at the intercell boundary positions $x_{i+\frac{1}{2}}$.

As we have seen, the solution of the Riemann problem will depend on $\bar{x}/\bar{t}$, where $\bar{x}, \bar{t}$ are in local coordinates to the specific Riemann problem under consideration.

For a time step Δt that is sufficiently small (to avoid wave interactions), one can define a global solution $\hat{\underline{U}}(x,t)$ in the strip $0 \le x \le L$, $t^n \le t \le t^{n+1}$, $t^{n+1} - t^n = \Delta t$ as follows:

$$\hat{\underline{U}}(x,t) = \underline{U}_{i+\frac{1}{2}}(\bar{x}/\bar{t}) \quad , \quad x \in [x_i, x_{i+1}]$$

Note that this is an exact solution for the piece-wise constant data.

3.2 Godunov's Method

We start from the integral form:

$$\int_{x_1}^{x_2} \underline{U}(x,t_2)\,dx = \int_{x_1}^{x_2} \underline{U}(x,t_1)\,dx + \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_1,t))\,dt - \int_{t_1}^{t_2} \underline{F}(\underline{U}(x_2,t))\,dt$$

Applying it to the control volume

$$[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$$

we can use the fact that we have an exact solution for the piece-wise constant problem:

$$\hat{\underline{U}}(x_{i\pm 1/2}, t) = \underline{U}_{i\pm\frac{1}{2}}(\bar{x}/\bar{t}=0) \quad \text{in the control volume}$$

$$= \text{const}$$

(Remember that characteristics are straight lines, so for $t > 0$, the state doesn't change!)

Inside the control volume, the integral form of the conservation law reads:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x, t^{n+1})\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x, t^{n})\,dx + \int_{t^n}^{t^{n+1}} \underline{F}(\underline{U}(x_{i-1/2}, t))\,dt - \int_{t^n}^{t^{n+1}} \underline{F}(\underline{U}(x_{i+1/2}, t))\,dt$$

Using that $\underline{U}(x_{i\pm 1/2}, t^n \le t \le t^{n+1}) = \text{const}$, the flux part of the integrals simplifies:

$$\int_{t^n}^{t^{n+1}} \underline{F}(\underline{U}(x_{i-1/2}, t))\,dt - \int_{t^n}^{t^{n+1}} \underline{F}(\underline{U}(x_{i+1/2}, t))\,dt =$$

$$= \int_{t^n}^{t^{n+1}} \left[\underline{F}(\underline{U}_{i-1/2}) - \underline{F}(\underline{U}_{i+1/2})\right] dt = \left[\begin{array}{l}\text{in local coordinates:} \\ t^n = 0,\ t^{n+1} = \Delta t\end{array}\right]$$

$$= \Delta t \left[\underline{F}(\underline{U}_{i-1/2}) - \underline{F}(\underline{U}_{i+1/2})\right] \equiv \Delta t\,(\underline{F}_{i-1/2} - \underline{F}_{i+1/2})$$

Dividing the equation by Δx gives:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x, \Delta t)\,dx = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{U}(x, 0)\,dx + \frac{\Delta t}{\Delta x}\,(\underline{F}_{i-1/2} - \underline{F}_{i+1/2})$$

[allowed because we're in the control volume]

$$= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{\tilde{U}}(x, \Delta t)\,dx = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{\tilde{U}}(x, 0)\,dx + \frac{\Delta t}{\Delta x}\,(\underline{F}_{i-1/2} - \underline{F}_{i+1/2})$$

Finally, using the definition that

$$\underline{U}(x, t^n) = \underline{U}_i^n = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \underline{U}_{ini}(x, t^n)\, dx$$

and considering that the exact solution $\hat{\underline{U}}(x, t^n)$ is the initial data state for t^{n+1}, we get

$$\underline{U}^{n+1} = \underline{U}^n + \frac{\Delta t}{\Delta x}\left[\underline{F}_{i-\frac{1}{2}} - \underline{F}_{i+\frac{1}{2}}\right]$$

3.3 Choice of time step size

We mustn't allow for a wave to be able to travel further than one cell length between two timesteps; Otherwise we get bogus results. This is imposed by the CFL condition:

$$\Delta t \leq \frac{C_{CFL}\, \Delta x}{S_{max}^n}$$

where S_{max}^n is the highest wave propagation speed at the current time and $C_{CFL} \in [0, 1]$ is used for increased accuracy and stability.

