Smoothed Particle Hydrodynamics

Technique for approximating the continuum dynamics of fluids through the use of particles, which may also be viewed as interpolation points.
Principal idea: Treat hydrodynamics in a completely mesh-free fashion.

A kernel summation interpolant is used to estimate the density. For any field $F(\underline{r})$, we may define a smoothed interpolated version $F_s(\underline{r})$ through a convolution with a kernel $W(\underline{r}, h)$:

$$F_s(\underline{r}) = \int F(\underline{r}') W(\underline{r} - \underline{r}', h) \, d\underline{r}'$$

h: characteristic width of the kernel.
Kernel is normalised to unity; $W \to \delta$ for $h \to 0$.
Kernel is symmetric, twice differentiable. Most implementations are based on kernels with a finite support.

Provided the points sufficiently densely sample the kernel volume, we can approximate:

$$F_s(\underline{r}) \simeq \sum_j V_j F_j W(\underline{r} - \underline{r}_j, h) = \sum_j \frac{m_j}{\rho_j} F_j W(\underline{r} - \underline{r}_j, h)$$

Important: the estimate for $F_s(\underline{r})$ is defined <u>everywhere</u>, not just at the underlying points.

In general, the smoothing length $h = h(\underline{r}, t)$ can be made variable in space and time to account for variations in the sampling density.

2 options to introduce variability of h:

1) Scatter approach

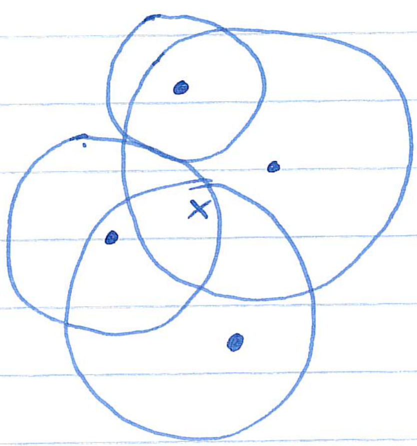
adopt $W(\underline{r} - \underline{r}_j, h(\underline{r}))$ as a kernel

$\Rightarrow$ particle i collects the contributions from all other particles j whose smoothing volumes h_j scatter onto location $\underline{r}_i$; $h \to h_j$

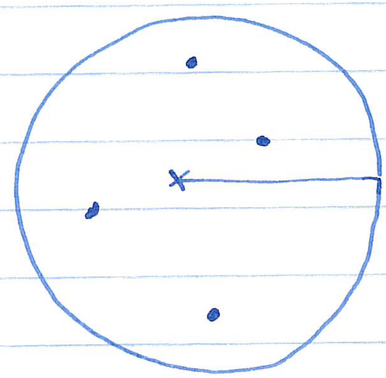
2) gather approach

use $W(\underline{r} - \underline{r}_j, h(\underline{r}_i))$

$\Rightarrow$ particle i gathers the contributions from all particles whose centers fall within the smoothing volume of i



scatter



gather

Equations of motion:

Euler equations for an inviscid ideal gas follow from the Lagrangian $\mathcal{L} = \int \rho \left(\frac{v^2}{2} - u \right) dV$

Assumptions:

- Thermal energy per unit mass of a particle can be expressed through an entropic function A_i. For an isentropic flow (no shocks, no mixing or thermal conduction) $A_i \approx \text{const}$
- EoS: $P_i = A_i \rho_i^{\gamma} = (\gamma - 1) \rho_i u_i$

  $\Leftrightarrow u_i(\rho_i) = A_i \frac{\rho_i^{\gamma-1}}{\gamma - 1}$
- $\rho_i h_i^3 = \text{const}$; This eqn defines h_i.

Discretize Lagrangian:

$$\mathcal{L} = \sum_i \left(\tfrac{1}{2} m_i \underline{v}_i^2 - m_i u_i \right)$$

