Circular Speed

Newton derived two useful theorems concerning shells of matter:

1) A body inside a spherical shell of matter experiences no net gravitational force from the shell.

2) A body lying outside a closed spherical shell experiences a gravitational force which is the same as if all the matter in the shell were concentrated at a point at its center.

A circular mass distribution can therefore be characterized in terms of a <u>circular speed</u> v_{circ}, which is the speed a test particle would have in a circular orbit at radius r.

With a spherically symmetric potential $\phi(\underline{r}) = \phi(r)$, and using polar coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \end{pmatrix}$$

Then the Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \phi(x,y)$$

$$= \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) - \phi(r)$$

The Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad , \quad q = r, \varphi$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2} 2\dot{r}\right) - \frac{1}{2} \cdot 2r \, \dot{\varphi}^2 + \frac{\partial \phi}{\partial r} = 0$$

$$\Rightarrow \boxed{\ddot{r} - r\dot{\varphi}^2 + \frac{\partial \phi}{\partial r} = 0}$$

$$\frac{d}{dt}\left(\frac{1}{2}r^2 \cdot 2\dot{\varphi}\right) - 0 = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(r^2\dot{\varphi}\right) = 0}$$

If we assume circular motion, we have

$$dr = 0 \Rightarrow \dot{r} = \frac{dr}{dt} = 0$$

Then from $\frac{d}{dt}(r^2\dot{\varphi}) = 0 = 2r \underbrace{\frac{dr}{dt}}_{=0} + r^2\ddot{\varphi}$

$$\Rightarrow \ddot{\varphi} = 0$$

[Alternatively, you can just demand that the rotation speed $\dot{\varphi}$ is constant, and skip the entire Euler-Lagrange step]

Then the expression for the circular speed can be obtained through

$$\underline{r}_c = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

$$\underline{v}_c = \dot{\underline{r}}_c = r\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}\dot{\varphi}$$

$$\underline{a}_c = \dot{\underline{v}}_c = r\begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix}\dot{\varphi}^2 + r\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}\underbrace{\ddot{\varphi}}_{=0}$$

$$= -\underline{r}_c\dot{\varphi}^2 = \frac{-\|(r\dot{\varphi})^2\|}{r}\underline{e}_r = -\frac{\|\underline{v}_c\|^2}{r}\underline{e}_r = -\frac{v_c^2}{r}\underline{e}_r$$

Using Newton's first law:

$$\underline{F} = m\underline{a} = -m\nabla\Phi = -m\frac{\partial\Phi}{\partial r} \quad \text{spherical symmetry}$$

$$\Rightarrow \underline{a} = -\frac{\partial\Phi}{\partial r} = \frac{v_c^2}{r}$$

For the gravitational potential in spherical symmetry, we find an expression for the $\frac{\partial\Phi}{\partial r}$ via the Poisson equation:

$$\Delta\Phi = 4\pi G\rho(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) \quad \text{in sph. sym.}$$

$$\Rightarrow \partial\left(r^2\frac{\partial\Phi}{\partial r}\right) = -4\pi G\rho(r)\,r^2\,\partial r$$

$$\Rightarrow r^2\frac{\partial\Phi}{\partial r} = -4\pi G\int\rho(r)r^2dr =$$

$$= -G\int 4\pi\rho(r)r^2dr = -G\int\rho(\underline{x})dV =$$

$$= -GM(<r)$$

$$\Rightarrow \frac{\partial\Phi}{\partial r} = \frac{-GM(<r)}{r^2}$$

This finally gives us:

$$\underline{a} = -\frac{\partial \phi}{\partial r} = \frac{v_c{}^2}{r} = +\frac{GM(<r)}{r^2}$$

$$\Rightarrow \boxed{v_c{}^2 = +\frac{GM(<r)}{r}}$$