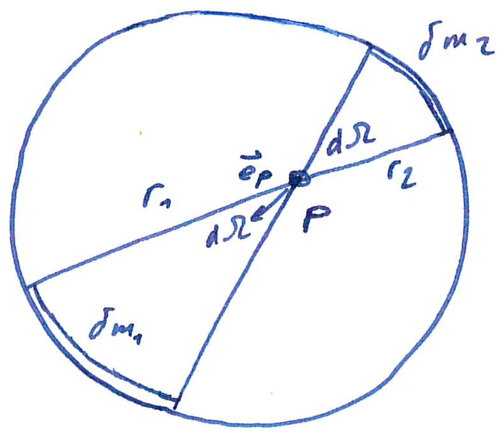
Newton's Theorems

First Theorem

A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.

Derivation:



Draw two lines (cones) arbitrarily through an arbitrary point P inside a uniform spherical shell of width dr and density ρ.

Then the mass enclosed in these cones is given by

$$\begin{cases} \delta m_1 = \rho \, d\Omega \, r_1^2 \, dr \\ \delta m_2 = \rho \, d\Omega \, r_2^2 \, dr \end{cases} \Rightarrow \quad \frac{\delta m_1}{\delta m_2} = \frac{r_1^2}{r_2^2}$$

The force extorted by the shell sections are

$$\vec{F}_1 = -\frac{G\, m\, \delta m_1}{r_1^2}\, \vec{e}_p \quad , \quad \vec{F}_2 = +\frac{G\, m\, \delta m_2}{r_2^2}\, \vec{e}_p$$

where m is the test particle's mass at the point p and $\vec{e}_p$ the unit vector pointing from p to the center of the shell section with mass δm_1.

Using our previous findings:

$$\frac{\delta m_1}{\delta m_2} = \frac{r_1^2}{r_2^2} \quad \Rightarrow \quad \frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}$$

$$\Rightarrow \vec{F}_1 = -\vec{F}_2$$

So the sum of all forces will always cancel out, in particular when we sum over all cone sections such that the entire shell is covered.

Corollary:

Since $\vec{F}_{tot} = 0$, $\quad \phi = \text{const}$

$$m \vec{\nabla} \phi = \vec{F}_{tot} \quad \Rightarrow \quad \vec{\nabla} \phi = 0 \quad \Rightarrow \quad \phi = \text{const}$$

What is the value of $\phi(\vec{x})$?

Since $\phi = \text{const}$, we can compute it at any point. It's easiest for $\phi(0)$:

$$\phi(\vec{x}) = - \int_V \frac{G \rho(x')}{|x - x'|} d^3x'$$

$$\phi(0) = - \int \frac{G \rho(x')}{|x'|} d^3x'$$

For a spherical shell:

$$\rho(r') = \frac{M}{4\pi r'^2} \delta(R - r)$$

[Reminder: δ has the inverse dimension of the argument it takes:

$$\int_V \delta(\underline{x}) \, dV = 1$$

and $M = 4\pi \int dr' \, r'^2 \rho(r') = 4\pi \int dr' \, r'^2 \frac{M}{4\pi r'^2} \delta(R - r) = M$]

Then

$$\phi(r) = - \int d\Omega \int dr' r'^2 \underbrace{\frac{GM}{4\pi r'^2} \delta(R-r')}_{\varrho(r)} \cdot \frac{1}{r'} =$$

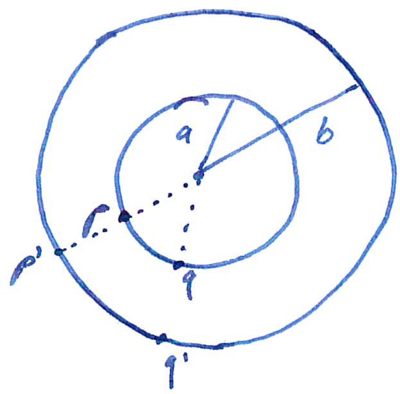
$$= - 4\pi \int dr' \frac{GM}{r'} \delta(R-r') = - \frac{GM}{R}$$

$$\Rightarrow \phi(x) = - \frac{GM}{R} \quad \forall \vec{x} \in \text{sphere}$$

Second Newton Theorem

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.

Consider two shells, an inner one with radius r and an outer one with radius r, both having the mass M.



Because we assume a uniform mass distribution, the mass within the solid angle $\delta\Omega$ is

$$\delta m = M \frac{\delta \Omega}{4\pi}$$

Furthermore, consider two pairs of points, p and q, and p' and q', which enclose the same solid angle on the outer and inner shell, respectively.

We now compare

1) The potential $\phi_i(\vec{p})$, which is the potential of the inner shell at an outside point $\vec{p}$ with distance r from the origin,

with

2) The potential $\phi_o'(\vec{p}\,')$, which is the potential of the outer shell at position $\vec{p}\,'$ which has the distance $a < r$ from the origin.

Consider the contribution $\delta\phi$ to the potential at $\vec{p}$ from the portion of the inner shell with solid angle $d\Omega$ located at $\vec{q}'$. We have:

$$1) \quad \delta\phi = -\frac{GM}{|\vec{p}-\vec{q}|}\frac{\delta\Omega}{4\pi}$$

Similarly, the contribution $\delta\phi'$ of the matter in the outer shell near $\vec{q}$ to the potential at $\vec{p}'$ is

$$2) \quad \delta\phi' = -\frac{GM}{|\vec{p}'-\vec{q}|}\frac{\delta\Omega}{4\pi}$$

As $|\vec{p}'-\vec{q}| = |\vec{p}-\vec{q}'|$, we have

$$\delta\phi = \delta\phi'$$

and by summing over all points $\vec{q}, \vec{q}'$:

$$\phi = \phi'$$

We already know from Newton's first theorem that the potential inside a spherical shell is constant, i.e.

$$\phi' = -\frac{GM}{r}$$

$$\Rightarrow \qquad \phi = -\frac{GM}{r}$$

which is exactly the potential that would be generated by concentrating the entire mass of the inner shell at its center.