Kelvin-Helmholtz timescale for a slow contraction

Assume slow contraction such that the star is still in equilibrium. Then

$$R \to R - \Delta R, \quad \Omega \to \Omega - \Delta\Omega, \quad U \to U - \Delta U$$

Since star is still in equilibrium, the virial theorem still applies:

$$2 E_{kin} + \Omega = 0$$

We relate the internal energy U to the kinetic energy. For an ideal gas:

$$U = c_v N m_u \mu T \quad , \quad E_{kin} = \frac{3}{2} N k T, \quad c_p - c_v = \frac{k}{m_u \mu}$$

$$c_p / c_v = \gamma$$

$$\Rightarrow U = c_v N m_u \mu T = c_v m_u \mu \frac{2}{3} E_{kin} \frac{1}{k}$$

$$= \frac{2}{3} \frac{c_v}{c_p - c_v} E_{kin} = \frac{2}{3} \frac{1}{\gamma - 1} E_{kin}$$

$$\Rightarrow 2 E_{kin} = 3(\gamma - 1) U = -\Omega$$

This also applies for the changes:

$$3(\gamma - 1) \Delta U + \Delta\Omega = 0$$

$$\Rightarrow \quad \Delta U = \frac{\Delta\Omega}{3(\gamma - 1)}$$

Clearly the gravitational energy change will not be fully converted into energy, the factor $[3(\gamma-1)]^{-1}$ is present.

Then the radiated away energy will be the difference of the lost gravitational energy and what gets absorbed into internal energy:

$$\Delta E_{rad} = (-\Delta\Omega) - \Delta U = -\Delta\Omega + \frac{\Delta\Omega}{3(\gamma-1)} = (-\Delta\Omega)\frac{3\gamma-4}{3\gamma-3}$$

We approximate:

$$E_{rad} = \overline{L}\,t\,, \qquad \Omega = -q\frac{GM^2}{R}$$

Then the timescale to radiate away the entire gravitational energy is

$$E_{rad} = \overline{L}\,t = \frac{3\gamma-4}{3\gamma-3}\,q\,\frac{GM^2}{R}$$

$$\Rightarrow \tau_{KH} \sim \frac{GM^2}{R\overline{L}}$$

It is the timescale during which a star can produce an average luminosity $\overline{L}$ at the expense of gravitational energy.