Second Derivative Centered Difference

Use a Taylor expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

We expand the function $f(x)$ at two points:

1) Let $x = x_0 + h$

$\Rightarrow \quad x - x_0 = h$

Then

$$f(x) = f(x_0 + h) =$$

$$= f(x_0) + f'(x_0)h + \frac{1}{2} f''(x_0)h^2 +$$

$$+ \frac{1}{6} f'''(x_0)h^3 + \mathcal{O}(h^4)$$

2) Let $x = x_0 - h$

$\Rightarrow \quad x - x_0 = -h$

Then

$$f(x) = f(x_0 - h) =$$

$$= f(x_0) - f'(x_0)h + \frac{1}{2} f''(x_0)h^2 -$$

$$- \frac{1}{6} f'''(x_0)h^3 + \mathcal{O}(h^4)$$

Now adding these expressions:

$$f(x_0+h)+f(x_0-h)=2f(x_0)+f''(x_0)h^2+O(h^4)$$

$$\Rightarrow f''(x_0)=\frac{1}{h^2}\left[f(x_0+h)-2f(x_0)+f(x_0-h)\right]+O(h^2)$$