Dynamical Timescale

Characterizes departures from mechanical equilibrium, and free-fall. In free-fall, the internal pressure is negligible.

$$\ddot{r} = \frac{1}{\rho}\frac{\partial P}{\partial \rho} - \frac{GM_r}{r^2} = -\frac{GM_r}{r^2}$$

i) By approximations:

$$\ddot{r} \sim \frac{R_C - R_S}{\tau_{dyn}^2} = \frac{-R}{\tau_{dyn}^2} \sim -\frac{GM}{R^2}$$

$$\Rightarrow \tau_{dyn} \sim \sqrt{\frac{R^3}{GM}} \sim \sqrt{\frac{1}{G\rho}}$$

ii) More exact: again $\frac{dP}{dr} \approx 0$

$$\ddot{r} = -\frac{GM_r}{r^2}$$

$$\Rightarrow \ddot{r}\dot{r} = -\frac{GM_r}{r^2}\dot{r}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) = \frac{d}{dt}\left(\frac{GM_r}{r}\right)$$

$$\Rightarrow \frac{1}{2}\dot{r}^2(t) - \dot{r}^2(t_{ini}) = GM_r\left(\frac{1}{r} - \frac{1}{r_{ini}}\right)$$

Assume $t_{ini} = 0$, $r(t_{ini}) = r_{ini} = \infty$, $\dot{r}(t_{ini}) = 0$,

$$M = \frac{4}{3}\pi r^3 \bar{\rho}$$

$$\Rightarrow \quad \frac{1}{2}\dot{r}^2(t) = \frac{G}{r}\,\frac{4}{3}\pi r^3 \bar{\rho}$$

$$\Rightarrow \quad \dot{r}^2(t) = \frac{8\pi G\bar{\rho}}{3}\, r^2$$

$$\Rightarrow \quad \frac{\dot{r}(t)}{r} = -\sqrt{\frac{8\pi G\bar{\rho}}{3}}$$ take minus for contraction

Assuming $M_r = \text{const} = 4\pi\bar{\rho} r^3$:

$$\frac{dM_r}{M_r} = 0 = \frac{4\pi r^3 d\rho}{4\pi r^3 \rho} + \frac{4\pi\rho\, 3r^2 dr}{4\pi r^3 \rho}$$

$$\Rightarrow \quad \frac{d\rho}{\rho} = -3\,\frac{dr}{r}$$

and

$$\frac{\dot{r}}{r} = \frac{1}{r}\,\frac{dr}{dt} = -\frac{1}{3}\,\frac{1}{\rho}\,\frac{d\rho}{dt} = -\sqrt{\frac{8\pi G\rho}{3}}$$

$$\Rightarrow \quad \frac{1}{\rho}\,\frac{d\rho}{dt} = \sqrt{24\pi G\rho}$$

We define

$$\tau_{ff} = \rho\,\frac{dt}{d\rho} = \frac{1}{\sqrt{24\pi G\rho}}$$