Get equations of motion from Euler-Lagrange variation technique:

$$\frac{d}{dt} \frac{\partial L}{\partial \underline{v}_i} - \frac{\partial L}{\partial \underline{r}_i} = 0$$

$$\frac{\partial L}{\partial \underline{v}_i} = \frac{\partial}{\partial \underline{v}_i}\left[\sum_j \left(\frac{1}{2} m_j \underline{v}_j^{\,2} - \cancelto{0}{m_j u_j}\right)\right]$$

$$= m_i \underline{v}_i$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \underline{v}_i} = m_i \frac{d\underline{v}_i}{dt}$$

$$\frac{\partial L}{\partial \underline{r}_i} = \frac{\partial}{\partial \underline{r}_i}\left[\sum_j \left(\cancelto{0}{\frac{1}{2} m_j \underline{v}_j^{\,2}} - m_j u_j\right]\right. =$$

$$= -\sum_j m_j \frac{\partial}{\partial \underline{r}_i} u_j = -\sum_j m_j \frac{\partial}{\partial \underline{r}_i} \frac{P_j}{(\gamma-1)\rho_j}$$

$$\frac{\partial}{\partial \underline{r}_i}\frac{P_j}{(\gamma-1)\rho_j} = \frac{\frac{\partial P_j}{\partial \underline{r}_i}\cancel{(\gamma-1)}\rho_j - P_j\cancel{(\gamma-1)}\frac{\partial \rho_j}{\partial \underline{r}_i}}{(\gamma-1)^{\cancel{2}}\rho_j^2}$$

$$\frac{\partial}{\partial \underline{r}_i} P_j = \frac{\partial}{\partial \underline{r}_i} A_j \rho_j^{\gamma} = \gamma A_j \rho_j^{\gamma-1}\frac{\partial \rho_j}{\partial \underline{r}_i} \quad [A_j \approx \text{const}]$$

$$= \gamma \cdot \frac{P_j}{\rho_j}\frac{\partial \rho_j}{\partial \underline{r}_i}$$

$$\Rightarrow \frac{\partial}{\partial \underline{r}_i}\frac{P_j}{(\gamma-1)\rho_j} = \frac{\gamma \cdot \frac{P_j}{\rho_j}\frac{\partial \rho_j}{\partial \underline{r}_i}\rho_j - P_j\frac{\partial \rho_j}{\partial \underline{r}_i}}{(\gamma-1)\rho_j^2}$$

$$= \frac{P_j \frac{\partial \rho_j}{\partial \underline{r}_i}}{\rho_j^2}$$

$$= -\sum_j m_j \frac{P_j}{\rho_j^2}\frac{\partial \rho_j}{\partial \underline{r}_i}$$

$$\Rightarrow m_i \frac{d\underline{v}_i}{dt} = \sum_j \frac{m_j P_j}{\rho_j^2}\frac{\partial \rho_j}{\partial \underline{r}_i}$$

Using $\rho_i h_i^3 = \text{const}$ and $\frac{\partial \rho_i}{\partial \underline{r}_j} \equiv \nabla_j \rho_i + \frac{\partial \rho_i}{\partial h_i}\frac{\partial h_i}{\partial \underline{r}_j}$:

$$\frac{\partial}{\partial \underline{r}_j}(\rho_i h_i^3) = \frac{\partial \rho_i}{\partial \underline{r}_j} h_i^3 + \rho_i \cdot 3 h_i^2 \frac{\partial h_i}{\partial \underline{r}_j} = 0$$

$$\Rightarrow \frac{\partial \rho_i}{\partial \underline{r}_j} h_i + 3\rho_i \frac{\partial h_i}{\partial \underline{r}_j} = 0 = \nabla_j \rho_i + \frac{\partial \rho_i}{\partial h_i}\frac{\partial h_i}{\partial \underline{r}_j} + \frac{3\rho_i}{h_i}\frac{\partial h_i}{\partial \underline{r}_j}$$

$$\Rightarrow -\nabla_j \rho_i = \frac{\partial h_i}{\partial \underline{r}_j}\left(\frac{\partial \rho_i}{\partial h_i} + 3\rho_i\right) = \frac{\partial h_i}{\partial \underline{r}_j}\frac{\partial \rho_i}{\partial h_i}\left(1 + \frac{3\rho_i}{h_i}\left(\frac{\partial \rho_i}{\partial h_i}\right)^{-1}\right)$$

$$\Rightarrow \frac{\partial h_i}{\partial \underline{r}_j}\frac{\partial \rho_i}{\partial h_i} = \left[1 + 3\frac{\rho_i}{h_i}\left(\frac{\partial \rho_i}{\partial h_i}\right)^{-1}\right]^{-1}(-\nabla_j \rho_i)$$

