Five Point Stencil Finite Difference

Given a square grid, the five-point stencil of a point in the grid is a stencil made up of the point itself together with its four neighbours. It is used to write finite difference approximations to derivatives at grid points.

To obtain the finite difference expression, we use the Taylor expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

This time on 4 points:

$$f(x \pm h) = f(x) \pm h f'(x) + \frac{h^2}{2} f''(x) \pm \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) + O(h^5)$$

$$f(x \pm 2h) = f(x) \pm 2h f'(x) + 2h^2 f''(x) \pm \frac{4}{3} h^3 f'''(x) + \frac{1}{3} h^4 f''''(x) + O(h^5)$$

Then:

$$f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{3} f'''(x) + O(h^5) \qquad [1]$$

$$f(x+2h) - f(x-2h) = 4h f'(x) + \frac{8h^3}{3} f'''(x) + O(h^5) \qquad [2]$$

To make the $f'''(x)$ term disappear, we compute $8\cdot([1]-[2])$:

$$8f(x+h)-8f(x-h)-f(x+2h)+f(x-2h) =$$

$$= 16hf' + \frac{8h^3}{3}f'''(x) + O(h^5) - 4hf' - \frac{8h^3}{3}f'''(x)$$

$$= 12hf' + O(h^5)$$

$$\Rightarrow f' = \frac{1}{12h}\left[8f(x+h)-8f(x-h)-f(x+2h)+f(x-2h)\right] +$$

$$+ \frac{1}{12h}O(h^5)$$

$$= \frac{1}{h}\left[\frac{2}{3}\big(f(x+h)-f(x-h)\big) - \frac{1}{12}\big(f(x+2h)-f(x-2h)\big)\right] +$$

$$+ O(h^4)$$