Reynolds Transport Theorem

Let $I = \int_{V(t)} \alpha \, d^3x$ be the integral form of an equation, where α is any scalar, vector or tensor field.

Then:

$$\frac{dI}{dt} = \frac{d}{dt}\left[\int_{V(t)} \alpha \, d^3x\right] = \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla}(\alpha \cdot \underline{v})\right] d^3x$$

Proof: Let $V(t)$ be some finite volume with $dV(t=0) = dV_0$. Furthermore, let $\Psi(\underline{x}) = \begin{cases} 1 & \underline{x} \in V(t) \\ 0 & \text{else} \end{cases}$

such that $I = \int_{V(t)} \alpha \, d^3x = \int_{\mathbb{R}^3} \alpha \Psi \, d^3x$

to move the time dependence of the volume into the integral.

By definition:

$$\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\partial t} + \underline{v} \underline{\nabla} \Psi = 0 \quad \text{because } \Psi \text{ is comoving.}$$

Then:

$$\frac{dI}{dt}\left[\int_{V(t)} \alpha \, d^3x\right] = \frac{d}{dt}\left[\int_{\mathbb{R}^3} \alpha \Psi \, d^3x\right] = \int_{\mathbb{R}^3} \frac{d}{dt}[\alpha \Psi] \, d^3x$$

$$= \int_{\mathbb{R}^3} \left[\frac{d\alpha}{dt} \Psi + \alpha \frac{d\Psi}{dt}\right] d^3x$$

Using $\frac{D\psi}{Dt} = 0 \Rightarrow \frac{\partial \psi}{\partial t} = -\underline{v} \cdot \underline{\nabla} \psi$

$$\Rightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3} \left[\frac{\partial \alpha}{\partial t} \psi - \alpha \underline{v} \cdot \underline{\nabla} \psi \right] d^3x$$

Using $\underline{\nabla} \cdot (\psi \alpha \underline{v}) = \psi \underline{\nabla} \cdot (\alpha \underline{v}) + \alpha \underline{v} \underline{\nabla} \psi$

$$\Rightarrow -\alpha \underline{v} \cdot \underline{\nabla} \psi = \psi \underline{\nabla} (\alpha \underline{v}) - \underline{\nabla} (\psi \alpha \underline{v})$$

$$\Rightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3} \left[\frac{\partial \alpha}{\partial t} \psi + \psi \underline{\nabla} (\alpha \underline{v}) - \underline{\nabla} (\psi \alpha \underline{v}) \right] d^3x$$

Use $\int_{\mathbb{R}^3} \underline{\nabla} (\psi \alpha \underline{v}) d^3x = \int_{S(\mathbb{R}^3)} \alpha \psi \underline{v} \underline{n} \, dS = 0$

Surface of $\mathbb{R}^3$ is clearly outside the finite $V(t)$

so $\psi = 0$

$$\Rightarrow \frac{dI}{dt} = \int_{\mathbb{R}^3} \psi \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} (\alpha \underline{v}) \right] d^3x$$

$$\boxed{\frac{dI}{dt} = \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \cdot \underline{v}) \right] d^3x}$$