Multipole Expansion

$$\phi(\vec{r}) = -G \sum_i \frac{m_i}{|\vec{r}-\vec{p}_i|} = -G \sum_i \frac{m_i}{|\vec{r}-\vec{s}+\vec{s}-\vec{p}_i|} = -G \sum \frac{m_i}{|\vec{y}+\vec{s}-\vec{p}_i|}$$

$\vec{s}$: centre of mass of a particle group

$\vec{p}_i$: particle position

Expand $\frac{1}{|\vec{y}+\vec{s}-\vec{p}_i|}$ using Taylor:

$$f(x_1, \dots, x_n) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[\sum_{k=1}^{n} (x_k - a_k) \frac{\partial}{\partial x_k'} \right]^j f(x_1', \dots, x_n') \right\} \Big|_{x_1' = a_1, \dots, x_n' = a_n}$$

In this case: $n = 3$

$$f(x,y,z) = f(\vec{y}) = \frac{1}{[(x+s_x-p_x)^2 + (y+s_y-p_y)^2 + (z+s_z-p_z)^2]^{1/2}}$$

Expand around $\vec{a} = \vec{y} - \vec{s} + \vec{p}$

Let $\vec{s} - \vec{p} \equiv \vec{c}$

$\Rightarrow \quad \vec{a} = \vec{y} - \vec{c}$

$$f(x,y,z) = \frac{1}{[(x+c_x)^2 + (y+c_y)^2 + (z+c_z)^2]^{1/2}}$$

Preparation: Derivative Polynomials

$$P[j] = \left[(x-a_x)\frac{\partial}{\partial x} + (y-a_y)\frac{\partial}{\partial y} + (z-a_z)\frac{\partial}{\partial z} \right]^j$$

We defined: $\vec{a} = \vec{y} - \vec{c}$

$$\Rightarrow P[j] = \left[c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z} \right]^j$$

$j=0$: $\quad P[0] = 1$

$j=1$: $\quad P[1] = c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z}$

$j=2$: $\quad P[2] = c_x^2 \frac{\partial^2}{\partial x^2} + c_y^2 \frac{\partial}{\partial y^2} + c_z^2 \frac{\partial}{\partial z^2} +$

$$+ 2 c_x c_y \frac{\partial^2}{\partial x \partial y} + 2 c_x c_z \frac{\partial^2}{\partial x \partial z} + 2 c_y c_z \frac{\partial^2}{\partial y \partial z}$$

$j=3$: $\quad P[3] = c_x^3 \frac{\partial^3}{\partial x^3} + c_y^3 \frac{\partial^3}{\partial y^3} + c_z^3 \frac{\partial^3}{\partial z^3} +$

$$+ 3 c_x^2 c_y \frac{\partial^3}{\partial x^2 \partial y} + 3 c_x c_y^2 \frac{\partial^3}{\partial x \partial y^2} +$$

$$+ 3 c_x^2 c_z \frac{\partial^3}{\partial x^2 \partial z} + 3 c_x c_z^2 \frac{\partial^3}{\partial x \partial z^2} +$$

$$+ 3 c_y^2 c_z \frac{\partial^3}{\partial y^2 \partial z} + 3 c_y c_z \frac{\partial^3}{\partial y \partial z^2}$$

$$+ 6 c_x c_y c_z \frac{\partial^3}{\partial x \partial y \partial z}$$

Preparation: Derivatives

$$f(x,y,z) = \frac{1}{[(x+c_x)^2+(y+c_y)^2+(z+c_z)^2]} = f(\vec{y}) = \frac{1}{|\vec{y}+\vec{c}|}$$

Need derivatives at point $\vec{y}_0 = \vec{y} - \vec{c}$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\frac{1}{|\vec{y}\,'+\vec{c}|^3}\cdot 2(x'+c_x)$$

$$= \boxed{-\frac{x'+c_x}{|\vec{y}\,'+\vec{c}|^3}}$$

$$\frac{\partial f}{\partial y} = \boxed{-\frac{y'+c_y}{|\vec{y}\,'+\vec{c}|^3}}$$

$$\frac{\partial f}{\partial z} = \boxed{-\frac{z'+c_z}{|\vec{y}\,'+\vec{c}|^3}}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial}{\partial x'}\left(\frac{x'+c_x}{|\vec{y}\,'+\vec{c}|^3}\right) = -\frac{|\vec{y}\,'+\vec{c}|^3 - (x'+c_x)\frac{3}{2}|\vec{y}\,'+\vec{c}|\cdot 2(x'+c_x)}{|\vec{y}\,'+\vec{c}|^6}$$

