Finite Differences

Three forms are commonly considered:

1) Forward difference:

$$\Delta_h [f](x) = f(x+h) - f(x)$$

2) Backward difference

$$\Delta_h [f](x) = f(x) - f(x-h)$$

3) Central difference

$$\Delta_h [f](x) = f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h)$$

Orders of Accuracy

By decreasing the difference h, we obtain more accurate expressions for the differentials, getting closer to the analytical function.
But how accurate are these expressions?

We use the Taylor-expansion of an analytical function $f(x)$ to demonstrate:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

1) Forward difference

Let $x = x_0 + h \Rightarrow x - x_0 = h$

$$\Rightarrow f(x_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n$$

$$= f(x_0) + f'(x_0) h + \frac{f''}{2} h^2 + O(h^3)$$

$$= f(x_0) + f'(x_0) h + O(h^2)$$

$$\Rightarrow f'(x_0) = \frac{1}{h}\left(f(x_0 + h) - f(x_0)\right) + \frac{1}{h} O(h^2)$$

$$= \frac{f(x_0 + h) - f(x_0)}{h} + O(h)$$

[the plus or minus sign of $O(h)$, $O(h^2)$ is of no importance here]

2) <u>Backward difference</u>

Let $x = x_0 - h \Rightarrow x - x_0 = -h$

$$f(x) = f(x_0 - h) =$$

$$= f(x_0) + f'(x_0)(-h) + O(h^2)$$

$$\Rightarrow f'(x_0) = \frac{-1}{h}\left(f(x_0-h) - f(x_0)\right) + \frac{1}{h}O(h^2)$$

$$= \frac{f(x_0) - f(x_0-h)}{h} + O(h)$$

3) <u>Central Difference</u>

Here, we use:

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \\ + \frac{1}{6}f'''(x_0)h^3 + O(h^4)$$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \\ - \frac{1}{6}f'''(x_0)h^3 + O(h^4)$$

Subtracting these expressions gives:

$$f(x_0+h) - f(x_0-h) = 2f'(x_0)h + \frac{1}{3}f'''(x_0)h^3 + \\ + \mathcal{O}(h^5)$$

$$\Rightarrow f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{1}{6}f'''(x_0)h^2 + \frac{\mathcal{O}(h^5)}{h}$$

$$= \frac{1}{H}\left[f(x_0+\tfrac{1}{2}H) - f(x_0-\tfrac{1}{2}H)\right] + \mathcal{O}(H^2)$$

with $H \equiv 2h$

$\Rightarrow$ The centered difference formula introduces a smaller error than the forward or backward difference.