Method of Characteristics

First order PDEs can be tackled with the method of characteristics. The main idea is to find a parametrisation for your variables such that the derivative of the required quantity is an ODE, not a PDE any longer.

Consider the advection equation with constant speed

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0$$

Let us perform a change of variables:

$$\xi = x + ct$$

$$\eta = x - ct$$

Now using the chain rule gives

$$\frac{\partial}{\partial x} = \underbrace{\frac{\partial \xi}{\partial x}}_{=1}\frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial x}}_{=1}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = \underbrace{\frac{\partial \xi}{\partial t}}_{=c}\frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial t}}_{=-c}\frac{\partial}{\partial \eta} = c\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)$$

Hence the advection equation can be written as

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = c\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)u + c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)u$$

$$= 2c\frac{\partial u}{\partial \xi} = 0$$

This can be easily integrated to

$$\frac{\partial u}{\partial \xi} = 0 \quad \Rightarrow u = f(\eta) = f(x - ct)$$

But where did we get the change of variables from?

The line $x - ct = \text{const.}$ is a line on the $x-t$ plane along which u is constant. So if we parametrise this line

$$x = x(r), \quad t = t(r)$$

then moving along that line by changing the line parameter r leaves u constant, or

$$\frac{du}{dr} = 0$$

This is the underlying principle of the method of characteristics.

Non-linear advection

Now let's solve

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0$$

We want a parametrisation of the form

$$x = x(r), \quad t = t(r)$$

such that

$$\frac{du}{dr} = 0$$

The chain rule gives us

$$\frac{du}{dr} = \frac{\partial u}{\partial x}\frac{dx}{dr} + \frac{\partial u}{\partial t}\frac{dt}{dr}$$

We want this to be zero. This can be achieved when we compare this expression to the original expression:

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0$$

$$\frac{dt}{dr}\frac{\partial u}{\partial t} + \frac{dx}{dr}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{dt}{dr} = 1, \quad \frac{dx}{dr} = c(x,t)$$

with an actual function $c(x,t)$, the ODE $\frac{dx}{dr} = c(x,t)$ can be solved. We will have made an actual coordinate transformation from x, t to r, C, where C is an integration constant.

Then, since

$$\frac{du}{dr} = 0$$

it follows that

$$u = f(C)$$

and we have found a general solution.

Example: Let us solve

$$\frac{\partial u}{\partial t} + 2xt\frac{\partial u}{\partial x} = 0$$

We start off by comparing

$$\frac{dt}{dr}\frac{\partial u}{\partial t} + \frac{dx}{dr}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{dt}{dr} = 1 \qquad \Rightarrow t = r$$

we skip adding the integration constant here. The origin of r is not set a priori and doesn't need to coincide with t.

But we will add one when integrating $\frac{dx}{dr}$:

$$\frac{dx}{dr} = 2xt = 2xr$$

$$\Rightarrow \ln(x) = r^2 + C \quad \Rightarrow x = x_0 e^{r^2}$$

We can now invert that expression to find the integration constant x_0:

$$x_0 = x e^{-r^2} = x e^{-t^2}$$

With this, we have the solution

$$\frac{du}{dr} = 0 \quad \Rightarrow u = f(x_0) = f(x e^{-t^2})$$

Inhomogeneous Case

For the inhomogeneous case, we can apply the same principles. The only difference is that the value of the function that we are looking for is not constant any longer, but is defined by an ODE.

Example: Let us solve

$$\frac{\partial u}{\partial t} + 2xt\frac{\partial u}{\partial x} = u$$

We again get

$$t = r$$
$$x = x_0 e^{r^2}$$

and end up with the ODE

$$\frac{du}{dr} = u \quad \Rightarrow \quad \ln(u) = r + C$$
$$\Rightarrow \quad u = u_0 e^{r} = u_0(x_0)e^{r}$$
$$= u_0\left(x e^{-t^2}\right)e^{t}$$