$$= \boxed{\frac{|\vec{y}\,'+\vec{c}|^2 - 3(x'+c_x)^2}{|\vec{y}\,'+\vec{c}|^5}}$$

$$\frac{\partial^2 f}{\partial y^2} = \boxed{-\frac{|\vec{y}\,'+\vec{c}|^2 - 3(y'+c_y)^2}{|\vec{y}\,'+\vec{c}|^5}}$$

$$\frac{\partial^2 f}{\partial y^2} = \boxed{-\frac{|\vec{y}\,'+\vec{c}|^2 - 3(z'+c_z)^2}{|\vec{y}\,'+\vec{c}|^5}}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -\frac{\partial}{\partial y}\left(\frac{x'+c_x}{|\vec{y}'+\vec{c}|^3}\right) = +\frac{3}{2}\,\frac{x'+c_x}{|\vec{y}'+\vec{c}|^5}\,2(y'+c_y)$$

$$= \boxed{3\,\frac{(x'+c_x)(y'+c_y)}{|\vec{y}'+\vec{c}|^5}}$$

$$\frac{\partial^2 f}{\partial x\,\partial z} = \boxed{3\,\frac{(x'+c_x)(z'+c_z)}{|\vec{y}'+\vec{c}|^5}}$$

$$\frac{\partial^2 f}{\partial y\,\partial z} = \boxed{3\,\frac{(y'+c_y)(z'+c_z)}{|\vec{y}'+\vec{c}|^5}}$$

$$\frac{\partial^3 f}{\partial x^3} = -\frac{\partial}{\partial x}\left(\frac{|\vec{y}'+\vec{c}|^2 - 3(x'+c_x)^2}{|\vec{y}'+\vec{c}|^5}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{1}{|\vec{y}'+\vec{c}|^3} - \frac{3(x'+c_x)^2}{|\vec{y}'+\vec{c}|^5}\right)$$

$$= -\Big[-\frac{3}{2}\,\frac{2(x'+c_x)}{|\vec{y}'+\vec{c}|^5} - \frac{6(x'+c_x)|\vec{y}'+\vec{c}|^5 - 3(x'+c_x)^2\cdot\frac{5}{2}|\vec{y}'+\vec{c}|^3\,2(x'}{|\vec{y}'+\vec{c}|^{10}}$$

$$= 3\Big[\frac{(x'+c_x)}{|\vec{y}'+\vec{c}|^5} + \frac{2(x'+c_x)|\vec{y}'+\vec{c}|^2 - 5(x'+c_x)^3}{|\vec{y}'+\vec{c}|^7}\Big]$$

$$= \frac{+3(x'+c_x)|\vec{y}'+\vec{c}|^2 - 15(x'+c_x)^3}{|\vec{y}'+\vec{c}|^7}$$

$$= \boxed{+3(x'+c_x)\,\frac{|\vec{y}'+\vec{c}|^2 - 5(x'+c_x)^2}{|\vec{y}'+\vec{c}|^7}}$$

$$\frac{\partial^3 f}{\partial y^3} = \boxed{+3\,(y'+c_y)\,\frac{|\vec{y}'+\vec{c}|^2 - (y'+c_y)^2\cdot 5}{|\vec{y}'+\vec{c}|^7}}$$

$$\frac{\partial^3 f}{\partial z^3} = \boxed{+3\,(z'+c_z)\,\frac{|\vec{y}'+\vec{c}|^2 - (z'+c_z)^2\cdot 5}{|\vec{y}'+\vec{c}|^7}}$$

$$\frac{\partial^3 f}{\partial x^2\partial y} = -\frac{\partial}{\partial y}\left(\frac{|\vec{y}'+\vec{c}|^2 - 3(x'+c_x)^2}{|\vec{y}'+\vec{c}|^5}\right)$$

$$= -\frac{\partial}{\partial y}\left(\frac{1}{|\vec{y}'+\vec{c}|^3} - \frac{3(x'+c_x)^2}{|\vec{y}'+\vec{c}|^5}\right)$$

$$= \frac{3(y'+c_y)}{|\vec{y}'+\vec{c}|^5} - \frac{3(x'+c_x)^2\cdot\frac{5}{2}(y'+c_y)\cdot 2}{|\vec{y}'+\vec{c}|^7}$$

