On Smoothing Kernels

A kernel-smoothed representation of a function can be obtained via

$$\langle f\rangle(\underline{r}) = \int f(\underline{r}')\, W(\underline{r}-\underline{r}'; h)\, dr'^D$$

where $W(\underline{x}, h)$ is a smoothing kernel.

The width of the kernel function is determined by the quantity h ("smoothing length"). The kernel has dimension of an inverse volume and should be normalized to unity.

$$\int W(\underline{r}', h)\, dr'^D = 1$$

and it shoul posess the δ-property:

$$\lim_{h\to 0} \langle A\rangle(\underline{r}) = A(\underline{r})$$

We assume that the kernels are spherically symmetric:

$$W(\underline{r}) = W(r)$$

which leads to the useful property

$$\nabla_a W(|\underline{r}_a - \underline{r}_b|, h) = -\nabla_b W(|\underline{r}_b - \underline{r}_a|, h)$$

since $\nabla_a W(|\underline{r}_a - \underline{r}_b|, h) = \frac{\partial W}{\partial r} \frac{dr}{dr_a} =$

$$= \frac{\partial W}{\partial r} \frac{d}{dr_a} |\underline{r}_a - \underline{r}_b| = \frac{\partial W}{\partial r} \frac{d}{dr_a} \sqrt{(r_a - r_b)^2}$$

$$= \frac{\partial W}{\partial r} \frac{1}{2} \frac{1}{\sqrt{(\underline{r}_a - \underline{r}_b)^2}} \cdot 2 \cdot \frac{dr_a}{dr_a} = \frac{1}{|\underline{r}_a - \underline{r}_b|} \frac{\partial W}{\partial r}$$

and $\nabla_b W(|\underline{r}_a - \underline{r}_b|, h) = \frac{\partial W}{\partial r} \frac{dr}{dr_b}$

$$= \frac{\partial W}{\partial r} \frac{1}{2} \frac{1}{\sqrt{(\underline{r}_a - \underline{r}_b)^2}} \cdot 2 \left(-\frac{dr_b}{dr_b}\right) = \frac{-1}{|\underline{r}_a - \underline{r}_b|} \frac{\partial W}{\partial r}$$

Let us assume that a function A is known at the positions $\underline{r}_b$, $A_b = A(\underline{r}_b)$.

One can then approximate:

$$\langle A \rangle (\underline{r}) = \int A \, W(\underline{r} - \underline{r}', h) \, dr'^D \approx \sum_b A_b V_b W(\underline{r} - \underline{r}_b, h)$$

Where V_b is the volume associated with the particle at $\underline{r}_b$. In nearly all SPH formulations in astrophysics the particle volume is estimated as $V_b = m_b / \rho_b$

We write the normalized SPH kernels in the form

$$W(|\underline{r}-\underline{r}'|, h) = \frac{\sigma}{h^D} w(q)$$

σ: normalisation

h: compact support radius of the kernel

D: Numbers of dimensions

q: $\dfrac{|\underline{r}-\underline{r}'|}{h}$ dimensionless quantity

The normalisations σ are obtained from

$$\int_{\mathbb{R}^D} W \, dr^D = \int_{\mathbb{R}^D} \frac{\sigma}{h^D} w(q) \, dr^D \stackrel{!}{=} 1$$

Using $\quad q = \frac{r}{h} \quad \Rightarrow \quad dq = \frac{dr}{h} \quad \Rightarrow dq^D = \frac{dr^D}{h^D}$

Then $\quad \int_{\mathbb{R}^D} W \, dr^D = \int_{\mathbb{R}^D} \frac{\sigma}{h^D} w(q) \, dr^D = \int_{\mathbb{R}^D} \sigma w(q) \, dq^D \overset{!}{=} 1$

$$\Rightarrow \sigma^{-1} = \int_{\mathbb{R}^D} w(q) \, dq^D = \int_{-H}^{H} w(q) \, dq^D$$

where H is the compact support radius of the kernel.

Since we assume W to be radial, we can write:

$$\sigma^{-1} = \begin{cases} 2 \int_0^H w(q) \, dq & \text{for } 1D \\ 2\pi \int_0^H w(q) \, q \, dq & \text{for } 2D \\ 4\pi \int_0^H w(q) \, q^2 \, dq & \text{for } 3D \end{cases}$$

Example: Cubic Spline kernel in 3D

$$
w(q) = \begin{cases} \frac{1}{4}(2-q)^3 - (1-q)^3 & 0 \le q < 1 \\ \frac{1}{4}(2-q)^3 & 1 \le q < 2 \\ 0 & \text{else} \end{cases}
$$

This gives us:

$$
\frac{1}{4\pi\sigma} = \int_0^1 \left[\tfrac{1}{4}(2-q)^3 - (1-q)^3\right] q^2\,dq + \int_1^2 \left[\tfrac{1}{4}(2-q)^3\right] q^2\,dq
$$

$$
= \int_0^1 \left[\tfrac{1}{2}(8 - 12q + 6q^2 - q^3) - 1 + 3q - 3q^2 + q^3\right] q^2\,dq + \int_1^2 \tfrac{1}{4}(8 - 12q + 6q^2 - q^3)\,q^2\,dq
$$

$$
= \int_0^1 \left[1 - \tfrac{3}{2}q^2 + \tfrac{3}{4}q^3\right] q^2\,dq + \int_1^2 \left[2 - 3q + \tfrac{3}{2}q^2 - \tfrac{q^3}{4}\right] q^2\,dq
$$

$$
= \left[\frac{q^3}{3} - \frac{3}{10}q^5 + \frac{1}{8}q^6\right]_0^1 + \left[\frac{2}{3}q^3 - \frac{3}{4}q^4 + \frac{3}{10}q^5 - \frac{q^6}{24}\right]_1^2
$$

$$
= \left(\frac{1}{3} - \frac{3}{10} + \frac{1}{8}\right) + \left(\frac{16}{3} - 12 + \frac{96}{10} - \frac{8}{3}\right) - \left(\frac{2}{3} - \frac{3}{4} + \frac{3}{10} - \frac{1}{24}\right)
$$

$$
= \frac{1}{4}
$$

$$
\Rightarrow \boxed{\sigma = 1/\pi}
$$

The desirable properties of an SPH kernel $W(\underline{x}, h)$ are:

i) $W(\underline{x}, h)$ should be isotropic in $\underline{x}$

ii) $W(\underline{x}, h)$ should be positive and decrease monotonically

iii) $W(\underline{x}, h)$ should be twice differentiable

iv) $W(\underline{x}, h)$ should have a finite support and be cheap to compute.