Smoothing Matters

Dehnen & Aly, 2012
Schaller & Borrow, SWIFT theory documentation

1. Smoothing Scale

SPH smoothing kernels are usually isotropic and can be written as

$$W(\underline{x}, h) = h^{-\nu} \tilde{w}(|\underline{x}|, h)$$

with a dimensionless function $\tilde{w}(r)$, which specifies the functional form and normalisation $\int d^{\nu}\underline{x}\, \tilde{w}(|\underline{x}|) = 1$.

The re-scaling

$$h \to \alpha h$$

$$\tilde{w}(r) \to \alpha^{\nu} \tilde{w}(\alpha r)$$

leaves the functional form of W unchanged.
The normalisation is still satisfied:
Let $s = \alpha x \Rightarrow dx = \frac{ds}{\alpha}$
Then $\int dx^{\nu} \alpha^{\nu} \tilde{w}(\alpha r) = \int \frac{ds^{\nu}}{\alpha^{\nu}} \alpha^{\nu} \tilde{w}(s) = 1$

However, this rescaling alters the meaning of h.

In order to avoid this ambiguity, a definition of the smoothing length in terms of the kernel, i.e. via a functional $h = h[W(\underline{x})]$, must be specified.

In their study, they use two scales:

- h: smoothing scale, defined later
- H: kernel support radius: the largest $|x|$ for which $W(\underline{x}) > 0$

Smoothing kernels have in practice compact support and hence finite H.

For such kernels, we have

$$W(\underline{x}, H) = H^{-2} w(|\underline{x}|/H)$$

where $w(r) = 0$ for $r \geq 1$ and $w(r) > 0$ for $r < 1$.

H is related to the average number N_H of neighbours within the smoothing sphere by $N_H = V_\nu H_i^\nu \frac{\hat{\rho}_i}{m_i}$
V_ν is here the volume of the unit sphere in ν-dim, and $\hat{\rho}_i$ is the estimated fluid density, estimated from the masses m_i and positions $\underline{x}_i$ of the particles:

$$\rho(\underline{x}_i) \approx \hat{\rho}_i \equiv \sum_j m_j W(\underline{x}_i - \underline{x}_j, h_i)$$

Then $\frac{\hat{\rho}_i}{m_i}$ is the estimated number density at position $\underline{x}_i$, and $V_\nu H^\nu$ the volume of a sphere with radius H, such that

$V_\nu H_i^\nu \frac{\hat{\rho}_i}{m_i}$ = estimated number of particles in H-sphere

An appropriate definition of h in terms of the smoothing kernel, i.e. $h = h[W(x)]$, is lacking.

Possible definitions include the kernel standard deviation

$$\sigma^2 = \frac{1}{2} \int d^2\underline{x}\ \underline{x}^2\, W(\underline{x}, h)$$

with e.g.

$$h = 2\sigma$$

which is convenient when working with sound waves, as the standard deviation is directly related to the numerical resolution of sound waves.

Other possibilities include for example the radius of the inflection point (which is the local maximum of $|\nabla W|$), or the ratio $W/|\nabla W|$ at the inflection point. However, these definitions are not suitable for all kernels. For example triangular kernels have no inflection point.

Why not use H as the smoothing scale?

As shown before, we can re-scale any kernel easily, i.e. rescale the compact support radius H arbitrarily. As such, just using H to compare between various kernels or estimate the resolution is not really meaningful.

Instead, we need a method to define what resolution in the SPH context even is.

Dehnen & Aly show (using the Nyquist criterium in Fourier space) that a kernel's standard deviation σ is directly proportional to the minimal wavelength of sound waves that can be resolved in a SPH scheme.

As such, it is a more physical and intuitive measure of resolution.

Due to their different functional form, the different kernels obviously will have different standard deviations, and, assuming all are scaled to the same compact support radius H, will have different ratios $\gamma = H/h$. [See table 1 in their paper].

In practice, the neighbour number N_H is often used as a convenient parameter, even though it holds little meaning by itself. A more meaningful quantity in terms of resolution is the average number N_h of particles within distance h, given by $N_h = \left(\frac{h}{H}\right)^2 N_H$ for kernels with compact support, or the ratio $h\,(\hat{\rho}/m)^{1/2}$ between h and the average particle separation $\left(\frac{m}{\hat{\rho}}\right)^{1/2}$.

Each kernel will have a different ratio $\gamma = \frac{H}{h}$. So for a fixed resolution h, one will have different kernel support sizes, H, and a different number of neighbours, N_{ngb}, to interact with.

See Dehnen & Aly table 1 for values for γ.

One would typically choose h for a simulation as a multiple η of the mean interparticle separation:

$$h = \eta \langle x \rangle = \eta \left(\frac{m}{\rho} \right)^{1/3}$$

The (weighted) number of neighbours within the kernel support radius is a useful quantity to use in implementations of SPH.

It is defined (in 3D) as

$$N_{ngb} = \frac{4}{3} \pi \left(\frac{H}{h} \eta \right)^3$$