$$= \boxed{\frac{3(y'+c_y)}{|\vec{y}'+\vec{c}|^7}\left(-5\,(x'+c_x)^2 + |\vec{y}'+\vec{c}|^2\right)}$$

$$\frac{\partial^3 f}{\partial x^2\partial z} = \boxed{\frac{3(z'+c_z)}{|\vec{y}'+\vec{c}|^7}\left(|\vec{y}'+\vec{c}|^2 - 5(x'+c_x)^2\right)}$$

$$\frac{\partial^3 f}{\partial y^2\partial x} = \boxed{\frac{3(x'+c_x)}{|\vec{y}'+\vec{c}|^7}\left(|\vec{y}'+\vec{c}|^2 - 5(y'+c_y)^2\right)}$$

$$\frac{\partial^3 f}{\partial y^2\partial z} = \boxed{\frac{3(z'+c_z)}{|\vec{y}'+\vec{c}|^7}\left(|\vec{y}'+\vec{c}|^2 - 5(y'+c_y)^2\right)}$$

$$\frac{\partial^3 f}{\partial z^2\partial x} = \boxed{\frac{3(x'+c_x)}{|\vec{y}'+\vec{c}|^7}\left(|\vec{y}'+\vec{c}|^2 - 5(z'+c_z)^2\right)}$$

$$\frac{\partial^3 f}{\partial z^2\partial y} = \boxed{\frac{3(y'+c_y)}{|\vec{y}'+\vec{c}|^7}\left(|\vec{y}'+\vec{c}|^2 - 5(z'+c_z)^2\right)}$$

$$\frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial z}\left(3\, \frac{(x'+c_x)(y'+c_y)}{|\vec{y}\,'+\vec{c}|^5} \right)$$

$$= -15\, \frac{(x'+c_x)(y'+c_y)(z'+c_z)}{|\vec{y}\,'+\vec{c}|^7}$$

Now inserting actual value $\left.\frac{\partial f}{\partial \ldots}\right|_{\vec{y}' = \vec{y}_0 = \vec{y} - \vec{c} = \vec{r}}$:

$$\Rightarrow \quad x' + c_x = x - c_x + c_x = x$$

$$y' + c_y = y$$

$$z' + c_z = z$$

$$|\vec{y}' + \vec{c}| = |\vec{y}|$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = -\frac{x}{|\vec{y}|^3}, \quad \frac{\partial f}{\partial y} = -\frac{y}{|\vec{y}|^3}, \quad \frac{\partial f}{\partial z} = -\frac{z}{|\vec{y}|^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-|\vec{y}|^2 + 3x^2}{|\vec{y}|^5}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{-|\vec{y}|^2 + 3y^2}{|\vec{y}|^5}, \quad \frac{\partial^2 f}{\partial z^2} = \frac{-|\vec{y}|^2 + 3z^2}{|\vec{y}|^5}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3\frac{xy}{|\vec{y}|^5}, \quad \frac{\partial^2 f}{\partial x \partial z} = 3\frac{xz}{|\vec{y}|^5}, \quad \frac{\partial^2 f}{\partial y \partial z} = 3\frac{yz}{|\vec{y}|^5}$$

$$\frac{\partial^3 f}{\partial x^3} = 3x\frac{|\vec{y}|^2 - 5x^2}{|\vec{y}|^7}, \quad \frac{\partial^3 f}{\partial y^3} = 3y\frac{|\vec{y}|^2 - 5y^2}{|\vec{y}|^7}, \quad \frac{\partial^3 f}{\partial z^3} = 3z\frac{|\vec{y}|^2 - 5z}{|\vec{y}|^7}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{3y}{|\vec{y}|^7}\left(|\vec{y}|^2 - 5x^2\right), \quad \frac{\partial^3 f}{\partial x^2 \partial z} = \frac{3z}{|\vec{y}|^7}\left(|\vec{y}|^2 - 5x^2\right)$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{3x}{|\vec{y}|^7}\left(|\vec{y}|^2 - 5y^2\right), \quad \frac{\partial^3 f}{\partial y^2 \partial z} = \frac{3z}{|\vec{y}|^7}\left(|\vec{y}|^2 - 5y^2\right)$$

$$\frac{\partial^3 f}{\partial z^2 \partial x} = \frac{3x}{|\vec{y}|^7}\left(|\vec{y}|^2 - 5z^2\right), \quad \frac{\partial^3 f}{\partial z^2 \partial y} = \frac{3y}{|\vec{y}|^7}\left(|\vec{y}|^2 - 5z^2\right)$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = -15\frac{xyz}{|\vec{y}|^7}$$