3.4 Accuracy of Godunov's Method

To demonstrate some problems of Godunov's method, let's have a look at a simple one-dimensional advection equation:

$$\partial_t q + u \partial_x q = 0$$

$$u = \text{const}$$

3.4.1 Numerical Diffusion

The diffusion equation is:

$$\partial_t q - D \partial_x^2 q = 0$$

for some constant diffusion coefficient D.
If we discretize the equation, assuming constant grid spacing, we get:

$$\frac{q^{n+1} - q^n}{\Delta t} - D \frac{1}{\frac{x_{i+1} - x_{i-1}}{2}} \left(\underbrace{\frac{q_{i+1}^n - q_i^n}{x_{i+1} - x_i}}_{\frac{d}{dx}} - \underbrace{\frac{q_i^n - q_{i-1}^n}{x_i - x_{i-1}}}_{\frac{d}{dx}} \right)$$

$$= \frac{q^{n+1} - q^n}{\Delta t} - D \frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{\Delta x^2} = 0$$

Now consider the upwind differencing scheme for the advection equation:

$$\partial_t q + u \partial_x q = 0 \Rightarrow \frac{q_i^{n+1} - q_i^n}{t_{n+1} - t_n} + u \frac{q_i^n - q_{i-1}^n}{x_i - x_{i-1}} = 0$$

$$\Rightarrow q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} u (q_i^n - q_{i-1}^n)$$

We now can rewrite the difference as follows:

$$\frac{q_i^n - q_{i-1}^n}{\Delta x} = \frac{1}{2} \frac{2q_i - 2q_{i-1}}{\Delta x} + \frac{1}{2} \frac{q_{i+1}}{\Delta x} - \frac{1}{2} \frac{q_{i+1}}{\Delta x}$$

$$= \frac{1}{2} \frac{q_{i+1} - q_{i-1}}{\Delta x} - \frac{1}{2} \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x}$$

$$= \frac{1}{2} \frac{q_{i+1} - q_{i-1}}{\Delta x} - \frac{\Delta x}{2} \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2}$$

$$= \frac{q_{i+1} - q_{i-1}}{2\Delta x} - \frac{D}{u} \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2}$$

with $D = \frac{\Delta x u}{2}$

$\Rightarrow$ The upstream difference method can be regarded as the same as the center difference scheme supplemented with a diffusion term.

The pure centred difference method is unstable.

In principle, numeric diffusion is unavoidable, but can be managed.

For $\Delta x \to 0$, we indeed get $D \to 0$.

3.4.2 Local Truncation Error and order of the algorithm

Let $q_e(x,t)$ be an exact solution to the advection equation, and let q_i^n be a discrete solution. The numerical algorithm is represented by a transport operator T:

$$q_i^{n+1} = T[q_i^n]$$

for upstream differencing: $T[q_i^n] = q_i^n - \frac{\Delta t}{\Delta x} u (q_i^n - q_{i-1}^n)$

Now define the One Step Error OSE:

$$OSE \equiv T[q_{e,i}^n] - q_{e,i}^{n+1}$$

by definition: $T[q_{e,i}^n] = q_{e,i}^n - \frac{\Delta t}{\Delta x} u (q_{e,i}^n - q_{i-1}^n)$

We can expand $q_{e,i}^{n+1}$ and $q_{e,i-1}^n$:

$$q_{e,i}^{n+1} = q_{e,i}^n + \frac{\partial q_i^n}{\partial t}\Big|_{t=t_n} \Delta t + \frac{1}{2}\frac{\partial^2 q}{\partial t^2}\Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$q_{e,i-1}^n = q_{e,i}^n - \frac{\partial q}{\partial x}\Big|_{x=x_i} \Delta x + \frac{1}{2}\frac{\partial^2 q}{\partial x^2}\Delta x^2 + \mathcal{O}(\Delta x^3)$$

Inserting into the OSE: and ignoring higher order terms:

$$OSE = T[q_{e,i}^n] - q_{e,i}^{n+1} = \Delta t\left[u\frac{\partial q}{\partial x} - \frac{\partial q}{\partial t} + \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)\right]$$

$$= \Delta t\left[\mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)\right]$$

$\Rightarrow$ with smaller Δt, the OSE gets smaller by a factor Δt^2. However, you'll need to do more steps to get to a fixed end time, thus introducing more errors again which may propagate for each time step.

