Discretized Logarithmic Derivative

Suppose you want to compute a logarithmic derivative, e.g. a logarithmic mass growth

$$f(t) = \frac{d \log M}{d \log t} = \frac{dM}{M} \frac{t}{dt}$$

assuming the mass M is a function of time t and is known at two points in time, $M(t = t_A) = M_A$ and $M(t = t_B) = M_B$.

First, consider the backward difference:

$$f(t_B) \approx \frac{M_B - M_A}{M_B} \frac{t_B}{t_B - t_A}$$

Second, consider the forward difference:

$$f(t_A) \approx \frac{M_B - M_A}{M_A} \frac{t_A}{t_B - t_A}$$

Those are valid expressions, but we can do better with a central difference:

Let M_* be $M(t = \frac{t_A + t_B}{2})$, the mass at the midpoint in time between $t = t_A$ and $t = t_B$.

Then

$$f\left(\frac{t_A + t_B}{2}\right) \approx \frac{M_B - M_A}{M_*} \frac{\frac{t_A + t_B}{2}}{t_B - t_A}$$

If we set $M_* = \frac{M_A + M_B}{2}$, i.e. to be the average of M_A and M_B, which is exact if M is linear between t_A and t_B or a very close approximation for a fine enough grid step. Then, for the central difference:

$$f\left(\frac{t_A + t_B}{2}\right) \approx \frac{M_B - M_A}{\frac{M_B + M_A}{2}} \frac{\frac{t_A + t_B}{2}}{t_B - t_A} =$$

$$\boxed{= \frac{M_B - M_A}{M_B + M_A} \frac{t_B + t_A}{t_B - t_A}}$$

Remember that the central difference is $O(\Delta t^2)$, while the forward and backward differences are $O(\Delta t)$.