Pole calculation

$$T_n = \frac{1}{n!}\left[\sum_{k=1}^{j}(x_j - a_j)\frac{\partial}{\partial x_j'}\right]^n f(x_1', \dots, x_n')\Big|_{x_1'=a_1, \dots, x_n'=a_n}$$

$$f(x,y,z) = \frac{1}{[(x+c_x)^2 + (y+c_y)^2 + (z+c_z)^2]^{1/2}} \Rightarrow f(\vec{y}' = \vec{a}) = \frac{1}{|\vec{y}|}$$

Zeroth Order

$$T_0 = 1 \cdot 1 \cdot f(\vec{y}')\Big|_{\vec{a}} = \boxed{\frac{1}{|\vec{y}|}}$$

First Order

$$T_1 = 1 \cdot \left[\quad c_x \frac{\partial}{\partial x'} + \quad c_y)\frac{\partial}{\partial y'} + \quad c_z \frac{\partial}{\partial z'}\right] f$$

$$= -c_x \frac{\partial f}{\partial x} - c_y \frac{\partial f}{\partial y} - c_z \frac{\partial f}{\partial z} = -c_x \frac{x}{|\vec{y}|^3} - c_y \frac{y}{|\vec{y}|^3} - c_z \frac{y}{|\vec{y}|^3}$$

$$\boxed{= -\frac{\vec{y}\cdot\vec{c}}{|\vec{y}|^3} = -\frac{\vec{y}\cdot(\vec{s}-\vec{\rho})}{|\vec{y}|^3}}$$

Second Order

$$T_2 = \frac{1}{2}\left[c_x^2 \frac{\partial^2}{\partial x^2} + c_y^2 \frac{\partial}{\partial y^2} + c_z^2 \frac{\partial}{\partial z^2} + 2c_x c_y \partial_x \partial_y + 2c_x c_z \partial_x \partial_z + 2c_y c_z \partial_y \partial_z\right]$$

$$= \frac{1}{2}\left[\frac{-c_x^2|\vec{y}|^2 + 3c_x^2 x^2}{|\vec{y}|^5} \quad \frac{-c_y^2|\vec{y}|^2 + 3c_y^2 y^2}{|\vec{y}|^5} \quad \frac{-c_z^2|\vec{y}|^2 + 3c_z^2 z^2}{|\vec{y}|^5}\right] +$$

$$+ 3\frac{c_x x\, c_y y}{|\vec{y}|^5} + 3\frac{c_x x\, c_z z}{|\vec{y}|^5} + 3\frac{c_y y\, c_z z}{|\vec{y}|^5} =$$

$$= \frac{1}{|\vec{y}|^5}\left[\frac{3}{2}\left(c_x^2 x^2 + c_y^2 y^2 + c_z^2 z^2 + 2c_x x c_y y + 2c_x x c_z z + 2c_y y c_z z\right) - \right.$$

$$\left. - \frac{|\vec{y}|^2}{2}(c_x^2 + c_y^2 + c_z^2)\right]$$

Use $\vec{c} \cdot \vec{c}^T = \begin{pmatrix} c_x^2 & c_x c_y & c_x c_z \\ c_x c_y & c_y^2 & c_y c_z \\ c_x c_z & c_y c_z & c_z^2 \end{pmatrix}$

Then $(\vec{c} \cdot \vec{c}^T)\vec{y} = \begin{pmatrix} c_x^2 x + c_x c_y y + c_x c_z z \\ c_x c_y y + c_y^2 y + c_y c_z z \\ c_x c_z x + c_y c_z y + c_z^2 z \end{pmatrix}$

Then $\vec{y}^T = (x, y, z)$

$$\vec{y}^T\left[(\vec{c} \cdot \vec{c}^T)\vec{y}\right] = c_x^2 x^2 + c_y^2 y^2 + c_z^2 z^2 + 2c_x c_y xy + 2c_x c_z xz + 2c_y c_z yz$$

Also: $|\vec{y}|^2 (c_x^2 + c_y^2 + c_z^2) = |\vec{y}|^2 |\vec{c}|^2 = \vec{y}^T \vec{y}\, |\vec{c}|^2 = \vec{y}^T |\vec{c}|^2 \vec{y}$

$$\Rightarrow \boxed{T_2 = \frac{1}{2} \frac{1}{|\vec{y}|^5} \vec{y}^T \left(3(\vec{c} \cdot \vec{c}^T) - |\vec{c}|^2 \right) \vec{y}}$$