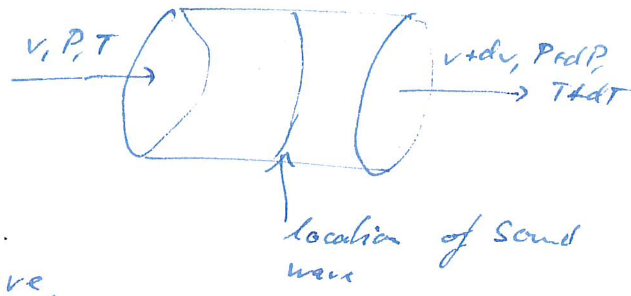
Speed of Sound in an ideal Gas

Consider a sound wave travelling with velocity v.



In the rest frame of the wave, the gas has a velocity v, and properties P, T. As the gas passes through the wave, it changes it's properties to $v+dv$, $P+dP$, $T+dT$.

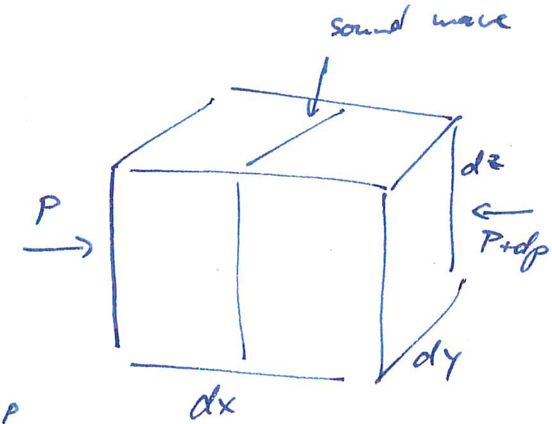
Consider a cross-sectional area perpendicular to the air flow with area A. The gas volume flowing through the area in the time dt is $A\,v\,dt$ with mass $\rho A v\,dt$.

Applying conservation of mass:

$$\rho A v = (\rho + d\rho) A (v + dv)$$

$$\rho v = \rho v + \rho\,dv + v\,d\rho + \underbrace{d\rho\,dv}_{\approx 0} \Rightarrow \boxed{\rho\,dv = -v\,d\rho}$$

Now consider a small volume element crossing through the sound wave.



The net force on the cube is $P\,dy\,dz - (P+dP)\,dy\,dz = -dP\,dy\,dz$, giving the acceleration $a = \frac{-dP\,dy\,dz}{\rho\,dx\,dy\,dz} = -\frac{dP}{\rho\,dx} = \frac{-1}{\rho}\frac{dP}{dx}$

Then $dv = a \cdot dt = a \cdot \frac{dx}{v} = -\frac{1}{\rho}\frac{dP}{dx}\frac{dx}{v} = -\frac{dP}{\rho v}$

$\Rightarrow \rho v \, dv = -dP$

using $\rho dv = -v d\rho$:

$v^2 d\rho = dP$

$\Rightarrow \boxed{v^2 = \frac{dP}{d\rho}}$

For an ideal gas: sound waves are adiabatic

$\rightarrow P V^\gamma = \text{const} = P\left(\frac{m}{\rho}\right)^\gamma$

$\Rightarrow P\rho^{-\gamma} = \text{const} \quad \Rightarrow dP\rho^{-\gamma} + (-\gamma) P \rho^{-\gamma-1} d\rho = 0$

$\Rightarrow \boxed{\frac{dP}{d\rho} = \frac{\gamma P}{\rho}}$