Then: $\frac{\partial \rho_i}{\partial \underline{r}_j} = \nabla_j \rho_i + \frac{\partial \rho_i}{\partial h_i}\frac{\partial h_i}{\partial \underline{r}_j} = \nabla_j \rho_i - \nabla_j \rho_i \left[1 + \frac{3\rho_i}{h_i}\left(\frac{\partial \rho_i}{\partial h_i}\right)^{-1}\right]^{-1}$

$$= \frac{\left(1 + \frac{3\rho_i}{h_i}\left(\frac{\partial \rho_i}{\partial h_i}\right)^{-1}\right)\nabla_j \rho_i - \nabla_j \rho_i}{1 + \frac{3\rho_i}{h_i}\left(\frac{\partial \rho_i}{\partial h_i}\right)^{-1}}$$

$$= \left(1 + \frac{h_i}{3\rho_i}\frac{\partial \rho_i}{\partial h_i}\right)^{-1}\nabla_j \rho_i \equiv f_i \nabla_j \rho_i$$

Now use that $\rho_j = \sum_i m_i W(\underline{r}_j - \underline{r}_i, h_j)$:

$$\nabla_i \rho_j = \nabla_i\left(\sum_j m_j W_{ij}\right) = m_i \nabla_i W_{ij} + \delta_{ij}\sum_k m_k \nabla_i W_{ki}$$

$$\Rightarrow \boxed{\frac{d\underline{v}_i}{dt} = -\sum_j m_j\left[f_i \frac{P_i}{\rho_i^2}\nabla_i W_{ij}(h_i) + f_j \frac{P_j}{\rho_j^2}\nabla_i W_{ij}(h_j)\right]}$$

Mass conservation equation is already taken care of
Total energy equation is already taken care of
Both because the particle masses and their specific entropies A_i stay constant for reversible gas flows.

In the variational formulation using entropy or thermal energy as the independent thermodynamical variable is equivalent and makes no difference.

Artificial Viscosity

Even when starting from perfectly smooth initial conditions, the gas dynamics described by the Euler equations may readily produce true discontinuities in the form of shock waves and contact discontinuities. At such fronts the differential form of the Euler equations breaks down, and their integral form needs to be used, which give the Rankine-Hugoniot jump conditions at a shock front. The specific entropy of the gas always increases at a shock front, implying that in the shock layer itself the gas dynamics can no longer be described as inviscid, and the previously derived SPH equations cannot correctly describe a shock because they strictly keep the entropy constant.

$\rightarrow$ Introduce artificial viscosity

The purpose of artificial viscosity is to dissipate kinetic energy into heat and to produce entropy in the process. The usual approach is to parametrize the artificial viscosity in terms of a friction force that damps the relative motion of particles.
Provided the viscosity is introduced into the dynamics in a conservative fashion, the conservation laws themselves ensure that the right amount of dissipation occurs at a shock front.
Bigger problem: How to get artificial viscosity to work only when shock's are present? If it also operates outside of shocks, the dynamics may begin to deviate from that of an ideal gas.

Most commonly: $$\left.\frac{d\mathbf{v}_i}{dt}\right|_{visc} = -\sum_j m_j \Pi_{ij} \nabla_i \overline{W_{ij}}$$

$$\overline{W_{ij}} = \frac{1}{2}\left[W_{ij}(h_i) + W_{ij}(h_j)\right]$$ symmetrized kernel

Provided viscosity factor Π_{ij} is symmetric in i and j, the viscous force between any pair of interacting particles will be antisymmetric and along the line joining the particles, hence linear momentum and angular momentum are still preserved.

In order to conserve total energy, we need to compensate work done against the viscous force in the thermal reservoir: $$\left.\frac{dA_i}{dt}\right|_{visc} = \frac{1}{2}\frac{\gamma-1}{\rho_i^{\gamma-1}}\sum_j m_j \Pi_{ij} \mathbf{v}_{ij} \cdot \nabla_i \overline{W_{ij}}$$

or

$$\left.\frac{du_i}{dt}\right|_{visc} = \frac{1}{2}\sum_j m_j \Pi_{ij} \mathbf{v}_{ij} \cdot \nabla_i \overline{W_{ij}}$$

$\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$

Advantages of SPH:

- Can provide a large dynamic range in spatial resolution and density
- automatically adaptive resolution
- excellent conservation properties for energy, momentum, angular momentum
- Galileian- invariant
- Free from advection errors
- Can't get unphysical values like negative densities by construction
- Can easily deal with complicated structures/settings

Disadvantages:

- limited accuracy in multi-dimensional flow