In the end, the final error goes down with Δt instead of Δt^2.

For this reason, it is convenient to define the Local Truncation Error LTE:

$$LTE \equiv \frac{1}{\Delta t} \, OSE = \frac{1}{\Delta t} \left(T[q_{e,i}^{n}] - q_{e,i}^{num} \right)$$

for this particular scheme, the LTE is

$$LTE = \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t)$$

4. Approximate Riemann Solvers

The method of Godunov requires the solution of the Riemann problem. In a practical computation, this is done billions of times.

Approximate, non-iterative solutions have the potential to provide the necessary information accurate enough for numerical purposes.

There are essentially two ways of extracting approximate information to be used in Godunov-type methods:

- find approximation to the numerical flux
- find approximation to a state and then evaluate the flux function at this state

4.1 Approximations based on the exact Solution

The exact solver is based on the solution of the pressure function $f(p, \underline{w}_L, \underline{w}_R)$:

$$f(p, \underline{w}_L, \underline{w}_R) = f_L(p, \underline{w}_L) + f_R(p, \underline{w}_R) + \Delta u = 0$$

The Two-Rarefaction Riemann Solver (TRRS)

assumes a priori that both non-linear waves are rarefactions, and with this assumption, the equation for p_* can be immediately solved:

$$p_* = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L / p_L^z + a_R / p_R^z} \right]^{1/z}, \quad z = \frac{\gamma-1}{2\gamma}$$

A similar approach is used with the Two-Shock Riemann Solver (TSRS), where both non-linear waves are assumed to be shocks. However, this does not lead to a closed form for the pressure. Further approximations must be constructed. A popular approach is to write the pressure function as

$$f(p) = (p - p_L) g_L(p) + (p - p_R) g_R(p) + u_R - u_L = 0$$

Then assume some guess for p, use it as the argument of $g_{L,R}(p) = \left[\frac{A_k}{p + B_k} \right]^{1/2}$ and solve the equation then:

$$p_* = \frac{g_L(p_0) p_L + g_R(p_0) p_R - (u_R - u_L)}{g_L(p_0) + g_R(p_0)} \quad \text{with } p_0 = \text{guess}$$

4.2 The HLL and HLLC Riemann Solvers

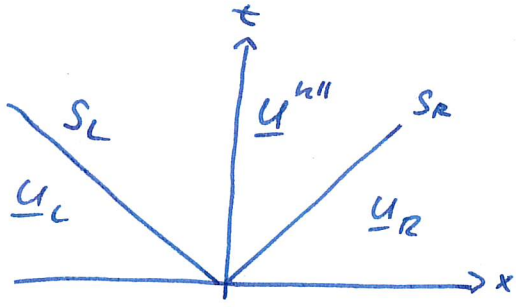
Original approach by Harten, Lax and van Leer.
The central idea is to assume a wave configuration for the solution that consists of two (three for HLLC) waves separating three (four) constant states.

Assuming that the wave speeds are given, the application of the integral form of the conservation laws gives a closed form approximate expression for the flux.

4.2.1 The HLL Solver

The solution of the HLL Solver is given as follows:

$$\underline{U}(x,t) = \begin{cases} \underline{U}_L & \text{if } \frac{x}{t} \leq S_L \\ \underline{U}^{**} & \text{if } S_L \leq \frac{x}{t} \leq S_R \\ \underline{U}_R & \text{if } \frac{x}{t} \geq S_R \end{cases}$$



Assuming the wave speeds are known, and that there are only two waves (which is incorrect for the Euler equations), it can be shown that

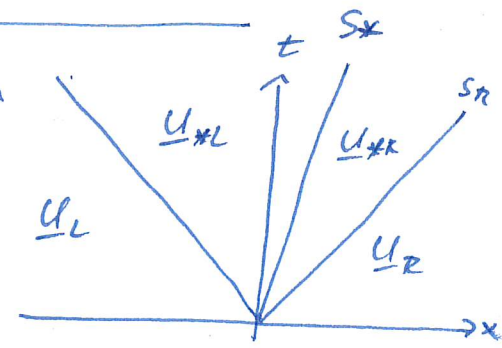
$$\underline{U}^{**} = \frac{S_R \underline{U}_R - S_L \underline{U}_L + \underline{F}_L - \underline{F}_R}{S_R - S_L}$$

