Curvilinear Coordinates

[Following W. Nolting's "Classical Mechanics"]

Let us look at what basis vectors may describe curvilinear coordinate systems.

We start from the cartesian coordinate system:

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then for the position vector $\underline{r}$:

$$\underline{r} = \sum_{j=1}^{3} x_j \underline{e}_j$$

$$\Rightarrow d\underline{r} = \underbrace{\sum_j \frac{d\underline{r}}{dx_j} dx_j}_{\text{chain rule}} = d\left[\sum_j x_j \underline{e}_j\right] \underset{d\underline{e}_j = 0}{=} \sum_j dx_j \underline{e}_j$$

$$\Rightarrow \underline{e}_j = \frac{\partial \underline{r}}{\partial x_j}$$

This allows us to compute the basis vectors for any curvilinear system.

Example: Cylindrical Coordinates

$$x_1 = r\cos\varphi$$
$$x_2 = r\sin\varphi$$
$$x_3 = z$$

Then $\underline{e}_r = \frac{\partial \underline{r}}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$

$$\underline{e}_\varphi = \left|\frac{\partial \underline{r}}{\partial \varphi}\right|^{-1} \frac{\partial \underline{r}}{\partial \varphi} = \left\|\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}\right\|^{-1} \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi}} \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\underline{e}_z = \frac{\partial \underline{r}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Transforming Differentials

Let $\phi(x', y', z') = \phi(\underline{r}')$ be a coordinate transformation:

$$\underline{r} = \phi(\underline{r}')$$

Then the transformation of the differentials is done via the functional determinant:

$$dr^n = |\det J| \cdot dr'^n$$

$$J_{ij} = \frac{\partial[\phi(\underline{r}')]_i}{\partial x_j}$$

Example:

1) In 1D:

Let $\phi(r') = ar' + b$

Then $dr = \frac{\partial \phi}{\partial r'} dr' = a\,dr'$

2) In 2 D:

Let $\Phi(\underline{r}') = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \end{pmatrix}$

Then:

$$dr^2 = \left| \frac{\partial \Phi_i}{\partial x_j} \right| dr'^2$$

$$\left| \frac{\partial \Phi_i}{\partial x_j} \right| = \begin{vmatrix} \frac{\partial \Phi_1}{\partial x_1} & \frac{\partial \Phi_1}{\partial x_2} \\ \frac{\partial \Phi_2}{\partial x_1} & \frac{\partial \Phi_2}{\partial x_2} \end{vmatrix} = \quad \text{with } x_1 = r, \; x_2 = \varphi$$

$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$= r\cos^2\varphi + r\sin^2\varphi = r$$

$$\Rightarrow dr^2 = r\, dr'^2 = r\, dr\, d\varphi$$

How do I get the single components of the differentials?

$\rightarrow$ Just use the chain rule:

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$\Rightarrow \quad dx = dr\cos\varphi - r\sin\varphi\, d\varphi$$
$$dy = dr\sin\varphi + r\cos\varphi\, d\varphi$$

$$\Rightarrow \quad dx^2 = dr^2\cos^2\varphi + r^2\sin^2\varphi\, d\varphi^2 - 2\sin\varphi\cos\varphi\, r\, dr\, d\varphi$$
$$dy^2 = dr^2\sin^2\varphi + r^2\cos^2\varphi\, d\varphi^2 + 2\sin\varphi\cos\varphi\, r\, dr\, d\varphi$$

$$\Rightarrow \quad dx^2 + dy^2 = dr^2(\sin^2\varphi + \cos^2\varphi) + r^2 d\varphi^2(\sin^2\varphi + \cos^2\varphi)$$
$$= dr^2 + r^2 d\varphi^2$$

Separating into basis vectors

Let $\underline{r} = \sum x_j \underline{e}_j$, $\underline{e}_j$ = Cartesian basis

then $d\underline{r} = \sum_j d(x_j \underline{e}_j) = \sum_j \underline{e}_j \, dx_j = \underbrace{\sum \frac{\partial \underline{r}}{\partial x_j} dx_j}$

by definition of total derivative

$$\Rightarrow \underline{e}_j = \frac{\partial \underline{r}}{\partial x_j} = \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_j}\right|} \frac{\partial \underline{r}}{\partial x_j}$$

for normalization; and because $\underline{e}_j$ is normalized

For polar coordinates, we get

$$\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \end{pmatrix}$$

$$\underline{e}_r = \frac{1}{|\ldots|} \frac{\partial \underline{r}}{\partial r} = \frac{1}{|\ldots|} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\underline{e}_\varphi = \frac{1}{|\ldots|} \frac{\partial \underline{r}}{\partial \varphi} = \frac{1}{|\ldots|} \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$$\Rightarrow \underline{r} = r \, \underline{e}_r$$

For the differential:

$$\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \end{pmatrix}$$

$$\Rightarrow d\underline{r} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dr \cos\varphi - r \sin\varphi \, d\varphi \\ dr \sin\varphi + r \cos\varphi \, d\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} dr + r \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} d\varphi$$

$$= \underline{e}_r \, dr + r \, \underline{e}_\varphi \, d\varphi$$