The corresponding flux is given by

$$\underline{F}^{**} = \underline{F}_L + S_L(\underline{U}^{**} - \underline{U}_L) = \underline{F}_R + S_R(\underline{U}^{**} - \underline{U}_R)$$

4.2.2 The HLLC Riemann Solver

The HLLC solver is a modification of the HLL scheme, whereby the missing contact wave in the Euler equations is restored. Now we additionally include a contact wave in the middle with speed S_*.



The solution is given by

$$\underline{U}(x,t) = \begin{cases} \underline{U}_L & \text{if } \frac{x}{t} \le S_L \\ \underline{U}_{*L} & \text{if } S_L \le \frac{x}{t} \le S_* \\ \underline{U}_{*R} & \text{if } S_* \le \frac{x}{t} \le S_R \\ \underline{U}_R & \text{if } S_R \le \frac{x}{t} \end{cases}$$

The relations across the contact wave still hold:

$$p_* = p_{*L} = p_{*R}, \quad u_* = u_{*L} = u_{*R}$$

It is therefore reasonable to set $S_* = u_*$

The solutions for the star regions are given by

$$S_* = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L (S_L - u_L) - \rho_R (S_R - u_R)}$$

$$p_* = p_{L,R} + \rho_{L,R} (S_* - u_{L,R})(S_{L,R} - u_{L,R})$$

$$\rho_{*L,R} = \rho_{L,R} \frac{u_{L,R} - S_{L,R}}{S_* - S_{L,R}}$$

$$\underline{F}_{*L,R} = \underline{F}_{L,R} + S_{L,R} (\underline{U}_{*L,R} - \underline{U}_{L,R})$$

4.2.3 Wave Speed Estimates

Both the HLL and HLLC solvers require estimates for the wave speeds S_L and S_R.
Unfortunately, simple choices like

$$S_L = u_L - a_L, \quad S_R = u_R + a_R$$

or $S_L = \min\{u_L - a_L, u_R - a_R\}$, $S_R = \max\{u_L + a_L, u_R + a_R\}$

are not good choices.
Instead, it's better to go with one of the following:

$$S_L = \hat{u} - \hat{a}, \quad S_R = \hat{u} + \hat{a}$$

$$\hat{u} = \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_R} + \sqrt{\rho_L}}, \quad \hat{a} = \left[(\gamma - 1)\left(\hat{H} - \frac{1}{2}\hat{u}^2\right)\right]^{1/2}$$

$$\hat{H} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad H = \frac{E + p}{\rho}$$

or $S_L = \bar{u} - \bar{d}, \quad S_R = \bar{u} + \bar{d}$

$$\bar{d}^2 = \frac{\sqrt{\rho_L}\, a_L^2 + \sqrt{\rho_R}\, a_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}} + \eta_2 (u_R - u_L)^2$$

$$\eta_2 = \frac{1}{2} \frac{\sqrt{\rho_L \rho_R}}{(\sqrt{\rho_L} + \sqrt{\rho_R})^2}$$

or $S_L = -S^+, \quad S_R = S^+$

$$S^+ = \max\{|u_L| + a_L, |u_R| + a_R\} \quad \text{or} \quad S^+ = S_{max}$$

Another possibility is to base the wave speeds on the pressure:

Suppose we have an estimate for the pressure p^*, e.g. $p_0 = \frac{1}{2}(p_L + p_R) - \frac{1}{2}(u_R - u_L)\bar{\rho}\bar{a}$ with

$$\bar{\rho} = \frac{1}{2}(\rho_L + \rho_R), \quad \bar{a} = \frac{1}{2}(a_L + a_R)$$

Then choose the following wave speeds:

$$S_L = u_L - a_L q_L, \quad S_R = u_R + a_R q_R$$

$$q_{L,R} = \begin{cases} 1 & \text{if } p_0 \leq p_{L,R} \\ \left[1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_0}{p_{L,R}} - 1\right)\right]^{1/2} & \text{if } p_0 > p_{L,R} \text{ (shock)} \end{cases}$$